On Matroids and Partial Sums of Binomial Coefficients

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Outline

Matroids: A Quick Introduction

Notation

- E : A finite set (groundset)
- $\rho : 2^E \to \mathbb{Z}_{\geq 0}$: An integer function (rank function)

Definition

$M(E, \rho)$ is a matroid if:

(R1) For all $X \subseteq E$, $0 \leq \rho(X) \leq |X|$.

(R2) For all $X \subseteq Y \subseteq E$, $\rho(X) \leq \rho(Y)$.

(R3) For all $X, Y \subseteq E$,
$\rho(X \cup Y) + \rho(X \cap Y) \leq \rho(X) + \rho(Y)$
(Submodularity).

Matroids: Introduction continued

Some Terminology

Independent Set: A set $X \subseteq E$ such that $\rho(X) = |X|$.

Circuit: A minimal non-independent set.

Spanning Set: A set $X \subseteq E$ such that $\rho(X) = \rho(E)$.

Basis: A set that is both independent and spanning.

Uniform Matroids: Introduction

Notation

$k, n \in \mathbb{Z}_{\geq 0}$ and $0 \leq k \leq n$.

Definition

A matroid $M(E, \rho) = U_{k,n}$ is a uniform matroid if:

- $|E| = n$, and
- For $X \subseteq E$,

$$\rho(X) = \begin{cases} |X| & \text{if } 0 \leq |X| \leq k, \\ k & \text{if } k < |X| \leq n. \end{cases}$$

Uniform Matroids: Introduction continued

$U_{k,n}$ Terminology

Independent Set: A set $X \subseteq E$ such that $|X| \leq k$.

Circuit: A set $X \subseteq E$ such that $|X| = k + 1$.

Spanning Set: A set $X \subseteq E$ such that $|X| \geq k$.

Basis: A set X such that $|X| = k$.

Whitney Rank Generating Function

Definition

$$R(M; x, y) = \sum_{X \subseteq E} x^{\rho(E)-\rho(X)} y^{|X|-\rho(X)}$$

Properties

- $R(M; 0, 0)$ counts the number of bases.
- $R(M; 0, 1)$ counts the number of spanning sets.
- $R(M; 1, 0)$ counts the number of independent sets.

Properties of $R(U_{k,n})$

$R(U_{k,n})$ Properties

$$R(U_{k,n}; 0, 0) = \text{Number of bases} = \binom{n}{k}$$

$$R(U_{k,n}; 0, 1) = \text{Number of spanning sets} = \sum_{i=k}^{n} \binom{n}{i}$$

$$R(U_{k,n}; 1, 0) = \text{Number of independent sets} = \sum_{i=0}^{k} \binom{n}{i}$$

Extended Submodularity

Preliminary Definitions

- Mutually disjoint sets $P_1, P_2, R \subseteq E$
- Set $S(P_1, P_2, R)$ is a collection of all $2^{|R|}$ partitions (X, Y) of the set $P_1 \cup P_2 \cup R$ under the constraints $P_1 \subseteq X$ and $P_2 \subseteq Y$.

$$S(P_1, P_2, R) = \{(P_1 \cup C, P_2 \cup (R \setminus C)) : C \subseteq R\}.$$

Examples

- $S(P_1, P_2, \phi) = \{(P_1, P_2)\}$.
- $S(P_1 \cup P_2, \phi, \{r\}) = \{(P_1 \cup P_2 \cup \{r\}, \phi), (P_1 \cup P_2, \{r\})\}$.

Rank Dominations in Matroids

Notation

- $P_1, P_2, Q_1, Q_2, R \subseteq E$.
- P_1, P_2, R are mutually disjoint.
- Q_1, Q_2, R are mutually disjoint.

Definition

We say $S(P_1, P_2, R)$ is rank dominated by $S(Q_1, Q_2, R)$ in matroid $M(E, \rho)$ (written as $S(P_1, P_2, R) \leq_M S(Q_1, Q_2, R)$) if there exists a bijection $\pi : S(P_1, P_2, R) \to S(Q_1, Q_2, R)$ such that whenever $\pi(W, Z) = (X, Y)$ we have $\rho(W) + \rho(Z) \leq \rho(X) + \rho(Y)$.

Extended Submodularity

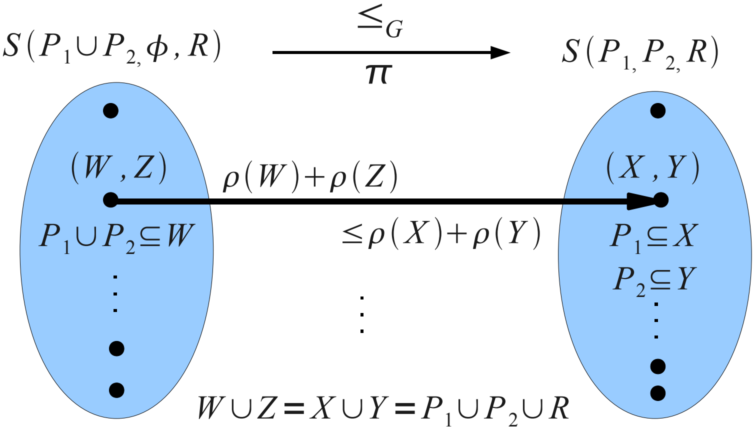
Submodularity

For all subsets $P_1, P_2 \subseteq E$ and all matroids M, we have $S(P_1 \cup P_2, \phi, \phi) \leq_M S(P_1, P_2, \phi)$.

Extended Submodularity

- Given a matroid M, for what mutually disjoint sets $P_1, P_2, R \subseteq E$ do we have $S(P_1 \cup P_2, \phi, R) \leq_M S(P_1, P_2, R)$?
- If true, then M is said to have the extended submodular property on sets P_1, P_2, R.

Extended Submodularity: Definition



$$W \cup Z = X \cup Y = P_1 \cup P_2 \cup R$$

$$W \cap Z = X \cap Y = \phi$$

Extended Submodularity: Uniform Matroids

Lemma

Let $M(E, \rho) = U_{k,n}$. Then for all mutually disjoint $P_1, P_2, R \subseteq E$, $S(P_1 \cup P_2, \phi, R) \leq_M S(P_1, P_2, R)$.

Proof Steps (Induction on $|P_1|$.)

- Base Case (Non-trivial): For all $P, R \subseteq E$, there exists a bijection $\pi_0 : S(P, \phi, R) \to S(\phi, P, R)$ such that whenever $\pi_0(W, Z) = (X, Y)$:

  (1) $\rho(W) + \rho(Z) \leq \rho(X) + \rho(Y)$, and
  (2) $|W| \geq |X|$.

- Inductive Hypothesis: Let $\pi : S(P_1 \cup P_2, \phi, R) \to S(P_1, P_2, R)$ be a bijection satisfying both (1) and (2) above.

Extended Submodularity in $U_{k,n}$: Proof continued

Proof Steps (continued)

- Inductive Step: For $p \in E \setminus (P_1 \cup P_2 \cup R)$, define $\pi' : S(P_1 \cup P_2 \cup \{p\}, \phi, R) \rightarrow S(P_1 \cup \{p\}, P_2, R)$ as

$$\pi'(W \cup \{p\}, Z) = (X \cup \{p\}, Y),$$

  whenever $\pi(W, Z) = (X, Y)$.
- Straightforward to check from (1) and (2) that $\rho(W \cup \{p\}) + \rho(Z) \leq \rho(X \cup \{p\}) + \rho(Y)$. Hence, $S(P_1 \cup P_2 \cup \{p\}, \phi, R) \leq_M S(P_1 \cup \{p\}, P_2, R)$. □

The Inequality Theorem

Notation

- $E_1, E_2 \subseteq E$.
- $r = \rho(E_1) + \rho(E_2) - \rho(E_1 \cup E_2) - \rho(E_1 \cap E_2)$.
- For $X \subseteq E$, $M|X$ is the matroid restriction of M to set X, defined as $M \setminus (E \setminus X)$.

Theorem

If $M(E, \rho) = U_{k,n}$, then for all $E_1, E_2 \subseteq E$,

$$x^r \cdot R(M|E_1 \cup E_2; x, y) \cdot R(M|E_1 \cap E_2; x, y) \leq R(M|E_1; x, y) \cdot R(M|E_2; x, y),$$

when $xy < 1$ and $x, y \geq 0$.

Partial Sums of Binomial Coefficients

Notation

k : a fixed non-negative integer.
For $n \geq 0$, let

$$A_n^k = \sum_{i=0}^{k} \binom{n+k}{i}.$$

A sequence $\{A_n\}$ is log-concave if $A_{n+1}A_{n-1} \leq A_n^2$ when $n \geq 1$.

Proposition [Semple and Welsh]

For all $k \geq 0$, the sequence $A_0^k, A_1^k, A_2^k, \cdots$ is log-concave.

Sequence A_n^k is Log-concave: An Injective Proof

Some Definitions

- $U_{k,n+1}$: Uniform matroid with $E = \{1, \cdots, n+1\}$.
- $E_1 = \{1, \cdots, n\}$
- $E_2 = \{2, \cdots, n+1\}$
- $E_1 \cap E_2 = \{2, \cdots, n\}$.

Injective Proof continued

Definitions continued

- $\mathcal{A}_{n+1}$: Set of all subsets of E of size at most k.
- $\mathcal{A}_{n-1}$: Set of all subsets of $E_1 \cap E_2$ of size at most k.
- $\mathcal{A}_n^1$: Set of all subsets of E_1 of size at most k.
- $\mathcal{A}_n^2$: Set of all subsets of E_2 of size at most k.

The Proof Method

Show an injection $\sigma : \mathcal{A}_{n+1} \times \mathcal{A}_{n-1} \to \mathcal{A}_n^1 \times \mathcal{A}_n^2$.

Injective Proof continued

The Injection σ

- Let $(W, Z) \in \mathcal{A}_{n+1} \times \mathcal{A}_{n-1}$.
- Let $T = W \cap Z$.
- Let $W' = W \setminus T$, $Z' = Z \setminus T$.
- Let $P_1 = W' \setminus E_2$, $P_2 = W' \setminus E_1$ and $R = (W' \cup Z') \cap (E_1 \cap E_2)$.

Injective Proof continued

The Injection σ continued

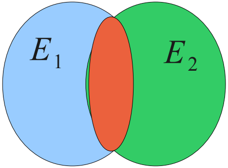
- Note 1: $(W', Z') \in S(P_1 \cup P_2, \phi, R)$.
- Note 2: The matroid $U_{k,n+1}/T$ is also uniform.
- Hence there exists a rank dominating bijection $\pi : S(P_1 \cup P_2, \phi, R) \rightarrow S(P_1, P_2, R)$ in $U_{k,n+1}/T$ (Extended Submodularity Property).
- Let $\pi(W', Z') = (X', Y')$.
- Let $X = X' \cup T$, $Y = Y' \cup T$.
- Then $(X, Y) \in 2^{E_1} \times 2^{E_2}$ and $\rho(W) + \rho(Z) \leq \rho(X) + \rho(Y)$.

Injective Proof continued

The Injection σ continued

- But $\rho(W) = |W|$, $\rho(Z) = |Z|$ and $|W| + |Z| = |X| + |Y|$.
- Hence $\rho(X) = |X|$ and $\rho(Y) = |Y|$.
- In other words, $(X, Y) \in \mathcal{A}_n^1 \times \mathcal{A}_n^2$.
- Define $\sigma(W, Z) = (X, Y)$. $\square$

Building the Injection σ: A 1000 Word Proof



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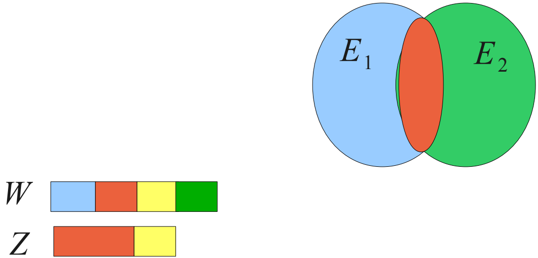
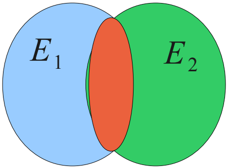
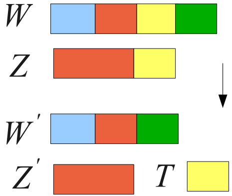
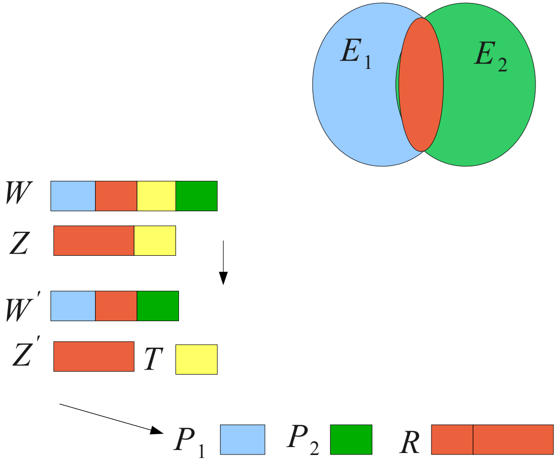


Building the Injection σ: A 1000 Word Proof





Building the Injection σ: A 1000 Word Proof



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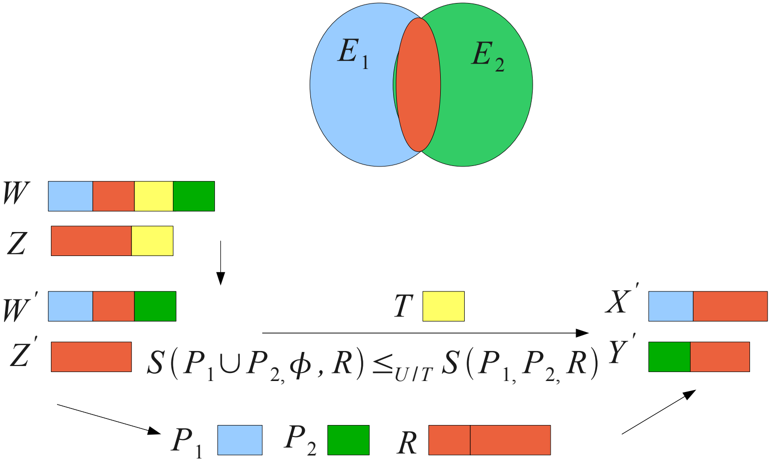
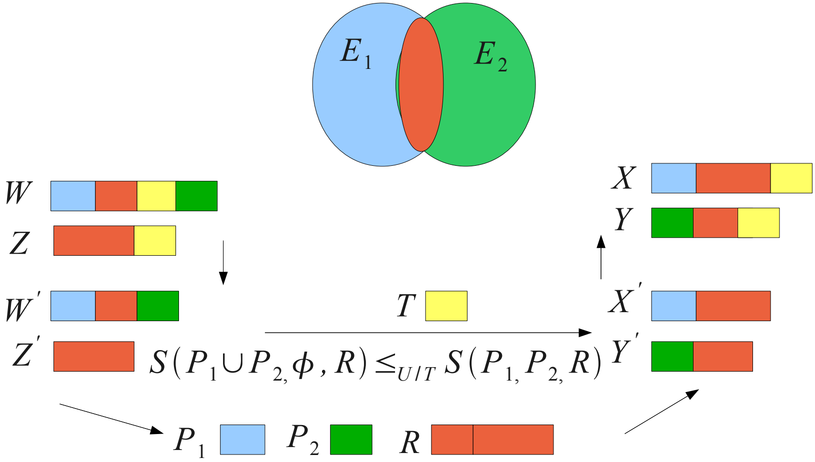
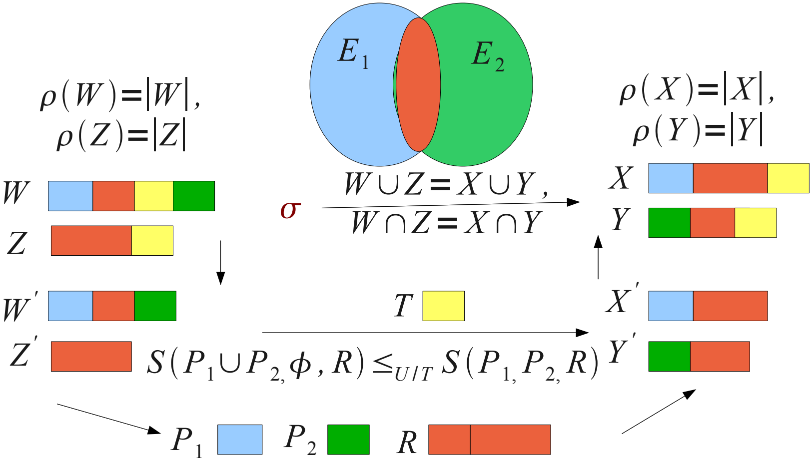


Building the Injection σ: A 1000 Word Proof



Building the Injection σ: A 1000 Word Proof



Log-concavity Results for Binomial Expansion of $(1+x)^n$

Notation

k : fixed non-negative integer.
$x > 0$: A positive real number.

Proposition

Let

$$B_n^{k,x} = \sum_{i=0}^{k} \binom{n+k}{i} x^i \text{ and } C_n^{k,x} = \sum_{i=0}^{n} \binom{n+k}{i} x^i.$$

For all $k \geq 0$, the sequences $B_0^{k,x}, B_1^{k,x}, \cdots$ and $C_0^{k,x}, C_1^{k,x}, \cdots$ are log-concave.

Concluding Remarks

Some Closing Observations

- Extended submodularity of matroids can be used to obtain injective proofs of some combinatorial inequalities.
- Only a few fully extended submodular matroid classes have been identified so far. Is there a characterization for all of them?
- Can the log-concavity results be used to approximate partial sum of binomial coefficients and binomial expansions quickly on a computer?