Extended Submodularity and Tutte Polynomial Inequalities for Graphs (Inequalities for Counting Problems in Graphs)

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Outline

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Introduction

Notations

- Graph $G$ 2-connected and synonymous with its edge set $E$
- $G$ synonymous with its cycle matroid $M(G)$
- For $X \subseteq E$, its rank $\rho(X) = \text{Size of maximal forest in } X$

Whitney-Tutte Polynomials of Graphs

$$R(G; x, y) = \sum_{X \subseteq E} x^{\rho(E) - \rho(X)} y^{|X| - \rho(X)}$$

$$T(G; x, y) = R(G; x - 1, y - 1)$$
Graph Polynomials and Counting

- $R(G; 0, 0) = T(G; 1, 1)$ counts the number of spanning trees of $G$
- $R(G; 1, 0) = T(G; 2, 1)$ counts the number of forests
- $R(G; 0, 1) = T(G; 1, 2)$ counts the number of connected spanning subgraphs
The Problem

Notation

- For $e \in E$, graph $G/e$ is obtained by contracting edge $e$ in $G$
- $\{e, f\} \subseteq E$ is not a cutset of $G$
- $x, y \in \mathbb{R}_{\geq 0}$

Graph Polynomial Inequalities

Is

$$R(G; x, y) \cdot R(G/e/f; x, y) \leq R(G/e; x, y) \cdot R(G/f; x, y),$$

in the region $xy < 1$ and $x, y \geq 0$?
What’s Known

- Studied at points \((0, 0), (1, 0)\) and \((0, 1)\) corresponding to spanning trees, forests and connected spanning subgraphs.
- True for all graphs at \((0, 0)\) [Tutte].
- True for Series-Parallel graphs at the points \((1, 0)\) and \((0, 1)\) [Semple and Welsh].
- Conjectured to be true at \((1, 0)\) and \((0, 1)\) for all graphs.
- Direction of inequality reversed in the region \(xy \geq 1\) and known to be true for all graphs (and matroids) [Seymour and Welsh].
The Problem (Cont’d)

Notation

- $E_1, E_2 \subseteq E$
- $k = \rho(E_1) + \rho(E_2) - \rho(E_1 \cup E_2) - \rho(E_1 \cap E_2)$

The Problem (Version 2)

Is

$$x^k \cdot R(E_1 \cup E_2; x, y) \cdot R(E_1 \cap E_2; x, y) \leq R(E_1; x, y) \cdot R(E_2; x, y),$$

when $xy < 1$ and $x, y \geq 0$?
Version Differences

- The “new” inequality true for all $E_1, E_2 \subseteq E \iff$ “old” inequality true for $G$ and all its minors.
- “New” version also lets us study validity for some subsets $E_1, E_2 \subseteq E$ even if other choices for $E_1, E_2$ are known to fail or simply hard to prove.
Our Approach

• Introduce a notion of extended submodularity for the rank function, $\rho$
• Extended submodularity of $G$ and its minors (restricted to the subsets $E_1, E_2$) $\Rightarrow$ “New” inequality
• Show Series-Parallel graphs have extended submodularity on all subsets $E_1, E_2 \subseteq E$
Submodularity

Notation

- $E_1, E_2 \subseteq E$
- For $X \subseteq E$, $\rho(X) =$ Size of maximal forest in $X$

Definition

$$\rho(E_1 \cup E_2) + \rho(E_1 \cap E_2) \leq \rho(E_1) + \rho(E_2)$$
Extended Submodularity

Preliminary Definitions

- Mutually disjoint sets $P_1, P_2, R \subseteq E$
- Set $S(P_1, P_2, R)$ is a collection of all $2^{|R|}$ partitions $(X, Y)$ of the set $P_1 \cup P_2 \cup R$ under the constraints $P_1 \subseteq X$ and $P_2 \subseteq Y$.

$$S(P_1, P_2, R) = \{(P_1 \cup C, P_2 \cup (R \setminus C)) : C \subseteq R\}$$

Examples

- $S(P_1, P_2, \emptyset) = \{(P_1, P_2)\}$
- $S(P_1 \cup P_2, \emptyset, \{r\}) = \{(P_1 \cup P_2 \cup \{r\}, \emptyset), (P_1 \cup P_2, \{r\})\}$
Rank Dominations in Graphs

Notation

- $P_1, P_2, Q_1, Q_2, R \subseteq E$
- $P_1, P_2, R$ are mutually disjoint
- $Q_1, Q_2, R$ are mutually disjoint

Definition

We say $S(P_1, P_2, R)$ is rank dominated by $S(Q_1, Q_2, R)$ in graph $G$ (written as $S(P_1, P_2, R) \leq_G S(Q_1, Q_2, R)$) if there exists a bijection $\pi : S(P_1, P_2, R) \rightarrow S(Q_1, Q_2, R)$ such that whenever $\pi(W, Z) = (X, Y)$ we have $\rho(W) + \rho(Z) \leq \rho(X) + \rho(Y)$
Extended Submodularity

Submodularity
For all subsets $E_1, E_2 \subseteq E$ and all graphs $G$, we have
$S(P_1 \cup P_2, \phi, \phi) \leq_G S(P_1, P_2, \phi)$

Extended Submodularity

- Given a graph $G$, for what mutually disjoint sets $P_1, P_2, R \subseteq E$ do we have
  $S(P_1 \cup P_2, \phi, R) \leq_G S(P_1, P_2, R)$?
- If true, then $G$ is said to have the extended submodular property on sets $P_1, P_2, R$
Extended Submodularity: Definition

\[ S(P_1 \cup P_2, \phi, R) \xrightarrow{\leq G} \xrightarrow{\pi} S(P_1, P_2, R) \]

\[ W \cup Z = X \cup Y = P_1 \cup P_2 \cup R \]

\[ W \cap Z = X \cap Y = \phi \]
Extended Submodularity: What’s Known?

- For all $P, R \subseteq E$ and all graphs $G$, it is easy to show $S(P, \phi, R) \leq_G S(P, \phi, R)$ and $S(P, \phi, R) \leq_G S(\phi, P, R)$ (For the second one use the map $\pi(X, Y) = (Y, X)$)
- For all $P_1, P_2, R \subseteq E$ and all graphs $G$, if $P_1 \cup P_2$ is a connected spanning subgraph then $S(P_1 \cup P_2, \phi, R) \leq_G S(P_1, P_2, R)$ [Noble]
- For all $P_1, P_2, R \subseteq E$ and all graphs $G$, if $|R| \leq 3$, $S(P_1 \cup P_2, \phi, R) \leq_G S(P_1, P_2, R)$ (Non-trivial)
Extended Submodularity: Counterexample

\[ P_1 = \{ e_1 \} \]
\[ P_2 = \{ e_6 \} \]
\[ R = \{ e_2, e_3, e_4, e_5 \} \]

\[ S(P_1 \cup P_2, \phi, R) \nsubseteq S(P_1, P_2, R) \]
Fully Extended Submodular Graphs

Notation
For a graph $G$ and subset $T \subseteq E$, the $T$-Minor Family is

$$\mathcal{MF}(G, T) = \{G/C \setminus (T \setminus C) : C \subseteq T\}$$

Definition
A graph $G$ is fully extended submodular if for all mutually disjoint subsets $P_1, P_2, R \subseteq E$, we have

$$S(P_1 \cup P_2, \emptyset, R) \leq_H S(P_1, P_2, R)$$

in all minors $H \in \mathcal{MF}(G, E \setminus (P_1 \cup P_2 \cup R))$
Fully Extended Submodular Graphs

Notation
Let $\mathcal{ESG}$ denote the class of graphs that are fully extended submodular

Properties
- If $G \in \mathcal{ESG}$ then so are its minors
- If $G \in \mathcal{ESG}$ then its (matroid) dual $G^* \in \mathcal{ESG}$
Properties of $\mathcal{ESG}$

Properties

- $K_4 \notin \mathcal{ESG}$ but every minor of $K_4$ belongs to $\mathcal{ESG}$
- If $\mathcal{SP}$ denotes the class of Series-Parallel graphs, then $\mathcal{SP} = \mathcal{ESG}$ (Yet another characterization of the class $\mathcal{SP}$)
- In other words, every graph without a $K_4$ minor belongs to $\mathcal{ESG}$
Definition
Graph $G'$ is a **parallel extension** of graph $G$ if $G'$ has a two edge cycle $\{e, f\}$ such that $G' \setminus f = G$, and a **series extension** of $G$ if it has a two edge minimum cutset $\{e, f\}$ such that $G' / f = G$.

Proof Steps

- Graph with one edge is trivially in $\mathcal{ESG}$
- If $G \in \mathcal{ESG}$ then show its parallel extensions are also in $\mathcal{ESG}$
- Using duality arguments show the series extensions of $G$ are also in $\mathcal{ESG}$, and so $\mathcal{SP} \subseteq \mathcal{ESG}$
- Equality follows because any graph that is not series-parallel is known to contain a $K_4$ minor
Parallel Extension in $\mathcal{ESG}$

**Notation**

- Let $G'$ be a parallel extension of $G \in \mathcal{ESG}$ with $G' \setminus f = G$ and $\{e, f\}$ a cycle.
- $N' \in \mathcal{MF}(G', E \setminus (P_1 \cup P_2 \cup R))$

**Proof Idea**

- $S(P_1 \cup P_2, \phi, R) \leq_{N'} S(P_1, P_2, R)$ if $f \notin P_1 \cup P_2 \cup R$ (Easy)
- If $e \notin P_1 \cup P_2 \cup R$ then easily $S(P_1 \cup P_2 \cup \{f\}, \phi, R) \leq_{N'} S(P_1 \cup \{f\}, P_2, R)$ and $S(P_1 \cup P_2, \phi, R \cup \{f\}) \leq_{N'} S(P_1, P_2, R \cup \{f\})$ because we know $f$ is just a parallel edge to $e$
Parallel Extension in $\mathcal{ESG}$

Proof (Cont’d)

- If $e \in P_1$ then $S(P_1 \cup P_2 \cup \{f\}, \phi, R) \leq_{N'} S(P_1 \cup \{f\}, P_2, R)$ because the rank sums of the individual partitions on LHS do not increase by adding parallel edge $f$
- Also $S(P_1 \cup P_2, \phi, R \cup \{f\}) \leq_{N'} S(P_1, P_2, R \cup \{f\})$ because
  1. $S(P_1 \cup P_2, \phi, R) \leq S(P_1, P_2, R)$ in $N'$, and
  2. $S((P_1 \cup P_2) \setminus \{e\}, \phi, R) \leq S(P_1 \setminus \{e\}, P_2, R)$ in $N' / e$
- And so on …
Proof (Cont’d)

\[
S(P_1 \cup P_2, \phi, R \cup \{f\}) \leq_{N'} S(P_1, P_2, R \cup \{f\})
\]

1. \(S(P_1 \cup P_2 \cup \{f\}, \phi, R) \leq_{N'} S(P_1 \cup \{f\}, P_2, R)\)

2. \(S(P_1 \cup P_2, \{f\}, R) \leq_{N'} S(P_1, P_2 \cup \{f\}, R)\)

\(e \in P_1, S(P_1 \cup P_2, \phi, R) \leq_{N'} S(P_1, P_2, R) \Rightarrow (1)\)

\(S((P_1 \cup P_2) \setminus e, \phi, R) \leq_{N'/e} S(P_1 \setminus e, P_2, R) \Rightarrow (2)\)
The Inequality Theorem

Notation
\[ k = \rho(E_1) + \rho(E_2) - \rho(E_1 \cup E_2) - \rho(E_1 \cap E_2) \]

The Theorem
If \( G \in \mathcal{EG} \) then for all \( E_1, E_2 \subseteq E \),

\[ x^k \cdot R(E_1 \cup E_2; x, y) \cdot R(E_1 \cap E_2; x, y) \leq R(E_1; x, y) \cdot R(E_2; x, y), \]

when \( xy < 1 \) and \( x, y \geq 0 \).
Concluding Remarks

- Therefore Series-Parallel graphs satisfy the “old” inequality at all points $x, y \geq 0$ such that $xy < 1$, and not just at $(0, 0), (1, 0)$ and $(0, 1)$.

- What more can be said about extended submodularity in graphs with a $K_4$ minor? For example, is it true that in all graphs $S(P_1 \cup P_2, \phi, R) \leq S(P_1, P_2, R)$ whenever $R$ is a forest? (This would imply “new” inequality is true whenever $E_1 \cap E_2$ is a forest.)

- Conjecture: The “new” inequality is true for all graphs and all subsets $E_1, E_2$ (and hence the “old” inequality for all graphs). But can extended submodularity be “further extended” to deal with the general case?