Computational aspects of finite p-groups
Question Sheet

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The questions on this sheet are to be discussed during the two tutorials and the practical sessions; some questions are to be done "by hand", others require to use GAP. In the following, p always denotes a prime and n is a positive integer.

Question 1 (tutorial)
Let G be a nontrivial finite p-group acting on a finite set Ω. Recall that the G-orbit of $\omega \in \Omega$ is defined as the subset $\omega^G = \{\omega^g \mid g \in G\} \subseteq \Omega$; its stabiliser is the subgroup $\mathrm{Stab}_G(\omega) = \{g \in G \mid \omega^g = \omega\} \leq G$.

a) Prove the Orbit-Stabiliser-Theorem, that is, show that $|G|/|\mathrm{Stab}_G(\omega)| = |\omega^G|$.
b) Denote by $\mathrm{Fix}_\Omega(G) = \{\omega \in \Omega \mid \forall g \in G : \omega^g = \omega\}$ the set of G-fixed points in Ω. Use a) to prove that $|\Omega| \equiv |\mathrm{Fix}_\Omega(G)| \bmod p$; in particular, if $|\Omega|$ is a p-power, then $|\mathrm{Fix}_\Omega(G)|$ is divisible by p.
c) Use b) to prove that the center $Z(G) = \{h \in G \mid \forall g \in G : h^g = h\}$ of G is non-trivial.
d) Let $H < G$ be a proper subgroup. Consider an action of H and use b) to prove that $N_G(H) > H$.

Question 2 (tutorial)
Let G be a finite p-group.

a) Prove that if $N \trianglelefteq G$ and G/N is cyclic, then $G' = [N, G]$
b) Prove that if $G/\gamma_2(G)$ is cyclic, then $\gamma_2(G) = \{1\}$ and G is abelian.
c) Prove that $\Phi(G) = G'G^p$; here $\Phi(G)$ is the Frattini subgroup of G (that is, the intersection of all maximal subgroups of G) and G^p is the subgroup of G generated by all p-th powers.

Question 3 (practical)
Use the SmallGroups Library of GAP to obtain a list of all p-groups G of size at most $\max\{p^6, 1000\}$ with the property that G admits a subgroup $A \leq G$ of size p^2 with $C_G(A) = A$; note that A necessarily contains the center of G. Do the following:

a) Let G and $A \leq G$ be as in the question and suppose G is nonabelian. Determine the orders of $N_G(A)$ and $Z(G)$; compute a few examples to see what these orders might be.
b) For the groups in your list, compare their nilpotency class with their order; based on your observations, make a conjecture about the structure of the groups.
c) *Challenge Question:* Prove your conjecture (for example, use a) and induction on the group order).

Question 4 (practical)
There are several ways to store and re-construct a pc-group in GAP; consult the manual at

`http://www.gap-system.org/Manuals/doc/ref/chap46.html`

for the following tasks.

a) Read about and use the commands `CodePcGroup` and `PcGroupCode`.
b) Prove that, indeed, every polycyclic presentation P can be encoded by a positive integer $c = c(P)$. Observe that if P has a generating set of cardinality n, then P has n power relations and $n(n-1)/2$ commutator relations. One (theoretical) way of encoding P as a number is to make use of the uniqueness of prime-power factorisations in $\mathbb{Z}$.
c) Read about and use the command `GapInputPcGroup`.

Question 5 (tutorial/practical)
Let $G = \mathrm{Sym}(9)$ be the symmetric group of rank 9.

a) By hand, determine a polycyclic series and a polycyclic presentation for the Sylow 3-subgroup of G.
b) Now do the same calculation with GAP; compare with your results for a). The following commands might be useful:

```
gap> G   := SymmetricGroup(9);;
gap> S   := SylowSubgroup(G,3);;
gap> iso := IsomorphismPcGroup(S);;
gap> Spc := Image(iso);;
gap> mypcgs := List(Pcgs(Spc),x->PreImagesRepresentative(iso,x));;
```

Question 6 (tutorial/practical)
Let $G = \langle g_1, g_2, g_3 \mid g_1^4 = g_3,\ g_2^4 = g_3,\ g_3^4 = 1,\ g_2^{g_1} = g_2,\ g_3^{g_1} = g_3^2,\ g_3^{g_2} = g_3 \rangle$.

a) By hand, show that this polycyclic presentation is not consistent.
b) By hand, find a consistent polycyclic presentation of G.
c) Construct G in GAP using the following commands:

```
gap> F:=FreeGroup(["g1","g2","g3"]);;
gap> AssignGeneratorVariables(F);
#I  Assigned the global variables [ g1, g2, g3 ]
gap> R:=[g1^4/g3, g2^4/g3, g3^4, Comm(g1,g2), g3^g1/g3^-2,g3^g2/g3];;
gap> G:=F/R;
```

Do `IsomorphismPcGroup`, `StructureDescription`, and `PcGroupFpGroup`. The last command will yield an error message; re-define G by using the consistent pc-presentation you have obtained in b).

Question 7 (tutorial)
For a positive integer n let $G(n) = \langle a, b \mid a^n, b^n, [a, b] = a \rangle$.

a) Prove by hand that if $n = p$ is a prime, then $G(p) \cong C_p$.
b) Does the same hold when n is not prime? (Maybe compute some examples with GAP.)

Question 8 (tutorial/practical)
Consider the dihedral group $G = \langle r, m \mid r^{2^{n-1}},\ m^2,\ r^m = r^{2^{n-1}-1} \rangle$.

a) Find the normal form of the element $w = rmr^2m^2r^3m^3$.
b) Find a polycyclic series of G whose associated PCGS has relative orders $[2, \ldots, 2]$.
c) Find a polycyclic presentation of G, associated to the PCGS you have found in b).
d) Write a GAP function `getDn(n)` which constructs this group using the presentation you have found in c), via `PcGroupFpGroup`.

Question 9 (tutorial)
By hand, compute a wpcp of the group

$$G = \langle a, b, c \mid a^9,\ b^9,\ c^9,\ [[b, a], a] = a^3,\ (aba)^9,\ (ba)^5 a = b,\ [a, c] \rangle;$$

you can use that G has order 3^3.

Question 10 (tutorial)
By hand, show that the nucleus of

$$Q_8 = \mathrm{Pc}\langle a, b, c \mid a^2 = c,\ b^2 = c,\ c^2 = 1,\ [b, a] = c \rangle$$

in Q_8^* is trivial, and deduce that Q_8 has no immediate 2-descendants.

Question 11 (practical)
Make sure the GAP package `Anupq` is installed and running; you might have to do `./configure` and `make` in `pkg/anupq` before you can load it in gap with `LoadPackage(``anupq'')`. Look up the manual and use ...

a) ... the command `Pq` to compute a wpcp of G,

b) ... the command `PqPCover` to compute the 2-covering group G^* of G

c) ... the command `PqDescendants` to compute all immediate descendants of G,

for each group G in the questions above, and for

$$G = \langle x, y \mid [[y,x],x] = x^2,\ (xyx)^4,\ x^4,\ y^4,\ (yx)^3 y = x\rangle \quad \text{with } p = 2.$$

Question 12 (tutorial)
For $n \in \mathbb{N}$ consider the cyclic group $G = C_{p^n} = \mathrm{Pc}\langle r \mid r^{(p^n)}\rangle$; compute G^* and show that G has immediate descendants.