Nowhere-zero 3-flows in arc-transitive graphs on nilpotent groups

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circulations

Definition

Let $D = (V(D), A(D))$ be a digraph and A an abelian group. A circulation in D over A is a function

$$f : A(D) \to A$$

such that

$$\sum_{a \in A^+(v)} f(a) = \sum_{a \in A^-(v)} f(a), \quad \text{for all } v \in V(D),$$

where $A^+(v)$ ($A^-(v)$, respectively) is the set of arcs of D leaving from v (entering into v, respectively).

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where $A^+(v)$ ($A^-(v)$, respectively) is the set of arcs of D leaving from v (entering into v, respectively).

We say that f is nowhere-zero if $f(a) \neq 0$ for every $a \in A(D)$, where 0 is the identity element of A.

Theorem

(W. Tutte 1954)
A plane digraph is k-face-colorable if and only if it admits a nowhere-zero circulation over $\mathbb{Z}_k$.

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Four-Color-Theorem Restated: Every planar graph admits a nowhere-zero circulation over $\mathbb{Z}_4$.

integer flows

Definition

A nowhere-zero circulation f over $\mathbb{Z}$ in a digraph D is called a (nowhere-zero) *k-flow* if

$$-(k-1) \leq f(a) \leq k-1, \quad \text{for all } a \in A(D)$$

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A graph admits a k-flow if and only if it admits a nowhere-zero circulation over $\mathbb{Z}_k$.

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Theorem

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A graph admits a k-flow if and only if it admits a nowhere-zero circulation over $\mathbb{Z}_k$.

Four-Color-Theorem Again: Every planar graph admits a 4-flow.

Theorem

A graph admits a 2-flow if and only if its vertices all have even degrees.

Theorem

A 2-edge-connected cubic graph admits a 3-flow if and only if it is bipartite.

Tutte's 5-flow conjecture

Tutte proposed three conjectures on integer flows (1954, 1968, 1972).

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Theorem

(The 6-flow theorem, P. Seymour 1981)
Every 2-edge-connected graph admits a 6-flow.

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Every 2-edge-connected graph with no Petersen graph minor admits a 4-flow.

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Theorem

(F. Jaeger 1979)
Every 4-edge-connected graph admits a 4-flow.

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Every 4-edge-connected graph admits a 3-flow.

Theorem

(M. Kochol 2001)
The 3-flow conjecture is true if and only if every 5-edge-connected graph admits a 3-flow.

recent breakthrough

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Every 8-edge-connected graph admits a 3-flow.

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Theorem
(L. M. Lovász, C. Thomassen, Y. Wu and C. Q. Zhang 2013)
Every 6-edge-connected graph admits a 3-flow.

motivation

Theorem

(M. E. Watkins 1969; W. Mader 1970)
Every vertex-transitive graph of valency d is d-edge-connected.

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Every vertex-transitive graph of valency d is d-edge-connected.

Conjecture

(Vertex-transitive version of the 3-flow conjecture)
Every vertex-transitive graph of valency at least 4 admits a 3-flow.

It suffices to prove this for vertex-transitive graphs of valency 5.

3-flows in Cayley graphs on nilpotent groups

Theorem

(P. Potačnik 2005)
Every Cayley graph of valency at least 4 *on a finite abelian group admits a 3-flow.*

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Theorem

(M. Nánásiová and M. Škoviera 2009)
Every Cayley graph of valency at least 4 on a finite nilpotent group admits a 3-flow.

A finite group is nilpotent if it is the direct product of its Sylow subgroups.

an intermediate goal

Prove that every graph of valency at least 4 admitting a nilpotent vertex-transitive group of automorphisms admits a 3-flow.

As before it suffices to prove this for the case of valency 5.

symmetry of graphs

Definition

A graph Γ is G-vertex-transitive (G-edge-transitive, G-arc-transitive, respectively) if it admits G as a group of automorphisms such that G is transitive on the set of vertices (edges, arcs, respectively) of Γ, where an arc is an ordered pair of adjacent vertices.

result so far

Theorem

(X. Li and S. Zhou 2013)
Let G be a finite nilpotent group. Then every G-vertex-transitive and G-edge-transitive graph with valency at least 4 and not divisible by 3 admits a 3-flow.

result so far

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Let G be a finite nilpotent group. Then every G-vertex-transitive and G-edge-transitive graph with valency at least 4 and not divisible by 3 admits a 3-flow.

This together with the LTWZ theorem implies:

Corollary

Let G be a finite nilpotent group. Then every G-vertex-transitive and G-edge-transitive graph with valency at least 4 admits a 3-flow.

- Any G-arc-transitive graph without isolated vertices is G-vertex-transitive and G-edge-transitive
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Therefore, the corollary above is equivalent to the following:

Corollary

Let G be a finite nilpotent group. Then every G-arc-transitive graph with valency at least 4 admits a 3-flow.

nilpotent groups

Definition

The characteristic subgroups $\gamma_i(G)$ of a group G are inductively defined by:

$$\gamma_1(G) = G, \quad \gamma_{i+1}(G) = [\gamma_i(G), G],$$

where for $H, K \leq G$, $[H, K]$ is the subgroup of G generated by the commutators $h^{-1}k^{-1}hk$, $h \in H, k \in K$.

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A group G is nilpotent if there is an integer c such that $\gamma_{c+1}(G) = 1$; the least such c is called the nilpotency class of G, denoted by $c(G)$.

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Nilpotent groups with nilpotency class 1 are precisely abelian groups.

multicovers

Definition

Let Γ be a graph and $\mathcal{P}$ a partition of $V(\Gamma)$.

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Γ is a multicover of the quotient $\Gamma_{\mathcal{P}}$ if for each pair of adjacent $P, Q \in \mathcal{P}$, the subgraph $\Gamma[P, Q]$ of Γ induced by $P \cup Q$ is a t-regular bipartite graph with bipartition $\{P, Q\}$ for some integer $t \geq 1$ independent of P, Q.

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Lemma

Let $k \geq 2$ be an integer. If a graph admits a k-flow, then its multicovers all admit a k-flow.

normal quotients

Definition

Let Γ be a G-vertex-transitive graph, and let $N \trianglelefteq G$.

The set $\mathcal{P}_N$ of N-orbits on $V(\Gamma)$ is a G-invariant partition of $V(\Gamma)$, called a G-normal partition of $V(\Gamma)$.

Denote $\Gamma_N := \Gamma_{\mathcal{P}_N}$.

Lemma

(Praeger 1980's?)
Let Γ *be a connected* G*-vertex-transitive graph, and* $N \trianglelefteq G$ *be intransitive on* $V(\Gamma)$*. Then*

(a) Γ_N *is* G/N*-vertex-transitive under the induced action of* G/N *on* $\mathcal{P}_N$*;*

(b) *for* $P, Q \in \mathcal{P}_N$ *adjacent in* Γ_N*,* $\Gamma[P, Q]$ *is a regular subgraph of* Γ*;*

(c) *if in addition* Γ *is* G*-edge-transitive, then* Γ_N *is* G/N*-edge-transitive and* Γ *is a multicover of* Γ_N*.*

result so far

Theorem

(X. Li and S. Zhou 2013)
Let G be a finite nilpotent group. Then every G-vertex-transitive and G-edge-transitive graph with valency at least 4 and not divisible by 3 admits a 3-flow.

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- If $c(G) = 1$, then G is abelian and so is regular on $V(\Gamma)$; hence Γ is a Cayley graph on G and so the result holds by Potačnik's result.

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- Assume for some $c \geq 1$ the result holds for any finite nilpotent group of nilpotency class c.
- Let G be a finite nilpotent group with nilpotency class $c(G) = c + 1$. Let Γ be a connected G-vertex-transitive and G-edge-transitive graph such that $\mathtt{val}(\Gamma) \geq 4$ and $\mathtt{val}(\Gamma)$ is not divisible by 3.

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- If $\texttt{val}(\Gamma)$ is even, Γ has a 2-flow and so a 3-flow.

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- If $\mathtt{val}(\Gamma)$ is even, Γ has a 2-flow and so a 3-flow.
- Assume $\mathtt{val}(\Gamma) \geq 5$ is odd.

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- $\mathtt{val}(\Gamma_N)$ is a divisor of $\mathtt{val}(\Gamma)$ and so is not divisible by 3.
- If $\mathtt{val}(\Gamma_N) = 1$, then Γ is a regular bipartite graph of valency at least two and so admits a 3-flow.

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- Since Γ is a multicover of Γ_N, Γ admits a 3-flow.
- This completes the proof.

difficulty for vertex- but not edge-transitive graphs

A G-vertex- but not G-edge-transitive graph Γ may not be a multicover of its normal quotients Γ_N.

In fact, in this case blocks of a normal partition are not necessarily independent sets.

This makes a similar induction difficult.

Work in progress. Ideas are welcome.

thank you for your attention