A Correlation Inequality for Whitney-Tutte Polynomials

Arun Mani

Clayton School of Information Technology
Monash University

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Outline

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Question: Do counting problems have nice properties akin to the Chromatic polynomials of clique separable graphs?

$$P(G;\lambda) = \frac{P(G_1;\lambda)\cdot P(G_2;\lambda)}{P(K_r;\lambda)},$$

if G can be separated into graphs G_1 and G_2 both containing a common clique K_r.

Background

Graphs: Some Terminology

Graphs. A graph $G(V, E)$ has a vertex set V and an edge set E.
The set E may contain loops and/or parallel edges.
Throughout this discussion we only refer to the edge set E. (And restrict ourselves to matroid operations of a graph.)

Forests. A set $X \subseteq E$ is a forest of G if the edges in X do not contain a cycle in G.

Background

Graphs: Rank

An integer function $\rho : 2^E \rightarrow \mathbb{Z}_{\geq 0}$. Given $X \subseteq E$,

$$\rho(X) = \max\{|F| : F \subseteq X \text{ is a forest of } G\}$$

Rank Submodularity

For any two sets $X, Y \subseteq E$,

$$\rho(X \cup Y) + \rho(X \cap Y) \leq \rho(X) + \rho(Y).$$

Background

The Whitney Rank Generating Function

$$R(G; x, y) = \sum_{Z \subseteq E} \left(x^{(\rho(E) - \rho(Z))} \cdot y^{(|Z| - \rho(Z))} \right)$$
$$= x^{\rho(E)} \sum_{Z \subseteq E} y^{|Z|} (xy)^{-\rho(Z)}$$

The Tutte Polynomial

$$T(G; x, y) = R(G; x - 1, y - 1)$$

Background

Properties of $R(G; x, y)$

- $R(G; 1, 0) = T(G; 2, 1)$ counts the number of forests of G.
- $R(G; 0, 0) = T(G; 1, 1)$ counts the number of spanning forests of G.
- $R(G; 0, 1) = T(G; 1, 2)$ counts the number of spanning subgraphs of G.
- And much much more . . . (See Welsh, “Complexity: Knots, Colouring and Counting”, CUP, 1993)

The Correlation Inequalities for $R(G; x, y)$

The Basics

- Graphs G, G_1, G_2 and G_{12} are such that $E_1 \cup E_2 = E$ and $E_1 \cap E_2 = E_{12}$.
- Intuitively, the edge set of graph G is decomposed into smallers graphs G_1 and G_2 with their common edges forming the graph G_{12}.

The Inequalities (2 of 3)

The “Easy” Regions

Let $k = \rho(E_1) + \rho(E_2) - \rho(E) - \rho(E_{12})$. Then :

- Along the curve $xy = 1$,

$$x^k \cdot R(G; x, y) \cdot R(G_{12}; x, y) = R(G_1; x, y) \cdot R(G_2; x, y).$$

- In the region, $x, y > 0$, $xy > 1$,

$$x^k \cdot R(G; x, y) \cdot R(G_{12}; x, y) \geq R(G_1; x, y) \cdot R(G_2; x, y).$$

The Inequalities (3 of 3)

The "Hard" Region

$k = \rho(E_1) + \rho(E_2) - \rho(E) - \rho(E_{12})$:

- In the region, $x, y \geq 0$, $xy < 1$, whenever $|E_{12}| \leq 3$,

$$x^k \cdot R(G; x, y) \cdot R(G_{12}; x, y) \leq R(G_1; x, y) \cdot R(G_2; x, y).$$

- Conjecture: Above is true without the red text.

The Inequalities (Contd.,)

Corollaries

- $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_{12}$ be the set of forests of graphs G, G_1, G_2 and G_{12} respectively. Then if $|E_{12}| \leq 3$:

$$|\mathcal{F}| \leq \frac{|\mathcal{F}_1| \cdot |\mathcal{F}_2|}{|\mathcal{F}_{12}|}.$$

- $\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_{12}$ be the set of spanning forests of graphs G, G_1, G_2 and G_{12} respectively. If also $\rho(E) + \rho(E_{12}) = \rho(E_1) + \rho(E_2)$ and $|E_{12}| \leq 3$, then:

$$|\mathcal{T}| \leq \frac{|\mathcal{T}_1| \cdot |\mathcal{T}_2|}{|\mathcal{T}_{12}|}.$$

The Proof

The Idea

- Let $f(x,y) = R(G_1; x, y) \cdot R(G_2; x, y) - x^k \cdot R(G) \cdot R(G_{12})$.
- Show $f(x,y)$ is either zero or $(1 - xy)$ is a factor of $f(x,y)$ (easy).
- When non-zero, show that the function $\dfrac{f(x,y)}{1-xy}$ is positive in the region $x, y \geq 0, xy \neq 1$ (hard).

The Proof (Contd.,)

Step I

$$f(x,y) = x^{\rho(E_1)+\rho(E2)} \cdot g(x,y),$$

where,

$$g(x,y) = \sum_{(X,Y)\in 2^{E_1}\times 2^{E_2}} \left(y^{|X|+|Y|} \cdot (xy)^{-\rho(X)-\rho(Y)} \right) - \sum_{(W,Z)\in 2^{E}\times 2^{E_{12}}} \left(y^{|W|+|Z|} \cdot (xy)^{-\rho(W)-\rho(Z)} \right).$$

The Proof (Contd.,)

Step II

$$g(x,y) = \sum_{(P,Q)\in 2^{E}\times 2^{E_{12}}} y^{|P|+|Q|}\cdot h_{P,Q}(x,y),$$

where,

$$h_{P,Q}(x,y) = \sum_{\substack{(X,Y)\in 2^{E_1}\times 2^{E_2}\\ X\cup Y=P, X\cap Y=Q}} \left((xy)^{-\rho(X)-\rho(Y)}\right) - \sum_{\substack{(W,Z)\in 2^{E}\times 2^{E_{12}}\\ W\cup Z=P, W\cap Z=Q}} \left((xy)^{-\rho(W)-\rho(Z)}\right).$$

The Proof (Contd.,)

Step III

- Let $P_1 = P \cap (E_1 \setminus E_{12})$, $P_2 = P \cap (E_2 \setminus E_{12})$ and $R = (P \setminus Q) \cap E_{12}$.
- Suppose $S(P_1, P_2, Q, R)$ is the set of all $(X, Y) \in 2^{E_1} \times 2^{E_2}$ with $X \cup Y = P$ and $X \cap Y = Q$.
- Note $|S(P_1, P_2, Q, R)| = 2^{|R|}$.
- Similarly define $S(P, \phi, Q, R)$ to be the set of all $(W, Z) \in 2^{E} \times 2^{E_{12}}$ with $W \cup Z = P$ and $W \cap Z = Q$.

The Proof (Contd.,)

Step III (Contd.,)

Suppose $\pi : S(P_1, P_2, Q, R) \to S(P, \phi, Q, R)$ is a bijection such that whenever $\pi(X, Y) = (W, Z)$,

$$\rho(X) + \rho(Y) \geq \rho(W) + \rho(Z).$$

We know π exists whenever $|R| \leq 3$, and conjecture that it exists whenever R is a forest.

The Proof (Contd.,)

Step III (Contd.,)

Then,

$$h_{P,Q}(x,y) = \sum_{\substack{(X,Y)\in 2^{E_1}\times 2^{E_2} \\ X\cup Y=P, X\cap Y=Q \\ \pi(X,Y)=(W,Z)}} \left((xy)^{-\rho(X)-\rho(Y)} - (xy)^{-\rho(W)-\rho(Z)}\right).$$

- Each term in the sum is either zero or contains $(1-xy)$ as a factor.
- When the sum is non-zero, $\dfrac{h_{P,Q}(x,y)}{1-xy}$ is positive in the region $x, y \geq 0, xy \neq 1$.

Concluding Remarks

- The approach "shows" all three inequalities at once.
- (Partially) Extends the inequality to regions where applying the Four Function theorem is not possible.
- Can we get an approximation algorithm for counting problems?
- Maybe hard to prove cases where E_{12} contains a cycle.