From root systems to Dynkin diagrams

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Abstract. We describe root systems and their associated Dynkin diagrams; these notes follow closely the book of Erdman & Wildon ("Introduction to Lie algebras", 2006) and lecture notes of Willem de Graaf (Italy). We briefly describe how root systems arise from Lie algebras.

1. Root systems

1.1. Euclidean spaces. Let V be a finite dimensional **Euclidean space**, that is, a finite dimensional $\mathbb{R}$-space with inner product $(-,-)\colon V\times V\to\mathbb{R}$, which is bilinear, symmetric, and positive definite. The **length** of $v\in V$ is $||v|| = \sqrt{(v,v)}$; the **angle** α between two non-zero $v,w\in V$ is defined by $\cos\alpha = \frac{(v,w)}{||v||||w||}$.

If $v\in V$ is non-zero, then the **hyperplane** perpendicular to v is $H_v=\{w\in V \mid (w,v)=0\}$. The **reflection** in H_v is the linear map $s_v\colon V\to V$ which maps v to $-v$ and fixes every $w\in H_v$; recall that $V = H_v\oplus \mathrm{Span}_{\mathbb{R}}(v)$, hence

$$s_v\colon V\to V,\quad w\mapsto w-\frac{2(w,v)}{(v,v)}v.$$

In the following, for $v,w\in V$ we write

$$\langle w,v\rangle = \frac{2(w,v)}{(v,v)};$$

note that $\langle -,-\rangle$ is linear only in the first component. An important observation is that each s_u leaves the inner product invariant, that is, if $v,w\in V$, then $(s_u(v),s_u(w))=(v,w)$.

We use this notation throughout these notes.

1.2. Abstract root systems.

Definition 1.1. A finite subset $\Phi\subseteq V$ is a **root system** for V if the following hold:

- (R1) $0\notin\Phi$ and $\mathrm{Span}_{\mathbb{R}}(\Phi)=V$,
- (R2) if $\alpha\in\Phi$ and $\lambda\alpha\in\Phi$ with $\lambda\in\mathbb{R}$, then $\lambda\in\{\pm1\}$,
- (R3) $s_\alpha(\beta)\in\Phi$ for all $\alpha,\beta\in\Phi$,
- (R4) $\langle\alpha,\beta\rangle\in\mathbb{Z}$ for all $\alpha,\beta\in\Phi$.

The **rank** of Φ is $\dim(V)$. Note that each s_α permutes Φ, and if $\alpha\in\Phi$, then $-\alpha\in\Phi$.

Lemma 1.2. *If $\alpha,\beta\in\Phi$ with $\alpha\neq\pm\beta$, then $\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle\in\{0,1,2,3\}$.*

Proof. By (R4), the product in question is an integer. If $v,w\in V\setminus\{0\}$, then $(v,w)^2 = ||v||||w||\cos^2(\theta)$ where θ is the angle between v and w, thus $\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle = 4\cos^2\theta\leq 4$. If $\cos^2(\theta)=1$, then θ is a multiple of π, so α and β are linearly dependent, a contradiction. □

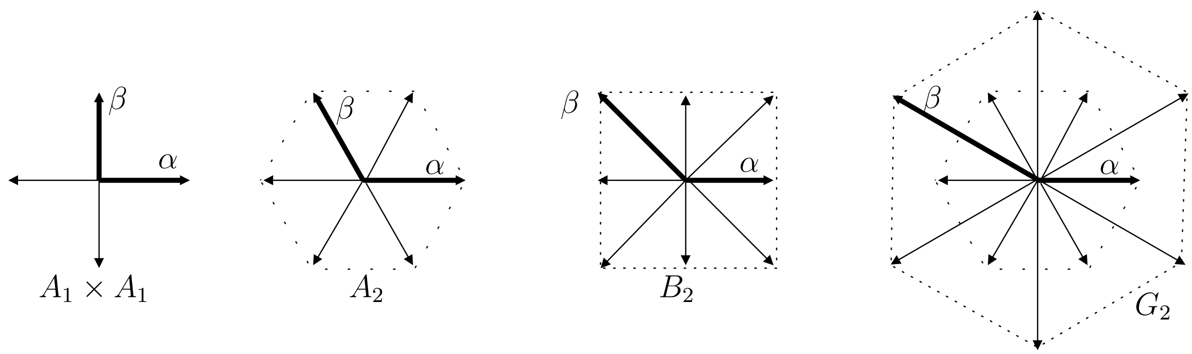
If $\alpha, \beta \in \Phi$ with $\alpha \neq \pm\beta$ and $||\beta|| \geq ||\alpha||$, then $|\langle \beta, \alpha \rangle| \geq |\langle \alpha, \beta \rangle|$; by the previous lemma, all possibilities are listed in Table 1.2.

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	θ	$\frac{(\beta,\beta)}{(\alpha,\alpha)}$
0	0	$\pi/2$	–
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

TABLE 1. Angles between root vectors.

Let $\alpha, \beta \in \Phi$ with $(\alpha, \beta) \neq 0$ and $(\beta, \beta) \geq (\alpha, \alpha)$. Recall that $s_\beta(\alpha) = \alpha - \langle \alpha, \beta \rangle \beta \in \Phi$, and $\langle \alpha, \beta \rangle = \pm 1$, depending on whether the angle between α and β is obtuse or acute, see Table 1. Thus, if the angle is $> \pi/2$, then $\alpha + \beta \in \Phi$; if the angle is $< \pi/2$, then $\alpha - \beta \in \Phi$.

EXAMPLE 1.3. We construct all root systems Φ of $\mathbb{R}^2$. Suppose $\alpha \in \Phi$ is of shortest length and choose $\beta \in \Phi$ such that the angle $\theta \in \{\pi/2, 2\pi/3, 3\pi/4, 5\pi/6\}$ between α and β is as large as possible. This gives root systems of *type* $A_1 \times A_1$, A_2, B_2, and G_2, respectively:



DEFINITION 1.4. A root system Φ is **irreducible** if it cannot be partitioned into two non-emtpy subsets Φ_1 and Φ_2, such that $(\alpha, \beta) = 0$ for all $\alpha \in \Phi_1$ and $\beta \in \Phi_2$.

LEMMA 1.5. *If* Φ *is a root system, then* $\Phi = \Phi_1 \cup \ldots \cup \Phi_k$, *where each* Φ_i *is an irreducible root system for the space* $V_i = \mathrm{Span}_{\mathbb{R}}(\Phi_i) \leq V$; *in particular,* $V = V_1 \oplus \ldots \oplus V_k$.

PROOF. For $\alpha, \beta \in \Phi$ write $\alpha \sim \beta$ if and only if there exist $\gamma_1, \ldots, \gamma_s \in \Phi$ with $\alpha = \gamma_1$, $\beta = \gamma_s$, and $(\gamma_i, \gamma_{i+1}) \neq 0$ for $1 \leq i < s$; then $\sim$ is an equivalence relation on Φ. Let $\Phi_1, \ldots, \Phi_k$ be the equivalence classes of this relation. Clearly, (R1), (R2), and (R4) are satisfied for each Φ_k and $V_k = \mathrm{Span}_{\mathbb{R}}(\Phi_k)$. To prove (R3), consider $\alpha \in \Phi_k$ and $\beta \in \Phi_k$; if $(\alpha, \beta) = 0$, then $s_\alpha(\beta) = \beta \in \Phi_k$. If $(\alpha, \beta) \neq 0$, then $(\alpha, s_\alpha(\beta)) \neq 0$ since s_α leaves the inner product invariant; thus, $s_\alpha(\beta) \sim \alpha$, and $s_\alpha(\beta) \in \Phi_k$. In particular, Φ_k is an irreducible root system of V_k. Clearly, every root appears in some V_i, and the sum of V_i spans V. If $v_1 + \ldots + v_k = 0$ with each $v_i \in V_i$, then $0 = (v_1 + \ldots + v_k, v_j) = (v_j, v_j)$ for all j, that is, $v_j = 0$ for all j. □

1.3. Bases of root systems. Let Φ be a root system for V.

DEFINITION 1.6. A subset $\Pi \subseteq \Phi$ is a **base** (or **root basis**) for Φ if the following hold:

(B1) Π is a vector space basis for V,

(B2) every $\alpha \in \Phi$ can be written as $\alpha = \sum_{\beta\in\Pi} k_\beta \beta$ with either all $k_\beta \in \mathbb{N}$ or all $-k_\beta \in \mathbb{N}$.

A root $\alpha \in \Phi$ is **positive with respect to** Π if the coefficients in (B2) are positive; otherwise α is negative. The roots in Π are called **simple roots**; the reflections s_β with $\beta \in \Pi$ are **simple reflections**.

We need the notion of root orders to prove that every root system has a base.

DEFINITION 1.7. A **root order** is a partial order "$>$" on V such that every $\alpha \in \Phi$ satisfies $\alpha > 0$ or $-\alpha > 0$, and "$>$" is compatible with addition and scalar multiplication.

LEMMA 1.8. *Let Φ be a root system of V.*

a) *Let $\{v_1, \ldots, v_\ell\}$ be a basis of V and write $v > 0$ if and only if $v = \sum_{i=1}^{\ell} k_i v_i$ and the first non-zero k_i is positive; define $v > w$ if $v - w > 0$. Then "$>$" is the* **lexicographic root order** *with respect to the ordered basis $\{v_1, \ldots, v_\ell\}$.*

b) *Choose $v_0 \in V$ outside the (finitely many) hyperplanes H_α, $\alpha \in \Phi$. For $u, v \in V$ write $u > v$ if and only if $(u, v_0) > (v, v_0)$. Then "$>$" is the* **root order defined by** *v_0.*

Let "$>$" be any root order; call $\alpha \in \Phi$ **positive** if $\alpha > 0$, and negative otherwise; α is **simple** if $\alpha > 0$ and it cannot be written as a sum of two positive roots. The proof of the following theorem shows that the set Π of all simple roots is a base for Φ:

THEOREM 1.9. *Every root system has a base.*

PROOF. Let "$>$" be a root order with set of simple roots Π; we show that Π is a base. First, let $\alpha, \beta \in \Pi$ with $\alpha \neq \beta$: If $\alpha - \beta$ is a positive root, then $\alpha = (\alpha - \beta) + \beta$, and α is not simple; if $\alpha - \beta$ is a negative root, then $\beta = -(\alpha - \beta) + \alpha$ is not simple; thus, $\alpha - \beta \notin \Phi$. This implies that the angle of α and β is at least $\pi/2$, hence $(\alpha, \beta) \leq 0$ as seen in Table 1.

Second, Π is linearly independent: if not, then there exist pairwise distinct $\alpha_1, \ldots, \alpha_k \in \Pi$ with $\alpha_1 = \sum_{i=2}^{k} k_i \alpha_i = \beta_+ + \beta_-$, where β_+ and β_- are the sums of all $k_i\alpha_i$ with k_i positive and negative, respectively. By construction, $(\beta_+, \beta_-) \geq 0$ since each $(\alpha_i, \alpha_j) \leq 0$. Note that $\beta_+ \neq 0$ since $\alpha_1 > 0$, thus $(\beta_+, \beta_+) > 0$ and $(\alpha_1, \beta_+) = (\beta_+, \beta_+) + (\beta_-, \beta_+) > 0$. On the other hand, $(\alpha_1, \alpha_j) \leq 0$ and the definition of β_+ imply that $(\alpha_1, \beta_+) \leq 0$, a contradiction.

Finally, we show that every positive root $\alpha \in \Phi$ is a linear combination of Π with coefficients in $\mathbb{N}$. Clearly, this holds if $\alpha \in \Pi$. If $\alpha \notin \Pi$, then $\alpha = \beta + \gamma$ for positive roots $\beta, \gamma \in \Phi$ with $\alpha > \beta, \gamma$. By induction, β and γ are linear combinations of Π with coefficients in $\mathbb{N}$. □

COROLLARY 1.10. *Let $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ be a root basis of Φ.*

a) *If "$>$" is the lexicographic order on V with respect to the basis Π of V, then Π is the set of simple roots with respect to "$>$"; thus, every base is defined by some root order.*

b) *If $v_0 \in V$ with $(v_0, \alpha) > 0$ for all $\alpha \in \Pi$, then Π is the set of simple roots with respect to the root basis defined by v_0.*

Proof. a) This is obvious.

b) Denote by ">" the root order defined by v_0. Let $\alpha_j \in \Pi$; clearly, $\alpha_j > 0$. Suppose $\alpha_j = \beta+\gamma$ for some $\beta, \gamma \in \Phi$ with $\beta, \gamma > 0$. Write $\beta = \sum_{i=1}^{\ell} k_i\alpha_i$ and $\gamma = \sum_{i=1}^{\ell} h_i\alpha_i$, thus $k_j + h_j = 1$ and $k_i = -h_i$ if $i \neq j$. Recall that either $k_1, \ldots, k_\ell \geq 0$ or $k_1, \ldots, k_\ell \leq 0$; by definition, each $(v_0, \alpha_k) > 0$, thus $(v_0, \beta) > 0$ implies $k_1, \ldots, k_\ell \geq 0$. Analogously, $(v_0, \gamma) > 0$ forces $h_1, \ldots, h_\ell \geq 0$. Thus, if $i \neq j$, then $h_i = -k_i$ implies $h_i = k_i = 0$. Now $\beta = k_j\alpha_j$ and $\gamma = h_j\alpha_j$ yield $k_j, h_j \in \{\pm 1\}$. But $h_j + k_j = 1$, which is not possible. Thus α_j must be simple. □

The proof of Theorem 1.9 also implies the following.

Corollary 1.11. *If $\alpha, \beta \in \Phi$ are distinct simple roots, then $(\alpha, \beta) \leq 0$.*

If Π is a root base of Φ, then $\alpha \in \Phi$ is **positive with respect to** Π if α is positive with respect to the root order which defines Π; write Φ^+ and Φ^- for the set of positive and negative roots, respectively. Note that $-\Phi^- = \Phi^+$ and $\Phi = \Phi^+ \cup \Phi^-$.

We remark that root bases can also be constructed geometrically: Fix a hyperplane in V which intersects Φ trivially; label the roots on one side of the hyperplane positive, the other negative. Define Π to be the set of positive roots which are nearest to the hyperplane.

1.4. Weyl group. Let Φ be a root system with ordered root base $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$.

Definition 1.12. The **Weyl group** of Φ is the subgroup of linear transformations of V generated by all reflections s_α with $\alpha \in \Phi$, that is, $W = W(\Phi) = \langle s_\alpha \mid \alpha \in \Phi \rangle$.

Lemma 1.13. *The Weyl group W of Φ is finite.*

Proof. By (R3), there is a group homomorphism $\varphi\colon W \to \mathrm{Sym}(\Phi)$. Since Φ contains a basis of V, the kernel of φ is trivial, hence $W \cong \varphi(W) \leq \mathrm{Sym}(\Phi)$ is finite. □

Theorem 1.14. *Let W_0 be the subgroup of W generated by the simple reflections $s_{\alpha_1}, \ldots, s_{\alpha_\ell}$.*

a) *Each s_{α_i} permutes the positive roots other than α_i.*
b) *If $\beta \in \Phi$, then $\beta = g(\alpha)$ for some $g \in W_0$ and $\alpha \in \Pi$.*
c) *We have $W = W_0$.*

Proof. a) Let $\beta \in \Pi$ with $\beta \neq \alpha_i$ be positive; write $\beta = \sum_{i=1}^{\ell} k_i\alpha_i$ with all $k_i \geq 0$. Since $\beta \neq \alpha_i$, there must be $k_j > 0$ for some $j \neq i$. The coefficient of α_j in $s_{\alpha_i}(\beta) = \beta - \langle \beta, \alpha_i \rangle \alpha_i$ still is $k_j > 0$, hence $s_\alpha(\beta)$ is positive.

b) We first consider $\beta \in \Phi^+$ and show that $\beta = g(\alpha)$ for some $g \in W_0$ and $\alpha \in \Pi$. The assertion follows by induction on the **height** of β defined as

$$\mathrm{ht}(\beta) = \sum\nolimits_{\gamma\in\Pi} k_\gamma \quad \text{where} \quad \beta = \sum\nolimits_{\gamma\in\Pi} k_\gamma\gamma.$$

If $\mathrm{ht}(\beta) = 1$, then choose $g = 1$ and $\alpha = \beta \in \Pi$; if $\mathrm{ht}(\beta) \geq 2$, then, by (R2), at least two k_γ must be strictly positive. Suppose $(\beta, \gamma) \leq 0$ for all $\gamma \in \Pi$. Then $(\beta, \beta) = \sum_{\gamma\in\Pi} k_\gamma(\beta, \gamma) \leq 0$, a contradiction to $\beta \neq 0$. Thus, there is $\gamma \in \Pi$ with $(\beta, \gamma) > 0$, and so

$$\mathrm{ht}(s_\gamma(\beta)) = \mathrm{ht}(\beta) - \langle \beta, \gamma \rangle < \mathrm{ht}(\beta).$$

Recall that $s_\gamma(\beta)$ is positive; by the induction hypothesis, $s_\gamma(\beta) = g'(\alpha)$ for some $g' \in W_0$ and $\alpha \in \Pi$, hence $\beta = g(\alpha)$ with $g = s_\gamma \circ g' \in W_0$. Negative roots are dealt with analogously.

c) We have to show that $s_\beta \in W_0$ for every $\beta \in \Phi$. Part b) shows that $\beta = g(\alpha)$ for some $g \in W_0$ and $\alpha \in \Pi$, and one can show that $s_\beta = g \circ s_\alpha \circ g^{-1}$, which lies in W_0. □

Corollary 1.15. *The root system* Φ *is completely determined by a base* Π.

Theorem 1.16. *If* Π *and* Π' *are two root bases of* Φ*, then* $g(\Pi) = \Pi'$ *for some* $g \in W$.

Proof. Consider Π and Π' as the simple roots with respect to root orders defined by $v_0 \in V$ and $v_0' \in V$, respectively. The **Weyl vector** with respect to Π is $\rho = \frac{1}{2}\sum_\beta \beta$ where β runs over all roots which are positive with respect to Π; similarly, ρ' is defined with respect to Π'. Since s_α with $\alpha \in \Pi'$ permutes the positive roots other than α, we have $s_\alpha(\rho') = \rho' - \alpha$. Since $W(\Phi)$ is finite, we can choose $w \in W(\Phi)$ such that $(w(v_0), \rho')$ is maximal.

Now, if $\alpha \in \Pi'$, then

$$\begin{aligned}
(w(v_0), \rho') &\geq (s_\alpha(w(v_0)), \rho') && \text{(by the choice of } w\text{)} \\
&= (w(v_0), s_\alpha(\rho')) && \text{(since } s_\alpha^2 = 1 \text{ and } s_\alpha \text{ preserves the inner product)} \\
&= (w(v_0), \rho' - \alpha) \\
&= (w(v_0), \rho') - (w(v_0), \alpha).
\end{aligned}$$

Thus, $(w(v_0), \alpha) \geq 0$ for all $\alpha \in \Pi'$. If $(w(v_0), \alpha) = 0$, then $(v_0, w^{-1}(\alpha)) = 0$, which is impossible as $(v_0, \beta) \neq 0$ for all $\beta \in \Phi$ by the definition of v_0. It follows from Corollary 1.10 that Π' is the set of simple roots with respect to the root order defined by $w(v_0)$. If $\beta \in \Pi$, then $(w(\beta), w(v_0)) = (\beta, v_0) > 0$. Thus, both $w(\Pi)$ and Π' are root bases with respect to the root order defined by $w(v_0)$. It follows that $w(\Pi) = \Pi'$. □

1.5. Cartan matrices. Let Φ be a root system with ordered base $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$.

Definition 1.17. The **Cartan matrix** of Φ with respect to Π is the $\ell \times \ell$ matrix

$$C = (C_{i,j})_{1 \leq i,j \leq \ell} \quad \text{where} \quad C_{ij} = \langle \alpha_i, \alpha_j \rangle \in \{0, \pm 1, \pm 2, \pm 3\}.$$

Note that each diagonal entry of a Cartan matrix is 2. Since $(s_\alpha(u), s_\alpha(v)) = (u, v)$ for all $\alpha \in \Phi$, Theorem 1.16 shows that the Cartan matrix of Φ depends only on the ordering adopted with the chosen base Π, and not on the base itself. If C and C' are two Cartan matrices of a root system Π, then they are **equivalent** (and we write $C \sim C'$) if and only if there is a permutation $\sigma \in \mathrm{Sym}(n)$ with $C_{i,j} = C_{\sigma(i),\sigma(j)}$ for all $1 \leq i, j \leq \ell$. We show that a Cartan matrix basically determines the root system; we first need more notation.

Definition 1.18. Let Φ and Φ' be root systems of V and V', respectively. Then Φ and Φ' are **isomorphic** (and we write $\Phi \cong \Phi'$) if there is a vector space isomorphism $\varphi\colon V \to V'$ such that $\varphi(\Phi) = \Phi'$ and $\langle \alpha, \beta \rangle = \langle \varphi(\alpha), \varphi(\beta) \rangle$ for all $\alpha, \beta \in \Phi$.

An isomorphism of root systems preserves angles between root vectors; it does not necessarily preserve distances as the map $v \mapsto \lambda v$ induces an isomorphism between Φ and $\{c\alpha \mid \alpha \in \Phi\}$.

Lemma 1.19. *Let* Φ *and* Φ' *be root systems with Cartan matrices* C *and* C'*, respectively. Then* $\Phi \cong \Phi'$ *if and only if* $C \sim C'$.

Proof. Let Π and Π' be root bases which define C and C', respectively. First, suppose there is an isomorphism $\varphi\colon \Phi \to \Phi'$ of root systems. Since $\varphi(\Pi)$ is a base of Φ', there is $w \in W(\Phi')$ with $\varphi(\Pi) = w(\Pi')$, see Theorem 1.16. Clearly, Π and $\varphi(\Pi)$ define the same Cartan matrix C, and the Cartan matrix of $w(\Pi')$ is equivalent to the Cartan matrix C' of Π', thus $C \sim C'$.

Second, suppose $C \sim C'$. Up to reordering the simple roots, we can assume that $C = C'$, defined by $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ and $\Pi' = \{\alpha'_1, \ldots, \alpha'_\ell\}$; thus, $\langle \alpha_i, \alpha_j\rangle = \langle \alpha'_i, \alpha'_j\rangle$ for all i, j. Let $\varphi\colon V \to V'$ be the linear map defined by $\varphi(\alpha_i) = \alpha'_i$ for all i. By definition, this is a vector space isomorphism which satisfies $\varphi(\Pi) = \Pi'$ and $\langle \alpha, \beta\rangle = \langle \varphi(\alpha), \varphi(\beta)\rangle$ for all $\alpha, \beta \in \Phi$. It remains to show that $\varphi(\Phi) = \Phi'$.

If $v \in V$ and $\alpha_i \in \Pi$, then $\langle v, \alpha_i\rangle = \langle \varphi(v), \alpha'_i\rangle$ follows from the definition of φ and the fact that $\langle -, -\rangle$ is linear in the first component. This implies $\varphi(s_{\alpha_i}(v)) = \varphi(v) - \langle v, \alpha_i\rangle\alpha' = s_{\alpha'_i}(\varphi(v))$. Thus, the image under φ of the orbit of $v \in V$ under the Weyl group $W(\Phi)$ is contained in the orbit of $\varphi(v)$ under $W(\Phi')$. Since $\Phi = \{w(\alpha) \mid w \in W_0,\ \alpha \in \Pi\}$, see Theorem 1.14b), and $\varphi(\Pi) = \Pi'$, it follows that $\varphi(\Phi) \subseteq \Phi'$. The same argument applied to φ^{-1} shows $\varphi^{-1}(\Phi') \subseteq \Phi$, hence $\varphi(\Phi) = \Phi'$. In conclusion, $\Phi \cong \Phi'$. □

1.6. Dynkin diagrams. Let Φ be a root system with ordered base $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$.

Definition 1.20. The **Dynkin diagram** of Φ (with respect to Π) has vertices $\alpha_1, \ldots \alpha_\ell$, and there are $d_{ij} = \langle \alpha_i, \alpha_j\rangle\langle \alpha_j, \alpha_i\rangle \in \{0, 1, 2, 3\}$ edges between α_i and α_j with $i \neq j$; if $||\alpha_j|| > ||\alpha_i||$, then these edges are directed, pointing to the shorter root α_i. The same graph, but without directions, is the **Coxeter graph** of Φ.

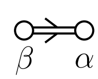
If there is a single edge between α and β, then $||\alpha|| = ||\beta||$ and the edge is undirected, see Table 1; if there are multiple edges between, then $||\alpha|| \neq ||\beta||$ and the edges are directed.

Theorem 1.21. *Two root systems are isomorphic if and only if their Dynkin diagrams are the same (up to relabeling the vertices).*

Proof. By Lemma 1.19, isomorphic root systems have similar Cartan matrices, and the entries of a Cartan matrix define the Dynkin diagram. Thus, up to relabeling the simple roots, the Dynkin diagrams are the same. Conversely, from a Dynkin diagram one can recover the values $\langle \alpha_i, \alpha_j\rangle$ for all $1 \leq, i, j \leq \ell$; recall that $\langle \alpha_i, \alpha_j\rangle < 0$, and Table 1 determines the angle between α_i and α_j, and their proportion of lengths. In particular, the Cartan matrix is determined. Together with Lemma 1.19, it follows that identical Dynkin diagrams define identical Cartan matrices, which define isomorphic root systems. □

Theorem 1.22. *A root system Φ is irreducible if and only if its Dynkin diagram is connected.*

Example 1.23. Consider Example 1.3b). We have $\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(2\alpha+\beta)\}$ with base $\Pi = \{\alpha, \beta\}$. The angle between α and β is $3\pi/4$, and $||\beta|| > ||\alpha||$. Table 1 shows that $\langle \alpha, \beta\rangle = -1$ and $\langle \beta, \alpha\rangle = -2$. Thus, the associated Cartan matrix and Dynkin diagram are

$C = \left(\begin{smallmatrix} 2 & -1 \\ -2 & 2 \end{smallmatrix}\right)$ and B_2 : 

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Conversely, from such a diagram we read off that $||\beta|| > ||\alpha||$ and $\langle \alpha, \beta\rangle\langle \beta, \alpha\rangle = 2$; Table 1 shows $\langle \alpha, \beta\rangle = -1$ and $\langle \beta, \alpha\rangle = -2$, recall that both values must be negative by Corollary

1.11. In particular, the angle between α and β is $3\pi/4$, and $||\beta|| = 2||\alpha||$. Note that we have recovered the Cartan matrix and, using Corollary 1.15, we can recover the root system by constructing the closure of $\{\pm\alpha, \pm\beta\}$ under simple reflections; the latter can be translated into an efficient algorithm (for arbitrary Dynkin diagrams).

In conclusion, we have seen how a root system is (up to isomorphism) uniquely determined by its Dynkin diagram.

2. Irreducible root systems

By Lemma 1.5, it suffices to study irreducible root systems; the associated Dynkin diagrams are classified in the following theorem.

Theorem 2.1. *The Dynkin diagram of an irreducible root system is either a member of one of the four families A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(c \geq 3)$, D_n $(n \geq 4)$ as shown in Table 2, where each diagram has n vertices, or one of the five exceptional diagrams E_6, E_7, E_8, G_2, F_4 as shown in Table 3. Each of the diagrams listed in Tables 2 and 3 occur as the Dynkin diagram of some irreducible root system.*

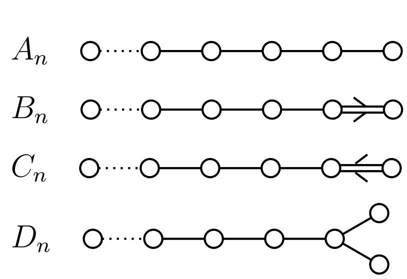

Table 2. Four infinite families of connected Dynkin diagrams.

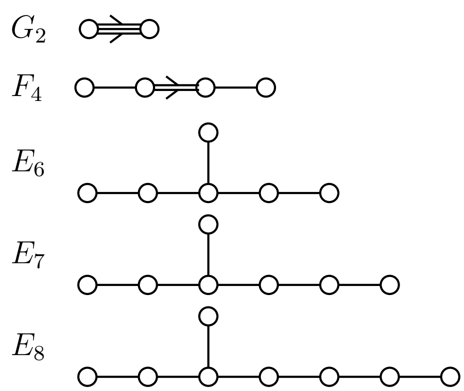

Table 3. Five exceptional Dynkin diagrams.

Sketch of Proof. Recall that the Coxeter diagram of an irreducible root system is the (connected) Dynkin diagram with all edges considered as undirected; the first step of the proof is to classify all connected Coxeter diagrams. For this, we consider **admissible sets** of an Euclidean space V with inner product $(-,-)$, that is, a set $A = \{v_1, \ldots, v_n\}$ of linearly independent unit vectors with $(v_i, v_j) \leq 0$ and $4(v_i, v_j)^2 \in \{0, 1, 2, 3\}$ if $i \neq j$. The associated graph Γ_A has vertices $v_1, \ldots, v_n$, and $d_{ij} = 4(v_i, v_j)^2$ edges between v_i and v_j if $i \neq j$. Every Coxeter diagram is the graph Γ_A for some admissible set A. We determine the structure of Γ_A for an admissible set A; we assume that Γ_A is connected and proceed as follows:

a) The number of vertices in Γ_A joined by at least one edge is at most $|A| - 1$:
$v = v_1 + \ldots + v_n \neq 0$ satisfies $(v, v) = n + 2\sum_{i<j}(v_i, v_j) > 0$, and so $n > \sum_{i<j} -2(v_i, v_j) = \sum_{i<j} \sqrt{d_{ij}} \geq N$, where N is the number of pairs $\{v_i, v_j\}$ such that $d_{ij} \geq 1$.

b) The graph Γ_A contains no cycles:
The vectors in a cycle of Γ_A form an admissible set A' which a contradicts a).

c) No vertex in Γ_A has more than four edges:
Let w be a vertex of Γ_A with adjacent vertices $w_1, \ldots, w_k$. Since there are no cycles, $(w_i, w_j) = 0$ for $i \neq j$. Let $U = \mathrm{Span}_{\mathbb{R}}(w_1, \ldots, w_k, w)$, and extend $\{w_1, \ldots, w_k\}$ to an orthonormal basis of U, say by adjoining w_0. Clearly, $(w, w_0) \neq 0$ and $w = \sum_{i=0}^{k}(w, w_i)w_i$. By assumption, w is a unit vector, so $1 = (w, w) = \sum_{i=0}^{k}(w, w_i)^2$. Since $(w, w_0)^2 > 0$, this shows that $\sum_{i=1}^{k}(w, w_i)^2 < 1$. Now, as A is admissible and $(w, w_i) \neq 0$, we know that $(w, w_i)^2 \geq 1/4$ for $1 \leq i \leq k$; hence $k \leq 3$.

d) If Γ_A has a triple edge, then Γ_A is the Coxeter graph of type G_2:
This follows from c) and the fact that Γ_A is assumed to be connected.

e) If Γ_A has a subgraph which is a line along $w_1, \ldots, w_k$ with single edges between w_i and w_{i+1}; let $A' = (A \setminus \{w_1, \ldots, w_k\}) \cup \{w\}$ where $w = w_1 + \ldots + w_k$. Then A' is admissible and the graph $\Gamma_{A'}$ is obtained from Γ_A by shrinking the line to a single vertex:
Clearly, A' is linearly independent, so we need only to verify the conditions on the inner products. By assumption, $2(w_i, w_{i+1}) = -1$ for $1 \leq i \leq k-1$ and $(w_i, w_j) = 0$ for $i \neq j$ otherwise, thus $(w, w) = k + 2\sum_{i=1}^{k-1}(w_i, w_{i+1}) = k - (k-1) = 1$. Suppose $v \in A$ and $v \neq w_i$ for $1 \leq i \leq k$; since there are no cycles, v is joint to at most one w_i. Thus, either $(v, w) = 0$ or $(v, w) = (v, w_i)$, and then $4(v, w) \in \{0, 1, 2, 3\}$, so A' is an admissible set; also $\Gamma_{A'}$ is determined.

f) A branch vertex is a vertex which is adjacent to three or more edges; by c), a branch vertex is adjacent to exactly three edges. The graph Γ_A has no more than one double edge, not both a double edge and a branch vertex, and no more than one branch vertex:
If Γ_A contains two or more double edges, then it has a subgraph which is a line along $w_1, \ldots, w_k$ with single edges between $w_2, \ldots, w_{k-1}$, and double edges between w_1 and w_2, and w_{k-1} and w_k. By e), we obtain an admissible set $\{w_1, v, w_k\}$ with each two edges between w_1 and v, and v and w_k, contradicting c). The other two parts of the claim are proved in a similar way.

g) If Γ_A has a subgraph which is a line along $w_1, \ldots, w_k$, then $(w, w) = k(k+1)/2$ where $w = w_1 + 2w_2 + \ldots + kw_k$:
The shape of the subgraph implies $2(w_i, w_{i+1}) = -1$ for $1 \leq i \leq k-1$, and $(w_i, w_j) = 0$ for $i \neq j$ otherwise; the claim follows.

h) If Γ_A has a double edge, then Γ_A is a Coxeter graph of type B_n or F_4:
Such a Γ_A is a line along $w_1, \ldots, w_p, u_q, u_{q-1}, \ldots, u_1$ with single edges and one double edge between w_p and u_q. By g), $(w, w) = p(p+1)/2$ and $(u, u) = q(q+1)/2$ for $w = \sum_{i=1}^{p} iw_i$ and $u = \sum_{i=1}^{q} iu_i$. From the graph, $4(w_p, u_q)^2 = 2$ and $(w_i, u_j) = 0$ otherwise, hence $(w, u)^2 = (pw_p, qu_q)^2 = p^2q^2/2$. As w and u are linearly independent, the Cauchy-Schwarz inequality implies $(w, u)^2 < (w, w)(u, u)$, which yields $2pq < (p+1)(q+1)$, hence $(p-1)(q-1) = pq - q - p < 2$. So either $q = 1$ or $p = q = 2$.

i) If Γ_A has a branch vertex, then Γ_A is of type D_n for some $n \geq 4$, or E_6, E_7, or E_8:
Such a graph consists of three lines $v_1, \ldots, v_p, z$ and $x_1, \ldots, x_r, z$ and $w_1, \ldots, w_q, z$, connected at the branch vertex z; we can assume $p \geq q \geq r$. We have to show that either $q = r = 1$ or $q = 2$, $r = 1$, and $p \leq 4$. Let $v = \sum_{i=1}^{p} iv_i$, $w = \sum_{i=1}^{q} iw_i$, and $x = \sum_{i=1}^{r} ix_i$. Note that x, w, x are pairwise orthogonal and $U = \mathrm{Span}_{\mathbb{R}}(v, w, x, z)$ has orthonormal basis $\{\hat{x}, \hat{v}, \hat{w}, z_0\}$ for a suitable z_0, where $\hat{u} = u/||u||$. Write $z = (z, \hat{v})\hat{v} + (z, \hat{w})\hat{w} + (z, \hat{x})\hat{x} + (z, z_0)z_0$; as z is a unit vector and $(z, z_0) \neq 0$, we get $(z, \hat{v})^2 + (z, \hat{w})^2 + (z, \hat{x})^2 < 1$. By g), the lengths of v, w, and x are known. Also, $(z, v)^2 = (z, pv_p)^2 = p^2/4$, and similarly $(z, w)^2 = q^2/4$ and $(z, x)^2 = r^2/4$. Substituting these in the previous inequality gives

$$\frac{2p^2}{4p(p+q)} + \frac{2q^2}{4q(q+1)} + \frac{2r^2}{4r(r+1)} < 1.$$

This is equivalent to $(p+1)^{-1} + (q+1)^{-1} + (r+1)^{-1} > 1$. Since $(p+1)^{-1} \leq (q+1)^{-1} \leq (r+1)^{-1} \leq 1/2$, we have $1 < 3/(r+1)$, and hence $r < 2$, so $r = 1$. Repeating this argument gives $q < 3$, so $q = 1$ or $q = 2$. If $q = 2$, then $p < 5$. On the other hand, if $q = 1$, then there is no restriction on p.

We have proved that the Coxeter diagram of an irreducible root system is a Coxeter diagram of type A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, or G_2; this proves that every connected Dynkin diagram occurs in Tables 2 and 3.

That every diagram in Tables 2 and 3 is indeed a Dynkin diagram of some root system follows from a direct construction. We omit the proof here; see Section 3 for more details. □

3. Root systems of Lie algebras

In this section we give a *very very brief* description of how the finite dimensional simple Lie algebras over $\mathbb{C}$ can be classified by Dynkin diagrams.

A **Lie algebra** over the complex numbers is a $\mathbb{C}$-vector space $\mathfrak{g}$ together with a multiplication (Lie bracket)

$$[-,-]\colon \mathfrak{g} \times \mathfrak{g}, \quad (g, h) \mapsto [g, h],$$

which is bilinear and for all $g, h, k \in \mathfrak{g}$ satisfies $[g, g] = 0$ and the Jacobi identity

$$[g, [h, k]] + [h, [k, g]] + [k, [g, h]] = 0.$$

The Lie algebra $\mathfrak{g}$ is **simple** if its only ideals are $\{0\}$ and $\mathfrak{g}$. Every $g \in \mathfrak{g}$ acts on $\mathfrak{g}$ via the linear transformation $\mathrm{ad}(g)\colon \mathfrak{g} \to \mathfrak{g}$, $h \mapsto [g, h]$; call $g \in \mathfrak{g}$ **semisimple** if $\mathrm{ad}(g)$ is diagonalisable. A subalgebra $\mathfrak{h} \leq \mathfrak{g}$ is a **Cartan subalgebra** if it is abelian, consists of semisimple elements of $\mathfrak{g}$, and is maximal with these properties; up to conjugacy, $\mathfrak{g}$ has a unique Cartan subalgebra.

Let $\mathfrak{h} \leq \mathfrak{g}$ be a Cartan subalgebra of a finite dimensional Lie algebra $\mathfrak{g}$ over the complex numbers. Since $\mathfrak{h}$ consists of pairwise commuting diagonalisable endomorphisms of $\mathfrak{g}$, there exists a $\mathbb{C}$-basis of $\mathfrak{g}$ such that, with respect to this basis, every $\mathrm{ad}(h)$, $h \in \mathfrak{h}$, is a diagonal matrix. Denote by $\mathfrak{h}^*$ the dual space of $\mathfrak{h}$, that is, the space of linear maps $\mathfrak{h} \to \mathbb{C}$. The **root space decomposition** of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha \quad \text{where} \quad \mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \forall h \in \mathfrak{h}\colon [h,x] = \alpha(h)x\}.$$

Let $\Phi \subseteq \mathfrak{h}^*$ be the set of $\alpha \in \mathfrak{h}^*$ with $\mathfrak{g}_\alpha \neq \{0\}$, thus, $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$. Each such $\mathfrak{g}_\alpha$ is 1-dimensional, spanned by a common eigenvector for each $\mathrm{ad}(h)$, $h \in \mathfrak{h}$. It turns out the $V = \mathrm{Span}_{\mathbb{R}}(\Phi)$ can be furnished with an inner product $(-,-)$ such that Φ is a **root system** of V. *(We omit the technical details here*[1]*; proving that Φ satisfies the axioms of a root system is technical and requires significant effort.)* This root system is irreducible if and only if $\mathfrak{g}$ is simple. Moreover, there is a one-to-one correspondence between the isomorphism types of finite dimensional simple Lie algebras over the complex numbers and the isomorphism types of irreducible root systems. Thus, such Lie algebras can be classified up to isomorphism by the different types of connected Dynkin diagrams. In particular, it turns out that for each of the Dynkin diagram in Tables 2 and 3 there exists a Lie algebra whose root system has this Dynkin diagram. *This result completes the proof if Theorem 2.1.* The Dynkin diagrams of type A_n, B_n, C_n, D_n correspond to the classical Lie algebras $\mathfrak{sl}_{n+1}(\mathbb{C})$, $\mathfrak{so}_{2n+1}(\mathbb{C})$, $\mathfrak{sp}_{2n}(\mathbb{C})$, $\mathfrak{so}_{2l}(\mathbb{C})$.

4. More general: Coxeter groups

A **Coxeter group** is a group generated by finitely many involutions (elements of order 2), satisfying specific relations. More precisely, a Coxeter group is a group satisfying a presentation

$$\langle w_1, \ldots, w_k \mid (w_i w_j)^{n_{ij}} = 1 \rangle$$

where $n_{ii} = 1$ for all i and $n_{ji} = n_{ij} \geq 2$ if $i \neq j$. The corresponding **Coxeter matrix** is the symmetric $k \times k$ matrix with integer entries n_{ij}, $1 \leq i, j \leq k$. The associated **Coxeter diagram** has vertices $w_1, \ldots, w_k$ and, if $n_{ij} \geq 3$, then there are n_{ij} edges between w_i and w_j. If $n_{ij} \geq 4$, then this edge is labelled n_{ij}. Finite Coxeter groups can, up to isomorphism, be classified by their Coxeter diagrams; the list of possible Coxeter diagrams contains the Coxeter diagrams of the Dynkin diagrams in Tables 2 and 3.

The Weyl group of a root system is a so-called **reflection group** (a group generated by hyperplane reflections of an Euclidean space), which is a special type of Coxeter group.

[1]The Killing form of $\mathfrak{g}$ is defined by $\kappa(g,h) = \mathrm{tr}(\mathrm{ad}_{\mathfrak{g}}(g) \circ \mathrm{ad}_{\mathfrak{g}}(h))$ where $g, h \in \mathfrak{g}$; it is a bilinear symmetric form, and non-degenerate if and only if $\mathfrak{g}$ is semisimple. Also the restriction of κ to $\mathfrak{h} \times \mathfrak{h}$ is non-degenerate, hence it defines an isomorphism $\varphi\colon \mathfrak{h} \to \mathfrak{h}^*$, $h \mapsto \kappa(h,-)$. For $\alpha \in \Phi \subseteq \mathfrak{h}^*$ let $t_\alpha \in \mathfrak{h}$ with $\kappa(t_\alpha,-) = \alpha(-)$. Now, if $\alpha, \beta \in \Phi$, then $(\alpha,\beta) = \kappa(t_\alpha, t_\beta) = \alpha(t_\beta) \in \mathbb{Q}$ defines a real-valued inner product on $V = \mathrm{Span}_{\mathbb{R}}(\Phi)$; note that if $x_\beta \in \mathfrak{g}_\beta$, then $\mathrm{ad}_{\mathfrak{g}}(t_\theta)(x_\beta) = \beta(t_\theta)x_\beta$, which implies that $(\theta,\theta) = \kappa(t_\theta, t_\theta) = \sum_{\beta\in\Phi} \beta(t_\theta)^2 = \sum_{\beta\in\Phi} (\beta,\theta)^2 \geq 0$ is real. If $(\theta,\theta) = 0$, then $\beta(t_\theta) = 0$ for all roots β, hence $t_\theta = 0$ and $\theta = 0$.