Go to Presentations
Go to $p$-Group Generation
Things we have discussed in the first lecture:

- polycyclic groups, sequences, and series
- polycyclic generating sets (pcgs) and relative orders
- polycyclic presentations (pcp), power exponents, and consistency
- normal forms and collection
- consistency checks
- weighted polycyclic presentations (wpccp)
Conclusion Lecture 1

weighted polycyclic presentation (wpcp):
- all relative orders $p$
- induced polycyclic series is chief series
- relations are partitioned into definitions and non-definitions

Example
Consider

$$G = Pc\langle \ x_1, \ldots, x_5 \ \mid \ x_1^2 = x_4, \ x_2^2 = x_3, \ x_3^2 = x_5, \ x_4^2 = x_5, \ x_5^2 = 1 \ [x_2, x_1] = x_3, \ [x_3, x_1] = x_5 \rangle.$$ 

Here $\{x_1, x_2\}$ is a minimal generating set, and we choose $[x_2, x_1] = x_3$ and $x_1^2 = x_4$ and $[x_3, x_1] = x_5$ as definitions for $x_3, x_4, \text{ and } x_5$, respectively.

Lecture 2: how to compute a wpcp?
Lower exponent \( p \) series

The lower exponent \( p \) series of a \( p \)-group \( G \) is

\[ G = P_0(G) > P_1(G) > \ldots > P_c(G) = 1 \]

where each \( P_{i+1}(G) = [G, P_i(G)]P_i(G)^p \); the \( p \)-class of \( G \) is \( c \).

Important properties

- each \( P_i(G) \) is characteristic in \( G \);
- \( P_1(G) = [G, G]G^p = \Phi(G) \), and \( G/P_1(G) \cong C_p^d \) with \( d = \text{rank}(G) \);
- each section \( P_i(G)/P_{i+1}(G) \) is \( G \)-central and elementary abelian;
- if \( G \) has \( p \)-class \( c \), then its nilpotency class is at most \( c \);
- if \( \theta \) is a homomorphism, then \( \theta(P_i(G)) = P_i(\theta(G)) \);
- \( G/N \) has \( p \)-class \( c \) if and only if \( P_c(G) \leq N \);
- weights: any \( \text{wpcp on } \{a_1, \ldots, a_n\} \) satisfies \( a_i \in P_{\omega(a_i)}(G) \setminus P_{\omega(a_i)+1}(G) \).
Lower exponent-$p$ series

Example 11
Consider

\[ G = D_{16} = Pc\langle a_1, a_2, a_3, a_4 \rangle \mid a_1^2 = 1, a_2^2 = a_3a_4, a_3^2 = a_4, a_4^2 = 1, \]
\[ [a_2, a_1] = a_3, [a_3, a_1] = a_4 \rangle. \]

Here we can read off:

- \( P_0(G) = G \)
- \( P_1(G) = [G, G]G^2 = \langle a_3, a_4 \rangle \)
- \( P_2(G) = [G, P_1(G)]P_1(G)^2 = \langle a_4 \rangle \)
- \( P_3(G) = [G, P_2(G)]P_2(G)^2 = 1 \)

So \( G \) has 2-class 3.
Computing a wpcp of a $p$-group

$p$-quotient algorithm

Input: a $p$-group $G = F/R = \langle x_1, \ldots, x_n \mid \mathcal{R} \rangle$

Output: a wpcp of $G$

Top-level outline:

1. compute wpcp of $G/P_1(G)$ and epimorphism $G \rightarrow G/P_1(G)$, then iterate:
2. given wpcp of $G/P_k(G)$ and epimorphism $G \rightarrow G/P_k(G)$, compute wpcp of $G/P_{k+1}(G)$ and epimorphism $G \rightarrow G/P_{k+1}(G)$;

For the second step, we use the so-called $p$-cover of $G/P_k(G)$.

More general: a “$p$-quotient algorithm” computes a consistent wpcp of the largest $p$-class $k$ quotient (if it exists) of any finitely presented group.

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**Computing a wpcp of** $G/P_1(G)$

Note that $G/P_1(G)$ is elementary abelian.

**Computing wpcp of** $G/P_1(G')$

**Input:** a $p$-group $G = F/R = \langle x_1, \ldots, x_n \mid \mathcal{R} \rangle$

**Output:** a wpcp of $G/P_1(G)$ and epimorphism $\theta: G \to G/P_1(G)$

**Approach:**

1. abelianise relations, take exponents modulo $p$, write these in matrix $M$
2. compute solution space of $M$ over $GF(p)$

**Then:**

- dimension $d$ of solution space is rank of $G$, that is, $G/P_1(G) \cong C_{p}^{d}$
- generating set of $G/P_1(G)$ lifts to subset of given generators;
  set $G/P_1(G) = Pc\langle a_1, \ldots, a_d \mid a_1^p = \ldots = a_d^p \rangle$ and define $\theta$ by

  $$\theta(x_i) = a_i \quad \text{for} \quad i = 1, \ldots, d;$$

  images of $\theta(x_j)$ with $j > d$ are determined accordingly.
Computing a wpcp of $G/P_1(G)$

**Example 12**

$G = \langle x_1, \ldots, x_6 \mid x_6^{10}, x_1x_2x_3, x_2x_3x_4, \ldots, x_4x_5x_6, x_5x_6x_1, x_1x_6x_2 \rangle$ and $p = 2$

Write coefficients of abelianised and mod-2 reduced equations as rows of matrix, use row-echelonisation, and determine that solution space has dimension 2:

$$
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix};
$$

Modulo $P_1(G)$, this shows that $x_1 = x_5x_6$, $x_2 = x_5$, $x_3 = x_6$, $x_4 = x_5x_6$, and **Burnside’s Basis Theorem** implies that $G = \langle x_5, x_6 \rangle$. Lastly, set

$$
G/P_1(G) = Pc\langle a_1, a_2 \mid a_1^2 = a_2^2 = 1 \rangle,
$$

and define $\theta: G \rightarrow G/P_1(G)$ via $x_5 \mapsto a_1$ and $x_6 \mapsto a_2$.

This determines $\theta(x_1) = a_1a_2$, $\theta(x_2) = a_1$, $\theta(x_3) = a_2$, and $\theta(x_4) = a_1a_2$. 
Compute \( w_{\text{pcp}} \) for \( G/P_{k+1}(G) \) from that of \( G/P_k(G) \)

Given:
- \( w_{\text{pcp}} \) of \( d \)-generator \( p \)-group \( G/P_k(G) \) and epimorphism \( \theta : G \rightarrow G/P_k(G) \)

Want:
- \( w_{\text{pcp}} \) of \( G/P_{k+1}(G) \) and epimorphism \( G \rightarrow G/P_{k+1}(G) \)

In the following:
- \( H = G/P_k(G) \) and \( K = G/P_{k+1}(G) \) and \( Z = P_k(G)/P_{k+1}(G) \)
- note that \( Z \) is elementary abelian, \( K \)-central, and \( K/Z \cong H \)

Approach: Construct a covering \( H^* \) of \( H \) such that every \( d \)-generator \( p \)-group \( L \) with \( L/M \cong H \) and \( M \leq L \) central elementary abelian, is a quotient of \( H^* \).

Thus, the next steps are:
1. define \( p \)-cover \( H^* \) and determine a \( \text{pcp} \) of \( H^* \); 
2. make this presentation consistent; 
3. construct \( K \) as quotient of \( H^* \) by enforcing defining relations of \( G \).
**Theorem 13: p-covering group**

Let $H$ be a $d$-generator $p$-group; there is a $d$-generator $p$-group $H^*$ with:

- $H^*/M \cong H$ for some central elementary abelian $M \leq H^*$;
- if $L$ is a $d$-generator $p$-group with $L/Y \cong H$ for some central elementary abelian $Y \leq L$, then $L$ is a quotient of $H^*$.

The group $H^*$ is unique up to isomorphism.

**Proof.**


Now $S/S^*$ is elementary abelian $p$-group, so $H^*$ is (finite) $d$-generator $p$-group.

Let $L$ be as in the theorem, and let $\psi: L \rightarrow H$ with kernel $Y$.

Let $\theta: F \rightarrow H$ with kernel $S$. Since $F$ is free, $\theta$ factors through $L$, that is,

$\theta: F \xrightarrow{\varphi} L \xrightarrow{\psi} H$, and so $\varphi(S) \leq \ker \psi = Y$. This implies that $\varphi(S^*) = 1$.

In conclusion, $\varphi$ induces surjective map from $H^* = F/S^*$ onto $L$.

If $H^*$ and $\tilde{H}^*$ are two such covers, then each is an image of the other.
**p-covering group: presentation**

**Given:** a wpcp $\text{Pc} \langle a_1, \ldots, a_m \mid S \rangle$ for $H = G/P_k(G) \cong F/S$ and epimorphism $\theta: G \to H$ with $\theta(x_i) = a_i$ for $i = 1, \ldots, d$

**Want:** a wpcp for $H^* \cong F/S^*$ where $S^* = [S, F]S^p$.

**Recall:** each of $a_{d+1}, \ldots, a_m$ occurs as right hand side of one relation in $S$; write $S = S_{\text{def}} \cup S_{\text{nondef}}$ with $S_{\text{nondef}} = \{s_1, \ldots, s_q\}$.

**Theorem 14**

Using the previous notation, $H^* = \text{Pc} \langle a_1, \ldots, a_m, b_1, \ldots, b_q \mid S^* \rangle$, where

$$S^* = S_{\text{def}} \cup \{s_1 b_1, \ldots, s_q b_q\} \cup \{b_1^p, \ldots, b_q^p\}.$$  

**Note:** $M = \langle b_1, \ldots, b_q \rangle \leq H^*$ is elementary abelian, central, and $H^*/M \cong H$.


**In practice:** fewer new generators are introduced.
**Example 15**

If $H = \text{Pc}\langle a_1, a_2 \mid a_1^2 = a_2^2 = 1 \rangle \cong C_2 \times C_2$, then

$$H^* = \text{Pc}\langle a_1, a_2, b_1, b_2, b_3 \mid a_1^2 = b_1, a_2^2 = b_2, [a_1, a_2] = b_3, b_1^2 = b_2^2 = b_3^2 = 1 \rangle;$$

indeed, $H^* \cong (C_4 \times C_2) : C_4$, thus we have found a consistent wpcp!

**Example 16**

If $H = \text{Pc}\langle a_1, a_2, a_3 \mid a_1^2 = a_3^2 = 1, a_2^2 = a_3, [a_2, a_1] = a_3 \rangle \cong D_8$, then

$$H^* = \text{Pc}\langle a_1, a_2, a_3, b_1, \ldots, b_5 \mid \mathcal{T} \cup \{b_1^2, \ldots, b_5^2\} \rangle$$

with

$$\mathcal{T} = \{a_1^2 = b_1, a_2^2 = a_3 b_2, a_3^2 = b_3, [a_2, a_1] = a_3, [a_3, a_1] = b_4, [a_3, a_2] = b_5\};$$

this pcp has power exponents $[2, 2, 2, 2, 2, 2, 2]$. However, $H^* \cong (C_8 \times C_2) : C_4$, so presentation is **not consistent**!

**Next step:** make the presentation of $H^*$ consistent.
**p-covering group: consistency algorithm**

By Theorem 8, the presentation $H^* = PC\langle u_1, \ldots, u_{m+q} \mid S^* \rangle$ with $(u_1, \ldots, u_{m+q}) = (a_1, \ldots, a_m, b_1, \ldots, b_q)$ is consistent if and only if

$$u_k(u_j u_i) = (u_k u_j) u_i$$

$$u_j^p u_i = u_j^{p-1} (u_j u_i)$$ and $$u_j (u_i^p) = (u_j u_i) u_i^{p-1}$$

$$(1 \leq i < j < k \leq m + q)$$

$$(1 \leq i < j \leq m + q)$$

$$(1 \leq j \leq m + q).$$

**Consistency Algorithm**

4: find consistent presentation for $H^*$

- If each pair of words in the above “consistency checks” collects to the same normal word, then the presentation is consistent.

- Otherwise, the quotient of the two different words obtained from one of these conditions is formed and equated to the identity word: this gives a new relation which holds in the group.

- The pcp for $H$ is consistent, so any new relation is an equation in the elementary abelian subgroup $M$ generated by the new generators $\{b_1, \ldots, b_q\}$, which implies that one of these generators is redundant.

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By Theorem 8, the presentation $H^* = \text{Pc}\langle u_1, \ldots, u_{m+q} \mid S^* \rangle$ with $(u_1, \ldots, u_{m+q}) = (a_1, \ldots, a_m, b_1, \ldots, b_q)$ is consistent if and only if

$$u_k(u_j u_i) = (u_k u_j) u_i \quad (1 \leq i < j < k \leq m + q)$$

$$(u_j^p) u_i = u_j^{p-1} (u_j u_i) \quad \text{and} \quad u_j (u_i^p) = (u_j u_i) u_i^{p-1} \quad (1 \leq i < j \leq m + q)$$

$$u_j (u_j^p) = (u_j^p) u_j \quad (1 \leq j \leq m + q).$$

**Example 17**

Consider $G = \text{Pc}\langle u_1, u_2, u_3 \mid u_1^2 = u_2, u_2^2 = u_3, u_3^2 = 1, [u_2, u_1] = u_3 \rangle$. The last test applied to $u_1$ yields

$$u_1^3 = (u_1^2) u_1 = u_2 u_1 = u_1 u_2 u_3 \quad \text{and} \quad u_1^3 = u_1 (u_1^2) = u_1 u_2,$$

so $u_3 = 1$ in $G$, hence $G = \text{Pc}\langle u_1, u_2 \mid u_1^2 = u_2, u_2^2 = 1 \rangle \cong C_4$. 
Construct $K$ from cover $H^*$ of $H$

So what have we got so far...

- $p$-group $G = F/R = \langle x_1, \ldots, x_n \mid \mathcal{R} \rangle$
- consistent wpcf of $H = G/P_k(G) = \langle a_1, \ldots, a_m \mid S \rangle$
- epimorphism $\theta: G \to H$ with $\theta(x_i) = a_i$ for $i = 1, \ldots, d$
- consistent wpcf of cover $H^* = \langle a_1, \ldots, a_m, b_1, \ldots, b_q \mid S^* \rangle$; note that $H^*/M \cong H$ where $M = \langle b_1, \ldots, b_q \rangle$

Want:
- consistent wpcf of $K = G/P_{k+1}(G)$ and epimorphism $G \to G/P_{k+1}(G)$

Know:
- $K/Z \cong H$ where $Z = P_k(G)/P_{k+1}(G)$ is elementary abelian, central
- $K$ is quotient of $H^*$

Idea:
- construct $K$ as quotient of $H^*$: add relations enforced by $G$ to $S^*$

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Construct $K$ from cover $H^*$ of $H$

So what have we got so far...

- $p$-group $G = F/R = \langle x_1, \ldots, x_n \mid R \rangle$
- consistent wpcp of $H = G/P_k(G) = \Pc \langle a_1, \ldots, a_m \mid S \rangle$
- epimorphism $\theta: G \to H$ with $\theta(x_i) = a_i$ for $i = 1, \ldots, d$
- consistent pcp of cover $H^* = \Pc \langle a_1, \ldots, a_m, b_1, \ldots, b_q \mid S^* \rangle$; note that $H^*/M \cong H$ where $M = \langle b_1, \ldots, b_q \rangle$

Enforcing relations of $G$:

- know that $K = G/P_{k+1}(G)$ is quotient of $H^*$
- lift $\theta: G \to H$ to $\hat{\theta}: F \to H^*$ such that $\hat{\theta}(x_i) = a_i$ for $i = 1, \ldots, d$
- for every relator $r \in R$ compute $n_r = \hat{\theta}(r) \in M$; let $L$ be the subgroup of $M$ generated by all these $n_r$
- by von Dyck’s Theorem $H^*/L \to K$ and $G \to H^*/L$ are surjective; since $K$ is the largest $p$-class $k + 1$ quotient of $G$, we deduce $K = H^*/L$

Finally: find consistent wpcp of $K = H^*/L$ and get epimorphism $G \to K$
Big example: $p$-quotient algorithm in action

Let $G = \langle x, y \mid [[y, x], x] = x^2, (xyx)^4, x^4, y^4, (yx)^3y = x \rangle$ and $p = 2$.

First round:

- compute $G/P_1(G)$ using abelianisation and row-echelonisation:
  
  obtain $H = G/P_1(G) \cong P\langle a_1, a_2 \mid a_1^2 = a_2^2 = 1 \rangle$
  
  and epimorphism $\theta: G \rightarrow H$, which is defined by $(x, y) \rightarrow (a_1, a_2)$.

- construct covering of $H$ by adding new generators and tails:
  
  $H^* = P\langle a_1, \ldots, a_5 \mid a_1^2 = a_3, a_2^2 = a_4, [a_2, a_1] = a_5, a_3^2 = a_4^2 = a_5^2 = 1 \rangle$

- the consistency algorithm shows that this presentation is consistent

- evaluate relations of $G$ in $H^*$:
  
  \begin{itemize}
  \item $1 = [[a_2, a_1], a_1] = \hat{\theta}([[y, x], x]) = \hat{\theta}(x^2) = a_1^2 = a_3$ forces $a_3 = 1$
  \item $(xyx)^4, x^4, y^4$ impose no conditions
  \item $a_1a_3 = \hat{\theta}((yx)^3y) = \hat{\theta}(x) = a_1$ also forces $a_3 = 1$
  \end{itemize}

- construct $G/P_2(G)$ as $H^*/\langle a_3 \rangle$; after renaming $a_4, a_5$:
  
  $G/P_2(G) \cong P\langle a_1, \ldots, a_4 \mid a_1^2 = 1, a_2^2 = a_4, [a_2, a_1] = a_3, a_3^2 = a_4^2 = 1 \rangle$

  and epimorphism $G \rightarrow G/P_2(G)$ defined by $(x, y) \rightarrow (a_1, a_2)$. 
**Big example: $p$-quotient algorithm in action**

\[ G/P_2(G) = \text{Pc}\langle a_1, \ldots, a_4 \mid a_1^2 = 1, a_2^2 = a_4, [a_2, a_1] = a_3, a_3^2 = a_4^2 = 1 \rangle \]

**Second round:**

- construct covering of \( H = G/P_2(G) \) by adding new generators and tails:

  \[ H^* = \text{Pc}\langle a_1, \ldots, a_12 \mid a_1^2 = a_{12}, a_2^2 = a_4, a_3^2 = a_{11}, a_4^2 = a_{10}, [a_2, a_1] = a_3, [a_3, a_1] = a_5, [a_3, a_2] = a_6, [a_4, a_1] = a_7, [a_4, a_2] = a_8, [a_4, a_3] = a_9, a_5^2 = \ldots = a_{12}^2 = 1 \rangle \]

- the consistency algorithm shows only the following inconsistencies:

  - \( a_2(a_2a_2) = a_2a_4 \) and \((a_2a_2) a_2 = a_4 a_2 = a_2a_4a_8 \) \( \implies a_8 = 1 \)
  - \( a_2(a_1a_1) = a_2a_{12} \) and \((a_2a_1) a_1 = a_1a_2a_3a_1 = \ldots = a_2a_5a_{11}a_{12} \) \( \implies a_5a_{11} = 1 \)
  - \( a_2(a_2a_1) = a_1a_2^2a_3^2a_6 = a_1a_4a_6a_{11} \) and \((a_2a_2) a_1 = a_1a_4a_7 \) \( \implies a_6a_7a_{11} = 1 \)
  - \( a_3(a_2a_2) = a_3a_4 \) and \((a_3a_2) a_2 = a_2a_3a_6a_2 = a_2^2a_3a_6a_2 = a_3a_4a_9 \) \( \implies a_9 = 1 \)

- removing redundant gens (and renaming), we obtain the consistent wpcp

\[ H^* = \text{Pc}\langle a_1, \ldots, a_8 \mid a_1^2 = a_8, a_2^2 = a_4, a_3^2 = a_7, a_4^2 = a_6, a_5^2 = \ldots = a_8^2 = 1 \]

\[ [a_2, a_1] = a_3, [a_3, a_1] = a_7, [a_3, a_2] = a_5a_7, [a_4, a_1] = a_5 \]
Big example: \( p \)-quotient algorithm in action

Still second round:

\[ G = \langle x, y \mid [y, x], x \rangle = x^2, (xyx)^4, x^4, y^4, (yx)^3y = x \rangle \text{ and } p = 2; \]

epimorphism \( \theta : G \rightarrow H \) onto \( H = G/P_2(H) \) defined by \( (x, y) \rightarrow (a_1, a_2) \)

\[ H^* = Pc\langle a_1, \ldots, a_8 \mid a_1^2 = a_8, a_2^2 = a_4, a_3^2 = a_7, a_4^2 = a_6, a_5^2 = \ldots = a_8^2 = 1 \]

\[ [a_2, a_1] = a_3, [a_3, a_1] = a_7, [a_3, a_2] = a_5a_7, [a_4, a_1] = a_5 \]

Evaluate relations of \( G \) in \( H^* \):

\[ a_7 = [[a_2, a_1], a_1] = \hat{\theta}([[y, x], x]) = \hat{\theta}(x^2) = a_1^2 = a_8 \text{ forces } a_7 = a_8 \]

\((xyx)^4 \text{ forces } a_6 = 1; \) \( x^4 \) and \( y^4 \) impose no condition

\( \hat{\theta}((yx)^3y) = \hat{\theta}(x) \text{ forces } a_7a_8 = 1 \)

Now construct \( G/P_3(G) \) as \( H^*/\langle a_7a_8, a_6 \rangle \); after renaming:

\[ G/P_3(G) = Pc\langle a_1, \ldots, a_6 \mid a_1^2 = a_6, a_2^2 = a_4, a_3^2 = a_6, a_4^2 = 1, a_5^2 = a_6^2 = 1, \]

\[ [a_2, a_1] = a_3, [a_3, a_1] = a_6, [a_3, a_2] = a_5a_6, [a_4, a_1] = a_5 \]

and the epimorphism \( G \rightarrow G/P_3(G) \) is defined by \( (x, y) \rightarrow (a_1, a_2) \).
Big example: \( p \)-quotient algorithm in action

In conclusion:
We started with

\[
G = \langle x, y \mid [[y, x], x] = x^2, (xyx)^4, x^4, y^4, (yx)^3 y = x \rangle
\]

and computed \( G/P_3(G) \) as

\[
Pc\langle a_1, \ldots, a_6 \mid a_1^2 = a_6, a_2^2 = a_4, a_3^2 = a_6, a_4^2 = a_5^2 = a_6^2 = 1, \\
[a_2, a_1] = a_3, [a_3, a_1] = a_6, [a_3, a_2] = a_5 a_6, [a_4, a_1] = a_5 \rangle
\]

with epimorphism \( G \to G/P_3(G) \) defined by \((x, y) \to (a_1, a_2)\).

One can check that \(|G| = |G/P_3(G)| = 2^6\), hence \( G \cong G/P_3(G) \).

In particular, we have found a consistent wpcp for \( G \).

In general: if our input group is a finite \( p \)-group, then the \( p \)-quotient algorithm constructs a consistent wpcp of that group.
Motivation and Application: Burnside problem

Burnside Problems

- **Generalised Burnside Problem (GBP), 1902:**
  Is every finitely generated torsion group finite?

- **Burnside Problem (BP), 1902:**
  Let $B(d, n)$ be the largest $d$-generator group with $g^n = 1$ for all $g \in G$.
  Is this group finite? If so, what is its order?

- **Restricted Burnside Problem (RBP), ~1940:**
  What is order of largest finite quotient $R(d, n)$ of $B(d, n)$, if it exists?

- Golod-Šafarevič (1964): answer to GBP is “no”; (cf. Ol’shanskii’s Tarski monster)

- Various authors: $B(d, n)$ is finite for $n = 2, 3, 4, 6$, but in no other cases with $d > 1$ is it known to be finite; is $B(2, 5)$ finite?

- Higman-Hall (1956): reduced (RBP) to prime-power $n$.

- Zel’manov (1990-91): $R(d, n)$ always exists! (Fields medal 1994)
Motivation and Application: Burnside problem

Burnside groups:
- $B(d, n) = \langle x_1, \ldots, x_d \mid g^n = 1 \text{ for all words } g \text{ in } x_1, \ldots, x_n \rangle$
- $R(d, n)$ largest finite quotient of $B(d, n)$; exists by Zel’manov

Recall: the $p$-quotient algorithm computes a consistent wpcp of the largest $p$-class $k$ quotient (if it exists) of any finitely presented group.

Implementations of the $p$-quotient algorithm have been used to determine the order and compute pcp's for various of these groups.

<table>
<thead>
<tr>
<th>Group</th>
<th>Order</th>
<th>Authors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B(3, 4)$</td>
<td>$2^{69}$</td>
<td>Bayes, Kautsky &amp; Wamsley (1974)</td>
</tr>
<tr>
<td>$R(2, 5)$</td>
<td>$5^{34}$</td>
<td>Havas, Wall &amp; Wamsley (1974)</td>
</tr>
<tr>
<td>$B(4, 4)$</td>
<td>$2^{422}$</td>
<td>Alford, Havas &amp; Newman (1975)</td>
</tr>
<tr>
<td>$B(5, 4)$</td>
<td>$2^{2728}$</td>
<td>Newman &amp; O’Brien (1996)</td>
</tr>
<tr>
<td>$R(2, 7)$</td>
<td>$7^{20416}$</td>
<td>O’Brien &amp; Vaughan-Lee (2002)</td>
</tr>
</tbody>
</table>