Extended Submodularity and Tutte Polynomial Inequalities for Graphs (Inequalities for Counting Problems in Graphs)

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Outline

Introduction

Notations

- Graph G 2-connected and synonymous with its edge set E
- G synonymous with its cycle matroid $M(G)$
- For $X \subseteq E$, its rank $\rho(X) =$ Size of maximal forest in X

Whitney-Tutte Polynomials of Graphs

$$R(G; x, y) = \sum_{X \subseteq E} x^{\rho(E)-\rho(X)} y^{|X|-\rho(X)}$$

$$T(G; x, y) = R(G; x-1, y-1)$$

Properties of Graph Polynomials

Graph Polynomials and Counting

- $R(G; 0, 0) = T(G; 1, 1)$ counts the number of spanning trees of G
- $R(G; 1, 0) = T(G; 2, 1)$ counts the number of forests
- $R(G; 0, 1) = T(G; 1, 2)$ counts the number of connected spanning subgraphs

The Problem

Notation

- For $e \in E$, graph G/e is obtained by contracting edge e in G
- $\{e, f\} \subseteq E$ is not a cutset of G
- $x, y \in \mathbb{R}_{\geq 0}$

Graph Polynomial Inequalities

Is

$$R(G; x, y) \cdot R(G/e/f; x, y) \leq R(G/e; x, y) \cdot R(G/f; x, y),$$

in the region $xy < 1$ and $x, y \geq 0$?

The Problem (Cont'd)

What's Known

- Studied at points $(0, 0)$, $(1, 0)$ and $(0, 1)$ corresponding to spanning trees, forests and connected spanning subgraphs
- True for all graphs at $(0, 0)$ [Tutte]
- True for Series-Parallel graphs at the points $(1, 0)$ and $(0, 1)$ [Semple and Welsh]
- Conjectured to be true at $(1, 0)$ and $(0, 1)$ for all graphs
- Direction of inequality reversed in the region $xy \geq 1$ and known to be true for all graphs (and matroids) [Seymour and Welsh]

The Problem (Cont'd)

Notation

- $E_1, E_2 \subseteq E$
- $k = \rho(E_1) + \rho(E_2) - \rho(E_1 \cup E_2) - \rho(E_1 \cap E_2)$

The Problem (Version 2)

Is

$$x^k \cdot R(E_1 \cup E_2; x, y) \cdot R(E_1 \cap E_2; x, y) \leq R(E_1; x, y) \cdot R(E_2; x, y),$$

when $xy < 1$ and $x, y \geq 0$?

The Problem (Cont'd)

Version Differences

- The "new" inequality true for all $E_1, E_2 \subseteq E \iff$ "old" inequality true for G and all its minors
- "New" version also lets us study validity for some subsets $E_1, E_2 \subseteq E$ even if other choices for E_1, E_2 are known to fail or simply hard to prove

Our Approach

- Introduce a notion of extended submodularity for the rank function, ρ
- Extended submodularity of G and its minors (restricted to the subsets E_1, E_2) $\implies$ “New” inequality
- Show Series-Parallel graphs have extended submodularity on all subsets $E_1, E_2 \subseteq E$

Submodularity

Notation

- $E_1, E_2 \subseteq E$
- For $X \subseteq E$, $\rho(X) =$ Size of maximal forest in X

Definition

$$\rho(E_1 \cup E_2) + \rho(E_1 \cap E_2) \leq \rho(E_1) + \rho(E_2)$$

Extended Submodularity

Preliminary Definitions

- Mutually disjoint sets $P_1, P_2, R \subseteq E$
- Set $S(P_1, P_2, R)$ is a collection of all $2^{|R|}$ partitions (X, Y) of the set $P_1 \cup P_2 \cup R$ under the constraints $P_1 \subseteq X$ and $P_2 \subseteq Y$.

$$S(P_1, P_2, R) = \{(P_1 \cup C, P_2 \cup (R \setminus C)) : C \subseteq R\}$$

Examples

- $S(P_1, P_2, \phi) = \{(P_1, P_2)\}$
- $S(P_1 \cup P_2, \phi, \{r\}) = \{(P_1 \cup P_2 \cup \{r\}, \phi), (P_1 \cup P_2, \{r\})\}$

Rank Dominations in Graphs

Notation

- $P_1, P_2, Q_1, Q_2, R \subseteq E$
- P_1, P_2, R are mutually disjoint
- Q_1, Q_2, R are mutually disjoint

Definition

We say $S(P_1, P_2, R)$ is rank dominated by $S(Q_1, Q_2, R)$ in graph G (written as $S(P_1, P_2, R) \leq_G S(Q_1, Q_2, R)$) if there exists a bijection $\pi : S(P_1, P_2, R) \rightarrow S(Q_1, Q_2, R)$ such that whenever $\pi(W, Z) = (X, Y)$ we have $\rho(W) + \rho(Z) \leq \rho(X) + \rho(Y)$

Extended Submodularity

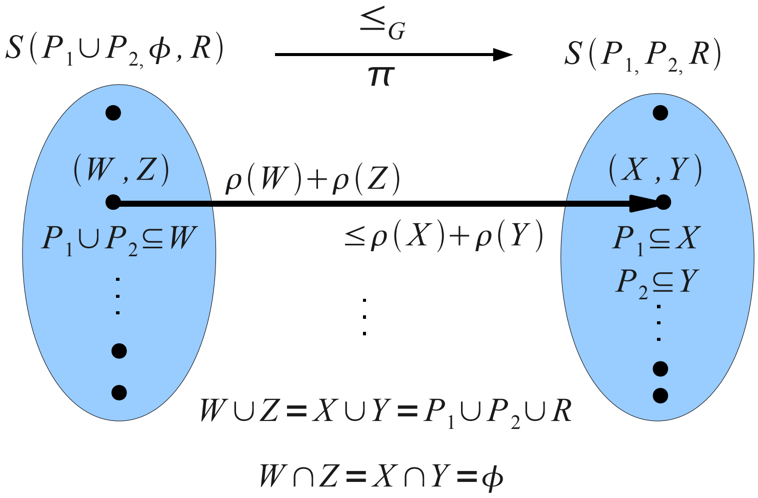
Submodularity

For all subsets $E_1, E_2 \subseteq E$ and all graphs G, we have $S(P_1 \cup P_2, \phi, \phi) \leq_G S(P_1, P_2, \phi)$

Extended Submodularity

- Given a graph G, for what mutually disjoint sets $P_1, P_2, R \subseteq E$ do we have $S(P_1 \cup P_2, \phi, R) \leq_G S(P_1, P_2, R)$?
- If true, then G is said to have the extended submodular property on sets P_1, P_2, R

Extended Submodularity: Definition



Extended Submodularity: What's Known?

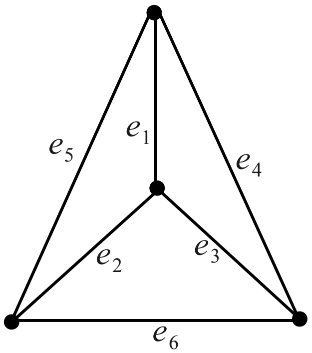
- For all $P, R \subseteq E$ and all graphs G, it is easy to show $S(P, \phi, R) \leq_G S(P, \phi, R)$ and $S(P, \phi, R) \leq_G S(\phi, P, R)$ (For the second one use the map $\pi(X, Y) = (Y, X)$)
- For all $P_1, P_2, R \subseteq E$ and all graphs G, if $P_1 \cup P_2$ is a connected spanning subgraph then $S(P_1 \cup P_2, \phi, R) \leq_G S(P_1, P_2, R)$ [Noble]
- For all $P_1, P_2, R \subseteq E$ and all graphs G, if $|R| \leq 3$, $S(P_1 \cup P_2, \phi, R) \leq_G S(P_1, P_2, R)$ (Non-trivial)

Extended Submodularity: Counterexample

$P_1 = \{e_1\}$

$P_2 = \{e_6\}$

$R = \{e_2, e_3, e_4, e_5\}$



$S(P_1 \cup P_2, \phi, R) \not\leq S(P_1, P_2, R)$

Graph K_4

Fully Extended Submodular Graphs

Notation

For a graph G and subset $T \subseteq E$, the T-Minor Family is

$$\mathcal{MF}(G, T) = \{G/C \setminus (T \setminus C) : C \subseteq T\}$$

Definition

A graph G is fully extended submodular if for all mutually disjoint subsets $P_1, P_2, R \subseteq E$, we have $S(P_1 \cup P_2, \phi, R) \leq_H S(P_1, P_2, R)$ in all minors $H \in \mathcal{MF}(G, E \setminus (P_1 \cup P_2 \cup R))$

Fully Extended Submodular Graphs

Notation

Let $\mathcal{ESG}$ denote the class of graphs that are fully extended submodular

Properties

- If $G \in \mathcal{ESG}$ then so are its minors
- If $G \in \mathcal{ESG}$ then its (matroid) dual $G^* \in \mathcal{ESG}$

Properties of $\mathcal{ESG}$

Properties

- $K_4 \notin \mathcal{ESG}$ but every minor of K_4 belongs to $\mathcal{ESG}$
- If $\mathcal{SP}$ denotes the class of Series-Parallel graphs, then $\mathcal{SP} = \mathcal{ESG}$ (Yet another characterization of the class $\mathcal{SP}$)
- In other words, every graph without a K_4 minor belongs to $\mathcal{ESG}$

$\mathcal{SP} = \mathcal{ESG}$

Definition

Graph G' is a parallel extension of graph G if G' has a two edge cycle $\{e, f\}$ such that $G' \setminus f = G$, and a series extension of G if it has a two edge minimum cutset $\{e, f\}$ such that $G'/f = G$

Proof Steps

- Graph with one edge is trivially in $\mathcal{ESG}$
- If $G \in \mathcal{ESG}$ then show its parallel extensions are also in $\mathcal{ESG}$
- Using duality arguments show the series extensions of G are also in $\mathcal{ESG}$, and so $\mathcal{SP} \subseteq \mathcal{ESG}$
- Equality follows because any graph that is not series-parallel is known to contain a K_4 minor

Parallel Extension in $\mathcal{ESG}$

Notation

- Let G' be a parallel extension of $G \in \mathcal{ESG}$ with $G' \setminus f = G$ and $\{e, f\}$ a cycle
- $N' \in \mathcal{MF}(G', E \setminus (P_1 \cup P_2 \cup R))$

Proof Idea

- $S(P_1 \cup P_2, \phi, R) \leq_{N'} S(P_1, P_2, R)$ if $f \notin P_1 \cup P_2 \cup R$ (Easy)
- If $e \notin P_1 \cup P_2 \cup R$ then easily $S(P_1 \cup P_2 \cup \{f\}, \phi, R) \leq_{N'} S(P_1 \cup \{f\}, P_2, R)$ and $S(P_1 \cup P_2, \phi, R \cup \{f\}) \leq_{N'} S(P_1, P_2, R \cup \{f\})$ because we know f is just a parallel edge to e

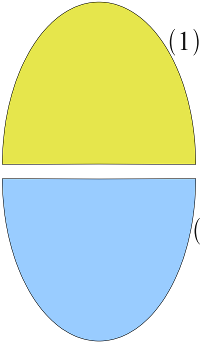
Parallel Extension in $\mathcal{ESG}$

Proof (Cont'd)

- If $e \in P_1$ then $S(P_1 \cup P_2 \cup \{f\}, \phi, R) \leq_{N'} S(P_1 \cup \{f\}, P_2, R)$ because the rank sums of the individual partitions on LHS do not increase by adding parallel edge f
- Also $S(P_1 \cup P_2, \phi, R \cup \{f\}) \leq_{N'} S(P_1, P_2, R \cup \{f\})$ because
  1. $S(P_1 \cup P_2, \phi, R) \leq S(P_1, P_2, R)$ in N', and
  2. $S((P_1 \cup P_2) \setminus \{e\}, \phi, R) \leq S(P_1 \setminus \{e\}, P_2, R)$ in N'/e
- And so on ...

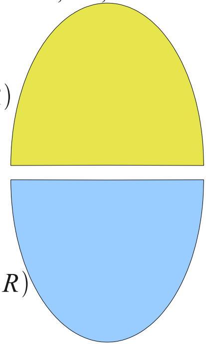
Proof (Cont'd)

$$S(P_1 \cup P_2, \phi, R \cup \{f\}) \leq_{N'} S(P_1, P_2, R \cup \{f\})$$



$$(1)\, S(P_1 \cup P_2 \cup \{f\}, \phi, R) \leq_{N'} S(P_1 \cup \{f\}, P_2, R)$$

$$(2)\, S(P_1 \cup P_2, \{f\}, R) \leq_{N'} S(P_1, P_2 \cup \{f\}, R)$$



$$e \in P_1,\ S(P_1 \cup P_2, \phi, R) \leq_{N'} S(P_1, P_2, R) \Rightarrow (1)$$

$$S((P_1 \cup P_2) \setminus e, \phi, R) \leq_{N'/e} S(P_1 \setminus e, P_2, R) \Rightarrow (2)$$

The Inequality Theorem

Notation

$k = \rho(E_1) + \rho(E_2) - \rho(E_1 \cup E_2) - \rho(E_1 \cap E_2)$

The Theorem

If $G \in \mathcal{ESG}$ then for all $E_1, E_2 \subseteq E$,

$$x^k \cdot R(E_1 \cup E_2; x, y) \cdot R(E_1 \cap E_2; x, y) \leq R(E_1; x, y) \cdot R(E_2; x, y),$$

when $xy < 1$ and $x, y \geq 0$.

Concluding Remarks

- Therefore Series-Parallel graphs satisfy the "old" inequality at all points $x, y \geq 0$ such that $xy < 1$, and not just at $(0,0), (1,0)$ and $(0,1)$
- What more can be said about extended submodularity in graphs with a K_4 minor? For example, is it true that in all graphs $S(P_1 \cup P_2, \phi, R) \leq S(P_1, P_2, R)$ whenever R is a forest? (This would imply "new" inequality is true whenever $E_1 \cap E_2$ is a forest.)
- Conjecture: The "new" inequality is true for all graphs and all subsets E_1, E_2 (and hence the "old" inequality for all graphs). But can extended submodularity be "further extended" to deal with the general case?