Rank Dominations in Matroids

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Outline

Introduction

Matroids

- Intuitively, an abstract notion of dependence
- Formally, a ground set E and a rank function $\rho : 2^E \rightarrow \mathbb{Z}_{\geq 0}$ define a matroid $M(E, \rho)$ if:

  (R1) For all $X \subseteq E$, $\rho(X) \leq |X|$,

  (R2) For all $X \subseteq Y \subseteq E$, $\rho(X) \leq \rho(Y)$, and

  (R3) (Submodularity) For all $X, Y \subseteq E$, $\rho(X \cup Y) + \rho(X \cap Y) \leq \rho(X) + \rho(Y)$.

Introduction: Matroids

Some Matroid Definitions

Independent Set. A set $X \subseteq E$ such that $\rho(X) = |X|$.

Basis. A set $X \subseteq E$ such that $\rho(X) = |X| = \rho(E)$.

Circuit. A non-empty set $X \subseteq E$ such that for every $x \in X$, $\rho(X \setminus \{x\}) = |X| - 1 = \rho(X)$.

Examples

- Cycle matroids of graphs. Here E is the edge set, and $\rho(X)$ is the maximum of size of all forests in G that can be formed with edges in X.

Introduction: Matroid Minors

Deletion. For any $X \subseteq E$, the matroid deletion of X gives a new matroid $M' = M \setminus X$ with ground set $E' = E \setminus X$ and rank function $\rho'(Y) = \rho(Y)$ for all $Y \subseteq E'$. M' may also be written as $M|E'$.

Contraction. For any $X \subseteq E$, the matroid contraction of X gives a new matroid $M'' = M/X$ with ground set $E'' = E \setminus X$ and rank function $\rho''(Y) = \rho(Y \cup X) - \rho(X)$ for all $Y \subseteq E''$.

Minor. A matroid N is a minor of M if $N = M/X \setminus Y$ for some disjoint sets $X, Y \subseteq E$.

Introduction: Rank Dominations

Notation

- Given mutually disjoint sets $P_1, P_2, R \subseteq E$, we define $S(P_1, P_2, R) = \{(X, Y) | X = P_1 \cup C_R, Y = P_2 \cup (R \setminus C_R), C_R \subseteq R\}$.
- In other words, $S(P_1, P_2, R)$ is the collection of all disjoint pairs $(X, Y) \in 2^E \times 2^E$ such that $X \cup Y = P_1 \cup P_2 \cup R$ with $P_1 \subseteq X$ and $P_2 \subseteq Y$.
- Clearly, $|S(P_1, P_2, R)| = 2^{|R|}$.

Introduction: Rank Dominations

We say $S(P_1, P_2, R)$ is rank dominated by $S(P'_1, P'_2, R)$ in matroid $M(E, \rho)$ if there exists a bijective map $\pi : S(P_1, P_2, R) \rightarrow S(P'_1, P'_2, R)$ such that whenever $\pi(W, Z) = (X, Y)$ we have

$$\rho(W) + \rho(Z) \leq \rho(X) + \rho(Y).$$

π is called a rank dominating bijection.
We write $S(P_1, P_2, R) \leq_\rho S(P'_1, P'_2, R)$.

The Problem

Submodularity

Equivalent to $S(P, \phi, \phi) \leq_\rho S(P_1, P_2, \phi)$ in any matroid for all $P, P_1, P_2 \subseteq E$, $P = P_1 \cup P_2$.

The Question

For all $P, P_1, P_2, R \subseteq E$, $P_1 \cup P_2 = P$, is it true that $S(P, \phi, R) \leq_\rho S(P_1, P_2, R)$?

Some Quick Answers

- $S(P, \phi, \phi) \leq_\rho S(P_1, P_2, \phi)$.
- $S(P, \phi, R) \leq_\rho S(P, \phi, R)$ and $S(P, \phi, R) \leq_\rho S(\phi, P, R)$.

The Theorem Statement

In any matroid $M(E, \rho)$, given $P, P_1, P_2, R \subseteq E$, $P_1 \cup P_2 = P$, we have $S(P, \phi, R) \leq_\rho S(P_1, P_2, R)$ whenever $|R| \leq 3$.

The Proof

For any $R \subseteq E$, a minor family
$\mathcal{MF}(M, R) = \{M/C_R \setminus (R \setminus C_R) | C_R \subseteq R\}$.

Lemma 1

If for every $N \in \mathcal{MF}(M, R)$, $\rho_N(W) + \rho_N(Z) \leq \rho_N(X) + \rho_N(Y)$ then $S(W, Z, R) \leq_\rho S(X, Y, R)$ in matroid M whenever $|R| \leq 3$.

The Proof(Contd.,)

Lemma 2

Suppose $\rho_N(W) + \rho_N(Z) \leq \rho_N(X) + \rho_N(Y)$ for every $N \in \mathcal{MF}(M, R)$.

Then whenever $1 \leq |R| \leq 3$, there exists an $r \in R$ and a bijection $\pi_r : S(W, Z, \{r\}) \to S(X, Y, \{r\})$ such that π_r is rank dominating in every $N \in \mathcal{MF}(M, R \setminus \{r\})$.

Proof of Lemma 2

Case $|R| = 1$

Let $R = \{r\}$. If $\rho_N(W) + \rho_N(Z) \leq \rho_N(X) + \rho_N(Y)$ for all $N \in \{M \setminus r, M/r\}$, then

$$\rho(W) + \rho(Z) \leq \rho(X) + \rho(Y), \tag{1}$$

and

$$\rho(W \cup \{r\}) + \rho(Z \cup \{r\}) \leq \rho(X \cup \{r\}) + \rho(Y \cup \{r\}). \tag{2}$$

(1) and (2) imply $S(W, Z, \{r\}) \leq_\rho S(X, Y, \{r\})$.

Proof of Lemma 2 (Contd.,)

Cases $|R| = 2$

- Let $R = \{r_1, r_2\}$. We know by the previous case, $S(W, Z, \{r_1\}) \leq_{\rho_N} S(X, Y, \{r_1\})$ for $N \in \{M \setminus r_2, M/r_2\}$.
- There are two possible choices for the rank dominating bijections in each of the matroids $M \setminus r_2$ and M/r_2.
- We show by contradiction that there must be at least one common rank dominating map between the two matroids.
- A similar proof also works for the case $|R| = 3$.

The Rank Domination Conjecture

The Conjecture

For all $P, P_1, P_2, R \subseteq E$, if $P_1 \cup P_2 = P$ then $S(P, \phi, R) \leq_\rho S(P_1, P_2, R)$ whenever R is independent.

Current Status

- Known to be true when $|R| = 4$.
- Known to be false when R is not independent and $|R| = 4$.
- Can be reduced to the case where P is independent.

A Consequence of Matroid Rank Dominations

A Correlation Inequality for Tutte Polynomials [Paper in preparation]

Let $M(E,\rho)$ be a matroid, $E = E_1 \cup E_2$, $E_X = E_1 \cap E_2$, $M_1 = M|E_1$, $M_2 = M|E_2$ and $M_X = M|E_X$.
Now, if for all $P \subseteq E \setminus E_X$, $P_1 \subseteq E_1$, $P_2 \subseteq E_2$, $P_1 \cup P_2 = P$ and $R \subseteq E_X$, we have $S(P,\phi,R) \leq_\rho S(P_1,P_2,R)$ then for any $x, y \geq 1$,

$$(x-1)^k \cdot T(M;x,y) \cdot T(M_X;x,y) \leq T(M_1;x,y) \cdot T(M_2;x,y),$$

where $k = \rho(E_1) + \rho(E_2) - \rho(E) - \rho(E_X)$.

Conclusion

- Matroid rank dominations extend rank submodularity.
- Useful in showing correlation inequalities of matroid polynomials.
- Yet to resolve conjecture when R is independent.