On Matroids and Partial Sums of Binomial Coefficients

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Outline

Introduction

Extended Submodularity in Matroids

The Inequalities

Conclusion
Matroids: A Quick Introduction

Notation

- \( E \) : A finite set (groundset)
- \( \rho : 2^E \to \mathbb{Z}_{\geq 0} \) : An integer function (rank function)

Definition

\( M(E, \rho) \) is a matroid if:

1. \((R1)\) For all \( X \subseteq E \), \( 0 \leq \rho(X) \leq |X| \).
2. \((R2)\) For all \( X \subseteq Y \subseteq E \), \( \rho(X) \leq \rho(Y) \).
3. \((R3)\) For all \( X, Y \subseteq E \),
   \[ \rho(X \cup Y) + \rho(X \cap Y) \leq \rho(X) + \rho(Y) \]
   (Submodularity).
Matroids: Introduction continued

Some Terminology

**Independent Set:** A set $X \subseteq E$ such that $\rho(X) = |X|$.

**Circuit:** A minimal non-independent set.

**Spanning Set:** A set $X \subseteq E$ such that $\rho(X) = \rho(E)$.

**Basis:** A set that is both independent and spanning.
Uniform Matroids: Introduction

Notation

$k, n \in \mathbb{Z}_{\geq 0}$ and $0 \leq k \leq n$.

Definition

A matroid $M(E, \rho) = U_{k,n}$ is a uniform matroid if:

- $|E| = n$, and
- For $X \subseteq E$,

$$\rho(X) = \begin{cases} |X| & \text{if } 0 \leq |X| \leq k, \\ k & \text{if } k < |X| \leq n. \end{cases}$$
Uniform Matroids: Introduction continued

$U_{k,n}$ Terminology

Independent Set: A set $X \subseteq E$ such that $|X| \leq k$.

Circuit: A set $X \subseteq E$ such that $|X| = k + 1$.

Spanning Set: A set $X \subseteq E$ such that $|X| \geq k$.

Basis: A set $X$ such that $|X| = k$. 
Whitney Rank Generating Function

**Definition**

\[ R(M; x, y) = \sum_{X \subseteq E} x^{\rho(E) - \rho(X)} y^{|X| - \rho(X)} \]

**Properties**

- \( R(M; 0, 0) \) counts the number of bases.
- \( R(M; 0, 1) \) counts the number of spanning sets.
- \( R(M; 1, 0) \) counts the number of independent sets.
Properties of $R(U_{k,n})$

$R(U_{k,n})$ Properties

$R(U_{k,n}; 0, 0) = \text{Number of bases} = \binom{n}{k}$

$R(U_{k,n}; 0, 1) = \text{Number of spanning sets} = \sum_{i=k}^{n} \binom{n}{i}$

$R(U_{k,n}; 1, 0) = \text{Number of independent sets} = \sum_{i=0}^{k} \binom{n}{i}$
Extended Submodularity

Preliminary Definitions

- Mutually disjoint sets \( P_1, P_2, R \subseteq E \)
- Set \( S(P_1, P_2, R) \) is a collection of all \( 2^{\left| R \right|} \) partitions \((X, Y)\)
of the set \( P_1 \cup P_2 \cup R \) under the constraints \( P_1 \subseteq X \) and \( P_2 \subseteq Y \).

\[
S(P_1, P_2, R) = \{ (P_1 \cup C, P_2 \cup (R \setminus C)) : C \subseteq R \}.
\]

Examples

- \( S(P_1, P_2, \phi) = \{ (P_1, P_2) \} \).
- \( S(P_1 \cup P_2, \phi, \{ r \}) = \{ (P_1 \cup P_2 \cup \{ r \}, \phi), (P_1 \cup P_2, \{ r \}) \} \).
Rank Dominations in Matroids

Notation

- $P_1, P_2, Q_1, Q_2, R \subseteq E$.
- $P_1, P_2, R$ are mutually disjoint.
- $Q_1, Q_2, R$ are mutually disjoint.

Definition

We say $S(P_1, P_2, R)$ is rank dominated by $S(Q_1, Q_2, R)$ in matroid $M(E, \rho)$ (written as $S(P_1, P_2, R) \leq_M S(Q_1, Q_2, R)$) if there exists a bijection $\pi : S(P_1, P_2, R) \rightarrow S(Q_1, Q_2, R)$ such that whenever $\pi(W, Z) = (X, Y)$ we have $\rho(W) + \rho(Z) \leq \rho(X) + \rho(Y)$. 
Extended Submodularity

Submodularity
For all subsets $P_1, P_2 \subseteq E$ and all matroids $M$, we have $S(P_1 \cup P_2, \phi, \phi) \leq_M S(P_1, P_2, \phi)$.

Extended Submodularity

- Given a matroid $M$, for what mutually disjoint sets $P_1, P_2, R \subseteq E$ do we have $S(P_1 \cup P_2, \phi, R) \leq_M S(P_1, P_2, R)$?
- If true, then $M$ is said to have the extended submodular property on sets $P_1, P_2, R$. 
Extended Submodularity: Definition

\[ S(P_1 \cup P_2, \phi, R) \leq_G \pi \leq_G \frac{\pi}{\pi} \rightarrow S(P_1, P_2, R) \]

\[ (W, Z) \]
\[ P_1 \cup P_2 \subseteq W \]
\[ \rho(W) + \rho(Z) \]
\[ W \cup Z = X \cup Y = P_1 \cup P_2 \cup R \]
\[ W \cap Z = X \cap Y = \phi \]

\[ P_1 \subseteq X \]
\[ P_2 \subseteq Y \]
\[ (X, Y) \]
Lemma
Let $M(E, \rho) = U_{k,n}$. Then for all mutually disjoint $P_1, P_2, R \subseteq E$, $S(P_1 \cup P_2, \phi, R) \leq_M S(P_1, P_2, R)$.

Proof Steps (Induction on $|P_1|$.)

- Base Case (Non-trivial): For all $P, R \subseteq E$, there exists a bijection $\pi_0 : S(P, \phi, R) \rightarrow S(\phi, P, R)$ such that whenever $\pi_0(W, Z) = (X, Y)$:
  
  1. $\rho(W) + \rho(Z) \leq \rho(X) + \rho(Y)$, and
  2. $|W| \geq |X|$.

- Inductive Hypothesis: Let $\pi : S(P_1 \cup P_2, \phi, R) \rightarrow S(P_1, P_2, R)$ be a bijection satisfying both (1) and (2) above.
Extended Submodularity in $U_{k,n}$: Proof continued

Proof Steps (continued)

- Inductive Step: For $p \in E \setminus (P_1 \cup P_2 \cup R)$, define
  $\pi' : S(P_1 \cup P_2 \cup \{p\}, \phi, R) \to S(P_1 \cup \{p\}, P_2, R)$ as

  \[ \pi'(W \cup \{p\}, Z) = (X \cup \{p\}, Y), \]

  whenever $\pi(W, Z) = (X, Y)$.

- Straightforward to check from (1) and (2) that
  $\rho(W \cup \{p\}) + \rho(Z) \leq \rho(X \cup \{p\}) + \rho(Y)$. Hence,
  $S(P_1 \cup P_2 \cup \{p\}, \phi, R) \leq M S(P_1 \cup \{p\}, P_2, R)$. □
The Inequality Theorem

Notation

- \( E_1, E_2 \subseteq E \).
- \( r = \rho(E_1) + \rho(E_2) - \rho(E_1 \cup E_2) - \rho(E_1 \cap E_2) \).
- For \( X \subseteq E \), \( M|X \) is the matroid restriction of \( M \) to set \( X \), defined as \( M \setminus (E \setminus X) \).

Theorem

If \( M(E, \rho) = U_{k,n} \), then for all \( E_1, E_2 \subseteq E \),

\[
x^r \cdot R(M|E_1 \cup E_2; x, y) \cdot R(M|E_1 \cap E_2; x, y) \leq R(M|E_1; x, y) \cdot R(M|E_2; x, y),
\]

when \( xy < 1 \) and \( x, y \geq 0 \).
Partial Sums of Binomial Coefficients

Notation
$k$ : a fixed non-negative integer.
For $n \geq 0$, let
\[
A_n^k = \sum_{i=0}^{k} \binom{n+k}{i}.
\]

A sequence $\{A_n\}$ is log-concave if $A_{n+1}A_{n-1} \leq A_n^2$ when $n \geq 1$.

Proposition [Semple and Welsh]
For all $k \geq 0$, the sequence $A_0^k, A_1^k, A_2^k, \cdots$ is log-concave.
Some Definitions

- $U_{k,n+1}$: Uniform matroid with $E = \{1, \ldots, n+1\}$.
- $E_1 = \{1, \ldots, n\}$
- $E_2 = \{2, \ldots, n+1\}$
- $E_1 \cap E_2 = \{2, \ldots, n\}$. 
Injective Proof continued

Definitions continued

- $\mathcal{A}_{n+1}$: Set of all subsets of $E$ of size at most $k$.
- $\mathcal{A}_{n-1}$: Set of all subsets of $E_1 \cap E_2$ of size at most $k$.
- $\mathcal{A}_n^1$: Set of all subsets of $E_1$ of size at most $k$.
- $\mathcal{A}_n^2$: Set of all subsets of $E_2$ of size at most $k$.

The Proof Method
Show an injection $\sigma : \mathcal{A}_{n+1} \times \mathcal{A}_{n-1} \rightarrow \mathcal{A}_n^1 \times \mathcal{A}_n^2$. 
Injective Proof continued

The Injection $\sigma$

1. Let $(W, Z) \in \mathcal{A}_{n+1} \times \mathcal{A}_{n-1}$.
2. Let $T = W \cap Z$.
3. Let $W' = W \setminus T$, $Z' = Z \setminus T$.
4. Let $P_1 = W' \setminus E_2$, $P_2 = W' \setminus E_1$ and $R = (W' \cup Z') \cap (E_1 \cap E_2)$. 
The Injection $\sigma$ continued

- **Note 1:** $(W', Z') \in S(P_1 \cup P_2, \phi, R)$.
- **Note 2:** The matroid $U_{k,n+1}/T$ is also uniform.
- Hence there exists a rank dominating bijection
  $\pi : S(P_1 \cup P_2, \phi, R) \rightarrow S(P_1, P_2, R)$ in $U_{k,n+1}/T$ (Extended Submodularity Property).
- Let $\pi(W', Z') = (X', Y')$.
- Let $X = X' \cup T$, $Y = Y' \cup T$.
- Then $(X, Y) \in 2^{E_1} \times 2^{E_2}$ and $\rho(W) + \rho(Z) \leq \rho(X) + \rho(Y)$.
The Injection $\sigma$ continued

- But $\rho(W) = |W|$, $\rho(Z) = |Z|$ and $|W| + |Z| = |X| + |Y|$.
- Hence $\rho(X) = |X|$ and $\rho(Y) = |Y|$.
- In other words, $(X, Y) \in A_n^1 \times A_n^2$.
- Define $\sigma(W, Z) = (X, Y)$.  \qed
Building the Injection $\sigma$: A 1000 Word Proof
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Building the Injection $\sigma$: A 1000 Word Proof

$E_1$ $E_2$

$W$
$Z$

$W'$
$Z'$ $T$
Building the Injection $\sigma$: A 1000 Word Proof
Building the Injection $\sigma$: A 1000 Word Proof

$W$ $W'$ $Z$ $Z'$ $P_1$ $P_2$ $R$

$E_1$ $E_2$

$W$ $W'$ $Z$ $Z'$ $P_1$ $P_2$ $R$

$S(P_1 \cup P_2, \phi, R) \leq_{U/T} S(P_1, P_2, R)$

$X'$ $Y'$
Building the Injection $\sigma$: A 1000 Word Proof

$W$  
$Z$  
$W'$  
$Z'$

$S(P_1 \cup P_2, \phi, R) \leq_{U/T} S(P_1, P_2, R)$

$P_1$  
$P_2$  
$R$
Building the Injection $\sigma$: A 1000 Word Proof

$$\rho(W) = |W|, \quad \rho(Z) = |Z|$$

$$\rho(X) = |X|, \quad \rho(Y) = |Y|$$

$$W \cup Z = X \cup Y, \quad W \cap Z = X \cap Y$$

$$S(P_1 \cup P_2, \phi, R) \leq_{U/T} S(P_1, P_2, R)$$

$$\pi$$
Log-concavity Results for Binomial Expansion of $(1 + x)^n$

Notation
$k$ : fixed non-negative integer.
$x > 0$ : A positive real number.

Proposition
Let

$$B_{n}^{k, x} = \sum_{i=0}^{k} \binom{n + k}{i} x^i \quad \text{and} \quad C_{n}^{k, x} = \sum_{i=0}^{n} \binom{n + k}{i} x^i.$$ 

For all $k \geq 0$, the sequences $B_{0}^{k, x}, B_{1}^{k, x}, \ldots$ and $C_{0}^{k, x}, C_{1}^{k, x}, \ldots$ are log-concave.
Some Closing Observations

- Extended submodularity of matroids can be used to obtain injective proofs of some combinatorial inequalities.
- Only a few fully extended submodular matroid classes have been identified so far. Is there a characterization for all of them?
- Can the log-concavity results be used to approximate partial sum of binomial coefficients and binomial expansions quickly on a computer?