A Correlation Inequality for Whitney-Tutte Polynomials

Arun Mani

Clayton School of Information Technology
Monash University

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Outline

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Kerri Morgan

Question: Do counting problems have nice properties akin to the Chromatic polynomials of clique separable graphs?

\[ P(G; \lambda) = \frac{P(G_1; \lambda) \cdot P(G_2; \lambda)}{P(K_r; \lambda)}, \]

if \( G \) can be separated into graphs \( G_1 \) and \( G_2 \) both containing a common clique \( K_r \).
Graphs: Some Terminology

**Graphs.** A graph $G(V, E)$ has a vertex set $V$ and an edge set $E$. The set $E$ may contain loops and/or parallel edges. Throughout this discussion we only refer to the edge set $E$. (And restrict ourselves to matroid operations of a graph.)

**Forests.** A set $X \subseteq E$ is a forest of $G$ if the edges in $X$ do not contain a cycle in $G$. 
Graphs: Rank
An integer function $\rho : 2^E \rightarrow \mathbb{Z}_{\geq 0}$. Given $X \subseteq E,$

$$\rho(X) = \max\{|F| : F \subseteq X \text{ is a forest of } G\}$$

Rank Submodularity
For any two sets $X, Y \subseteq E,$

$$\rho(X \cup Y) + \rho(X \cap Y) \leq \rho(X) + \rho(Y).$$
Background

The Whitney Rank Generating Function

\[ R(G; x, y) = \sum_{Z \subseteq E} \left( x^{(\rho(E)-\rho(Z))} \cdot y^{(|Z|-\rho(Z))} \right) \]

\[ = x^{\rho(E)} \sum_{Z \subseteq E} y^{|Z|} (xy)^{-\rho(Z)} \]

The Tutte Polynomial

\[ T(G; x, y) = R(G; x - 1, y - 1) \]
Properties of $R(G; x, y)$

- $R(G; 1, 0) = T(G; 2, 1)$ counts the number of forests of $G$.
- $R(G; 0, 0) = T(G; 1, 1)$ counts the number of spanning forests of $G$.
- $R(G; 0, 1) = T(G; 1, 2)$ counts the number of spanning subgraphs of $G$.
- And much much more . . . (See Welsh, “Complexity: Knots, Colouring and Counting”, CUP, 1993)
The Correlation Inequalities for \( R(G; x, y) \)

The Basics

- Graphs \( G, G_1, G_2 \) and \( G_{12} \) are such that \( E_1 \cup E_2 = E \) and \( E_1 \cap E_2 = E_{12} \).

- Intuitively, the edge set of graph \( G \) is decomposed into smaller graphs \( G_1 \) and \( G_2 \) with their common edges forming the graph \( G_{12} \).
The Inequalities (2 of 3)

The “Easy” Regions

Let $k = \rho(E_1) + \rho(E_2) - \rho(E) - \rho(E_{12})$. Then:

- Along the curve $xy = 1$,

  $$x^k \cdot R(G; x, y) \cdot R(G_{12}; x, y) = R(G_1; x, y) \cdot R(G_2; x, y).$$

- In the region, $x, y > 0$, $xy > 1$,

  $$x^k \cdot R(G; x, y) \cdot R(G_{12}; x, y) \geq R(G_1; x, y) \cdot R(G_2; x, y).$$
The “Hard” Region

\[ k = \rho(E_1) + \rho(E_2) - \rho(E) - \rho(E_{12}) : \]

- In the region, \( x, y \geq 0, \ xy < 1, \) whenever \( |E_{12}| \leq 3, \)

\[ x^k \cdot R(G; x, y) \cdot R(G_{12}; x, y) \leq R(G_1; x, y) \cdot R(G_2; x, y). \]

- **Conjecture:** Above is true without the red text.
The Inequalities (Contd.,)

Corollaries

• $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_{12}$ be the set of forests of graphs $G, G_1, G_2$ and $G_{12}$ respectively. Then if $|E_{12}| \leq 3$:

$$|\mathcal{F}| \leq \frac{|\mathcal{F}_1| \cdot |\mathcal{F}_2|}{|\mathcal{F}_{12}|}.$$ 

• $\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_{12}$ be the set of spanning forests of graphs $G, G_1, G_2$ and $G_{12}$ respectively. If also

$$\rho(E) + \rho(E_{12}) = \rho(E_1) + \rho(E_2)$$

and $|E_{12}| \leq 3$, then:

$$|\mathcal{T}| \leq \frac{|\mathcal{T}_1| \cdot |\mathcal{T}_2|}{|\mathcal{T}_{12}|}.$$
The Proof

The Idea

- Let \( f(x, y) = R(G_1; x, y) \cdot R(G_2; x, y) − x^k \cdot R(G) \cdot R(G_{12}) \).
- Show \( f(x, y) \) is either zero or \( (1 − xy) \) is a factor of \( f(x, y) \) (easy).
- When non-zero, show that the function \( \frac{f(x, y)}{1 − xy} \) is positive in the region \( x, y \geq 0, xy \neq 1 \) (hard).
The Proof (Contd.,)

Step I

\[ f(x, y) = x^{\rho(E_1)+\rho(E_2)} \cdot g(x, y), \]

where,

\[
g(x, y) = \sum_{(X,Y) \in 2^{E_1} \times 2^{E_2}} \left( y^{X|+|Y|} \cdot (xy)^{-\rho(X)-\rho(Y)} \right) \\
- \sum_{(W,Z) \in 2^{E} \times 2^{E_{12}}} \left( y^{W|+|Z|} \cdot (xy)^{-\rho(W)-\rho(Z)} \right). \]
The Proof (Contd.,)

Step II

\[ g(x, y) = \sum_{(P,Q) \in 2^E \times 2^{E_1 \times E_2}} y^{|P|+|Q|} \cdot h_{P,Q}(x, y), \]

where,

\[ h_{P,Q}(x, y) = \sum_{(X,Y) \in 2^{E_1} \times 2^{E_2}} \left( (xy)^{\rho(X)-\rho(Y)} \right) \]

\[ - \sum_{(W,Z) \in 2^E \times 2^{E_1 \times E_2}} \left( (xy)^{\rho(W)-\rho(Z)} \right). \]
The Proof (Contd.,)

Step III

- Let $P_1 = P \cap (E_1 \setminus E_{12})$, $P_2 = P \cap (E_2 \setminus E_{12})$ and $R = (P \setminus Q) \cap E_{12}$.
- Suppose $S(P_1, P_2, Q, R)$ is the set of all $(X, Y) \in 2^{E_1} \times 2^{E_2}$ with $X \cup Y = P$ and $X \cap Y = Q$.
- Note $|S(P_1, P_2, Q, R)| = 2^{|R|}$.
- Similarly define $S(P, \phi, Q, R)$ to be the set of all $(W, Z) \in 2^E \times 2^{E_{12}}$ with $W \cup Z = P$ and $W \cap Z = Q$. 


Step III (Contd.,)

Suppose \( \pi : S(P_1, P_2, Q, R) \rightarrow S(P, \phi, Q, R) \) is a bijection such that whenever \( \pi(X, Y) = (W, Z) \),

\[
\rho(X) + \rho(Y) \geq \rho(W) + \rho(Z).
\]

We know \( \pi \) exists whenever \( |R| \leq 3 \), and conjecture that it exists whenever \( R \) is a forest.
The Inequalities

The Proof

Step III (Contd.,)

Then,

\[ h_{P,Q}(x, y) = \sum_{(X,Y) \in 2^{E_1} \times 2^{E_2}} \left( (xy)^{-\rho(X)-\rho(Y)} - (xy)^{-\rho(W)-\rho(Z)} \right). \]

- Each term in the sum is either zero or contains \((1 - xy)\) as a factor.
- When the sum is non-zero, \(\frac{h_{P,Q}(x, y)}{1 - xy}\) is positive in the region \(x, y \geq 0, xy \neq 1\). 

Conclusion
Concluding Remarks

- The approach “shows” all three inequalities at once.
- (Partially) Extends the inequality to regions where applying the Four Function theorem is not possible.
- Can we get an approximation algorithm for counting problems?
- Maybe hard to prove cases where $E_{12}$ contains a cycle.