The questions on this sheet are to be discussed during the two tutorials and the practical sessions; some questions are to be done “by hand”, others require to use GAP. In the following, $p$ always denotes a prime and $n$ is a positive integer. This sheet also contains some comments on the solutions.

**Question 1** (tutorial)

Let $G$ be a nontrivial finite $p$-group acting on a finite set $\Omega$. Recall that the $G$-orbit of $\omega \in \Omega$ is defined as the subset $\omega^G = \{ \omega^g \mid g \in G \} \subseteq \Omega$; its stabiliser is the subgroup $\text{Stab}_G(\omega) = \{ g \in G \mid \omega^g = \omega \} \leq G$.

a) Prove the Orbit-Stabiliser-Theorem, that is, show that $|G|/|\text{Stab}_G(\omega)| = |\omega^G|$.

b) Denote by $\text{Fix}_G(\omega) = \{ \omega \in \Omega \mid \forall g \in G : \omega^g = \omega \}$ the set of $G$-fixed points in $\Omega$. Use a) to prove that $|\Omega| = |\text{Fix}_G(\Omega)| \mod p$; in particular, if $|\Omega|$ is a $p$-power, then $|\text{Fix}_G(\Omega)|$ is divisible by $p$.

c) Use b) to prove that the center $Z(G) = \{ h \in G \mid \forall g \in G : h^g = h \}$ of $G$ is non-trivial.

d) Let $H < G$ be a proper subgroup. Consider an action of $H$ and use b) to prove that $N_G(H) > H$.

**SOLUTION:**

a) The map $\psi : G \rightarrow \Omega$, $g \mapsto \omega^g$ induces $\hat{\psi} : \text{Stab}_G(\omega) \backslash G \rightarrow \omega^G$, $\text{Stab}_G(\omega)g \mapsto \omega^g$. Clearly, $\hat{\psi}$ is well-defined: if $\text{Stab}_G(\omega)g = \text{Stab}_G(\omega)h$, then $h = sg$ for some $s \in \text{Stab}_G(\omega)$, and so $\omega^h = \omega^sg = \omega^g$. By construction, $\hat{\psi}$ is surjective. If $\text{Stab}_G(\omega)g$ and $\text{Stab}_G(\omega)h$ are mapped to $\omega^h = \omega^g$, then $gh^{-1} \in \text{Stab}_G(\omega)$, and so $\text{Stab}_G(\omega)g = \text{Stab}_G(\omega)h$. This proves that $\hat{\psi}$ is a bijection, and therefore $|\omega^G| = |\text{Stab}_G(\omega)\backslash G|$; clearly, $|\text{Stab}_G(\omega)\backslash G| = |G|/|\text{Stab}_G(\omega)|$.

b) Note that $\omega \in \text{Fix}_G(\Omega)$ if and only if $\omega^G = \{ \omega \}$ is an orbit of size 1. Together with a), if $\omega \in \Omega$, then either $\omega \in \text{Fix}_G(\Omega)$ or $\omega^G$ has size divisible by $p$. Since the $G$-orbits in $\Omega$ partition $\Omega$, it follows that $|\Omega| = |\text{Fix}_G(\Omega)| \mod p$.

c) Let $G$ act on itself via conjugation. Then $h \in G$ lies in $\text{Fix}_G(G)$ if and only if $h^g = h$ for all $g \in G$, that is, if and only if $h \in Z(G)$; this shows that $\text{Fix}_G(G) = Z(G)$. Clearly, $1 \in Z(G)$, hence $|Z(G)| > 1$. Now b) implies that $|Z(G)| = |G| \mod p$, so $p \mid |Z(G)|$; together, it follows that $Z(G)$ is nontrivial.

d) The group $H$ acts via left multiplication on the left cosets $G/H = \{ gh \mid g \in G \}$. Since $|G/H| = |G|/|H|$ is a $p$-power, it follows from b) that $\text{Fix}_{G/H}(H)$ is divisible by $p$, and $1H \in \text{Fix}_{G/H}(H)$ yields $|\text{Fix}_{G/H}(H)| \geq p$. Thus, there is $g \in G \setminus H$ with $gh \in \text{Fix}_{G/H}(H)$, that is, $h^gH = gH$ for all $h \in H$. This implies $g^{-1}hg \in H$ for all $h \in H$, that is, $H^g = H$ and $g \in N_G(H) \setminus H$.

**Question 2** (tutorial)

Let $G$ be a finite $p$-group.

a) Prove that if $N \trianglelefteq G$ and $G/N$ is cyclic, then $G' = [N, G]$.

b) Prove that if $G/\gamma_2(G)$ is cyclic, then $\gamma_2(G) = \{ 1 \}$ and $G$ is abelian.

c) Prove that $\Phi(G) = G^{p'}$; here $\Phi(G)$ is the Frattini subgroup of $G$ (that is, the intersection of all maximal subgroups of $G$) and $G^{p'}$ is the subgroup of $G$ generated by all $p$-th powers.

**SOLUTION:**

a) If $G/N = \langle xN \rangle$ for some $x \in G$, then every $g \in G$ can be written as $g = x^i n$ for some $n \in N$ and $i \in \mathbb{Z}$. Thus the generators of $\gamma_2(G)$ are

$$[x^i n, x^j m] = [x^i, x^j m] n, x^j m]$$

$$= [x^i, m]^n [x^i, x^j m] n, m | n, x^j m]$$

$$= [x^i, m]^n [n, m] [n, x^j m] m \in [N, G].$$
which proves that $G' \leq [N, G]$. Clearly, $[N, G] \leq G'$, hence equality.

b) It follows from a) that $\gamma_2(G) = [G, G] = [G, \gamma_2(G)] = \gamma_3(G)$, which forces $\gamma_2(G) = 1$.

c) Every maximal subgroup $M \leq G$ has index $p$ and is normal in $G$, see also Q1d). Since $G/M$ is elementary abelian, it follows that $G'G^p \leq M$; thus $G'G^p \leq \Phi(G)$. Let $I$ be the intersection of all maximal subgroups of $G$ which contain $G'G^p$. Clearly, $\Phi(G) \leq I$. On the other hand, $G/G'G^p$ is elementary abelian, so $\Phi(G/G'G^p) = 1$, which implies that $I \leq G'G^p$, so $\Phi(G) \leq G'G^p$. The claim follows.

**Question 3** (practical)

Use the SmallGroups Library of GAP to obtain a list of all $p$-groups $G$ of size at most $\max\{p^6, 1000\}$ with the property that $G$ admits a subgroup $A \leq G$ of size $p^2$ with $C_G(A) = A$; note that $A$ necessarily contains the center of $G$. Do the following:

a) Let $G$ and $A \leq G$ be as in the question and suppose $G$ is nonabelian. Determine the orders of $N_G(A)$ and $Z(G)$; compute a few examples to see what these orders might be.

b) For the groups in your list, compare their nilpotency class with their order; based on your observations, make a conjecture about the structure of the groups.

c) **Challenge Question:** Prove your conjecture (for example, use a) and induction on the group order).

**SOLUTION:**

If $G = A$, then $G = A = Z(A) = N_G(A)$ is abelian of order $p^2$ and nilpotency class 1. Thus, with the GAP code below we only construct those groups with $G > A$, and stores them in a list $\text{res}$: an entry in $\text{res}$ has the form $L = [G, [A_1, \ldots, A_n]]$ where each $A_i < G$ is self-centralising:

```gap
gap> getGoodGroups := function(G,p)
> local c,A,cl;
> c := Center(G);
> if Size(c)>p^2 then return []; fi;
> cl := List(ConjugacyClassesSubgroups(G),Representative);
> cl := Filtered(cl,x->Size(x)=p^2 and IsSubgroup(x,c) and Centraliser(G,x)=x);
> return cl;
> end;;

gap> myprimes := Filtered(Primes,x->x^2<1000);;

gap> res := [];;

gap> for p in myprimes do
> for n in Filtered([3..6],i-> p^n<1000) do
> Print("start order ",p^n,"\n");
> Print("start order ",p^n,"\n");
> for nr in [1..NumberSmallGroups(p^n)] do
> G := SmallGroup(p^n,nr);
> grps := getGoodGroups(G,p);
> if Size(grps)>0 then Add(res,[G,grps]); fi;
> od; od; od;
>
> a) For each pair $G, A$ computed above, we inspect $|G|, |Z(G)|, |N_G(A)|$, and test whether $N_G(A)$ is abelian:

```gap
    gap> List(res, x -> [ Size(x[1]), Size(Center(x[1])),
        List(x[2], A -> [Size(Normaliser(x[1],A)),
        IsAbelian(Normaliser(x[1],A))])]);
    [output omitted]
```gap

The output indicates that $Z(G)$ is cyclic of order $p$ and $N_G(A)$ is nonabelian of order $p^3$; let us check this:

```gap
    gap> ForAll(res, x -> ForAll(x[2], A->
    > Order(Center(x[1]))=PrimePGroup(x[1])
    > and Size(Normaliser(x[1],A))=PrimePGroup(x[1])^3
    > and not IsAbelian(Normaliser(x[1],A))));
    true
```
Now we prove this. Since $G$ is nonabelian, $G > A$, and so $N_G(A) > A$ by Q1d). Note that $N_G(A)/C_G(A)$ embeds into $\text{Aut}(A)$. Since $A$ is either cyclic or elementary abelian of order $p^2$, it follows that $\text{Aut}(A)$ is either $C_p(p-1)$ or $\text{GL}_2(p)$. In both cases, $p$ divides $|\text{Aut}(A)|$, but $p^2$ does not. Since $C_G(A) = A$ has order $p^2$, it follows that $N_G(A)$ is nonabelian of order $p^2$. Now clearly $Z(G) \leq A \leq N_G(A)$; if $Z(G) = A$, then $Z(N_G(A)) = A$ has index $p$ in $N_G(A)$, which is not possible since $N_G(A)$ is nonabelian; this proves that $Z(G) < A$, hence $Z(G)$ is cyclic of order $p$.

b) Running through the examples, we see that every such group has maximal class, that is, order $p^n$ and nilpotency class $n - 1$; this can be seen, for example, with the command

```gap
List(res , x-> [ NilpotencyClassOfGroup(x[1]),
               Collected(FactorsInt(Size(x[1])))[[1]][2]]);
```

Thus we conjecture: if a $p$-group $G$ has a subgroup $A \leq G$ of order $p^n$ with $A = C_G(A)$, then $G$ has maximal class (that is, nilpotency class $n - 1$).

c) Let $G$ be a $p$-group with $A \leq G$ of order $p^n$ and $C_G(A) = A$. If $G = A$, then $G$ has maximal class; now suppose $G > A$; in particular, $N_G(A) > A$. We know from b) that $N_G(A)$ is nonabelian of order $p^3$ and that $Z = Z(G)$ is cyclic of order $p$. Now consider $\overline{G} = G/Z$ and $\overline{N} = N_G(A)/Z$. Note that $C_{\overline{G}}(\overline{N}) \leq C_{\overline{G}}(\overline{A}/Z) \leq N_G(A)/Z = N$, thus we can apply the induction hypothesis to $\overline{G}$ and $\overline{N}$ and obtain that $\overline{G}$ has maximal class. Since $\overline{G} = G/Z(G)$, it follows that $G$ has maximal class: consider the upper central series of $G$.

**Question 4** (practical)

There are several ways to store and re-construct a pc-group in GAP; consult the manual at

http://www.gap-system.org/Manuals/doc/ref/chap46.html

for the following tasks.

a) Read about and use the commands `CodePcGroup` and `PcGroupCode`.

b) Prove that, indeed, every polycyclic presentation $P$ can be encoded by a positive integer $c = c(P)$. Observe that if $P$ has a generating set of cardinality $n$, then $P$ has $n$ power relations and $n(n-1)/2$ commutator relations. One (theoretical) way of encoding $P$ as a number is to make use of the uniqueness of prime-power factorisations in $\mathbb{Z}$.

c) Read about and use the command `GapInputPcGroup`.

**SOLUTION:** For a) and c), just play around with GAP. For b), recall that there are infinitely many primes, say $p_0, p_1, \ldots$, where $p_i$ denotes the $i + 1$-th prime. Now suppose $P$ has a generating set of cardinality $n$, say $\{g_1, \ldots, g_n\}$; there are $n$ power relations and $n(n-1)/2$ commutator relations; choose a canonical order of these relations (for example, if $i \in \{1, \ldots, n\}$, then the $i$-th relation is the power relation of $g_i$, etc.) Each right hand side of a relation is a normalised word in $\{g_1, \ldots, g_n\}$, and therefore uniquely determined by the corresponding list of exponents; to have a uniform description, let every list of exponents be of length $n$, that is, we included zeroes. Thus, the whole presentation is uniquely determined by the number $e_0 = n$ and $n + n(n-1)/2$ lists of length $n$, that is, by a list of $m = 1 + n(n + 1)/2$ nonnegative integers, say $(e_0, e_1, \ldots, e_m)$. Associate with this list the positive integer $c(P) = \prod_{i=0}^m p_i^{e_i}$. Given such a number $c(P)$, its prime power factorisation reveals the list $(e_0, e_1, \ldots, e_m)$: for example, the exponent of the largest power of 2 dividing $c(P)$ is $e_0 = n$, which determines the number of generators in the presentation. *(This encoding is not efficient; see the description of `CodePcGroup` in the GAP manual to see how this can be done more efficiently.)*

**Question 5** (tutorial/practical)

Let $G = \text{Sym}(9)$ be the symmetric group of rank 9.

a) By hand, determine a polycyclic series and a polycyclic presentation for the Sylow 3-subgroup of $G$. 
b) Now do the same calculation with GAP; compare with your results for a). The following commands might be useful:

```gap
gap> G := SymmetricGroup(9);;
gap> S := SylowSubgroup(G,3);;
gap> iso := IsomorphismFpGroup(S);;
gap> Spc := Image(iso);;
gap> mypcgs := List(Pcgs(Spc),x->PreImagesRepresentative(iso,x));;
gap> Spc := Image(iso);;
gap> S := SylowSubgroup(G,3);;
gap> G := SymmetricGroup(9);;
```

**SOLUTION:** a) Note that \(|G| = 9!\), so every Sylow 3-subgroup of \(G\) has order \(3^4 = 81\). Note that
\[
S = \langle (1,2,3), (4,5,6), (7,8,9), (1,4,7)(2,5,8)(3,6,9) \rangle \leq G
\]
has order 81, hence \(S\) is a Sylow 3-subgroup of \(G\). Write \(g_1 = (1,4,7)(2,5,8)(3,6,9)\), \(g_2 = (1,2,3)\), \(g_3 = (4,5,6)\), and \(g_4 = (7,8,9)\); note that each \(|g_i| = 3\) and \(g_i^{g_j} = g_j\) for \(i, j \geq 2\). Also, \(g_2^{g_3} = g_3\), \(g_3^{g_2} = g_2\), and \(g_4^{g_3} = g_2\). Thus, if we define \(G = \langle g_i \rangle\), then \(G = G_1 \triangleright G_2 \triangleright G_3 \triangleright G_4\triangleright = \{\}\) is a polycyclic series and \(X = \langle g_1, g_2, g_3, g_4 \rangle\) is a PCGS with \(R(X) = [3,3,3,3]\). The corresponding pc presentation is determined as
\[
H = \langle x_1, x_2, x_3, x_4 \mid x_1^3 = x_2^3 = x_3^3 = x_4^3 = 1, x_2^x_1 = x_3, x_3^x_1 = x_4, x_4^x_1 = x_2, x_2^x_1 = x_j \text{ for } i, j \geq 2 \rangle.
\]

b) (Just play around with GAP.)

**Question 6** (tutorial/practical)
Let \(G = \langle g_1, g_2, g_3 \mid g_1^4 = g_3, g_2^4 = g_3, g_3^4 = 1, g_2^{g_3} = g_2, g_3^{g_2} = g_3, g_3^{g_2} = g_3 \rangle\).

a) By hand, show that this polycyclic presentation is not consistent.

b) By hand, find a consistent polycyclic presentation of \(G\).

c) Construct \(G\) in GAP using the following commands:

```gap
gap> F := FreeGroup(["g1","g2","g3"]);;
gap> AssignGeneratorVariables(F);
#I Assigned the global variables [ g1, g2, g3 ]
gap> R := [g1^4/g3, g2^4/g3, g3^4, Comm(g1,g2), g3^g1/g3^2, g3^g2/g3];;
gap> G := F/R;
```

Do \texttt{IsomorphismFpGroup}, \texttt{StructureDescription}, and \texttt{PcGroupFpGroup}. The last command will yield an error message; re-define \(G\) by using the consistent pc-presentation you have obtained in b).

**SOLUTION:** a) The exponents of this presentation are \((4,4,4)\) and the normalised words in the generators are \(\{g_1^{e_1}g_2^{e_2}g_3^{e_3} \mid 0 \leq e_1, e_2, e_3 \leq 3\}\). Consistency checks show that the presentation is not consistent. For example, the collections of \((g_3g_1)g_1^2\) and \(g_3(g_1^2)\) yield
\[
(g_3g_1)g_1^2 = g_1g_3g_1^2 = g_1g_3g_1g_2g_1 = g_1g_3g_1^2 = g_1^4 = g_3 \quad \text{and} \quad g_3(g_1^2) = g_3^2
\]
in \(G\). In particular, this shows that \(g_3 = g_3^2\) in \(G\) and, thus, \(g_3 = 1\) in \(G\).

b) Using that \(g_3 = 1\), we have that \(G = \langle g_1, g_2 \mid g_1^4, g_2^4, g_2^{g_1} = g_2\rangle\); obviously, this presentation is consistent and describes a group isomorphic to \(C_4 \times C_4\).

c) (Just play around with GAP.)

**Question 7** (tutorial)
For a positive integer \(n\) let \(G(n) = \langle a, b \mid a^n, b^n, [a, b] = a \rangle\).

a) Prove by hand that if \(n = p\) is a prime, then \(G(p) \cong C_p\).

b) Does the same hold when \(n\) is not prime? (Maybe compute some examples with GAP.)
SOLUTION: a) Write \( G = G(p) \). Note that \( G' = \langle [a, b] \rangle = \langle a \rangle \), which implies that \( G/G' \cong C_p \) or \( G/G' = C_2 \); since \( a^p = 1 \), this also implies that \( G' \cong C_p \) or \( G' = 1 \). Thus, \( |G| \in \{1, p, p^2\} \), so \( G \) is abelian. But this forces \( G' = 1 \) and thus \( a = 1 \). Now \( G = \langle b \mid b^p \rangle \cong C_p \).

b) The group \( G(n) \) is not always cyclic; here are some non-cyclic examples.

```gap
gap> Gn:=function(n)
> local F,R;
> F := FreeGroup(2);
> R := [F.1^n,F.2^n,Comm(F.1,F.2)/F.1];
> return F/R;
> end;

gap> for i in [1..50] do
> G:=Gn(i);
> if not IsCyclic(G) then
> G:=Gn(i);
> Display([i,StructureDescription(Image(IsomorphismPcGroup(G)))]);
> fi; od;

[6, "C3 \times S3"]
[12, "C3 \times (C3 : C4)"
[18, "(C9 : C9) : C2"
[20, "C5 \times (C5 : C4)"
[21, "C7 \times (C7 : C3)"
[24, "C3 \times (C3 : C8)"
[30, "C15 \times S3"
[36, "(C9 : C9) : C4"
[40, "C5 \times (C5 : C8)"
[42, "C7 \times (S3 \times (C7 : C3))"
[48, "C3 \times (C3 : C16)"

In general, \( G = G(n) \) satisfies \( G' = \langle a \rangle \cong C_m \) for some \( m \mid n \), and \( G/G' \cong C_n \) acts via \( a \mapsto a^2 \) on \( G' \). The latter requires that the order of \( a \) is odd. For example, if \( G = G(6) \), then \( a^3 = 1 \), so \( G' = \langle a \rangle \cong C_3 \).

**Question 8** (tutorial/practical)
Consider the dihedral group \( G = \langle r, m \mid r^{2n-1}, m^2, r^m = r^{2n-1}-1 \rangle \).

a) Find the normal form of the element \( w = rmr^5m \). Second, we use that \( rm = mr^{2n-1}-1 \) and get
\[
\begin{align*}
w &= rmr^5m = rm^2(r^{2n-1}-1)^5 = r^{5(2n-1)-1} = r^{-4} = r^{2n-1-4}.
\end{align*}
\]

b) Note that \( \langle r \rangle \leq G \) is a normal subgroup of index 2; thus it suffices to find a PCGS of \( \langle r \rangle \) with relative orders \( 2 \). However, the latter group is cyclic of order \( 2^{n-1} \), so we can choose \( g_1 = m, g_2 = r, g_3 = r^2, \ldots, g_n = r^{2n-1} \). If we define \( G_i = \langle g_1, \ldots, g_n \rangle \), then each \( G_{i+1} \) has index 2 in \( G_i \), which yields a polycyclic series \( G_1 > \ldots > G_n > 1 \) with sections of order 2. Thus, \( X = [g_1, \ldots, g_n] \) is a PCGS with relative orders \( R(X) = [2, \ldots, 2] \).

c) We use the notation of b). Note that all the elements \( g_2, \ldots, g_n \) commute pairwise, and \( g_i^2 = g_{i+1} \) for \( i = 2, \ldots, n-1 \). It remains to describe \( g_i^m \) for \( i = 2, \ldots, n \). Note that \( r^m = r^{-1} \), hence
\[
\begin{align*}
g_i^m &= (r^{2i-1})^m = r^{-2i-1} = g_i^{-1}.
\end{align*}
\]
Observe also that \( g_i g_{i+1} \cdots g_n = g_i^2 g_{i+2} \cdots g_n = \cdots = g_n^2 = 1 \), hence

\[
g_i^{g_i} = g_i^{-1} = g_i g_{i+1} \cdots g_n
\]

for every \( i = 2, \ldots, n \). This yields the following polycyclic presentation for \( G \):

\[
\langle g_1, \ldots, g_n \mid g_1 = 1, g_i^2 = g_{i+1} \text{ for } i = 2, \ldots, n-1, g_n^2 = 1, g_i^{g_i} = g_i \text{ for } 2 \leq j < i < n, g_i^{g_i} = g_i \cdots g_n \text{ for } i = 2, \ldots, n \rangle.
\]

**d)** Here is some GAP code for that task:

```gap
getDn := function(n)
local gens, i, j, F, R;
F := FreeGroup(n);
gens := GeneratorsOfGroup(F);
R := [gens[1]^2];
for i in [2..n-1] do Add(R,gens[i]^2/gens[i+1]); od;
Add(R,gens[n]^2);
for j in [2..n-1] do for i in [j+1..n] do
  Add(R,Comm(gens[i],gens[j]));
  od; od;
for i in [2..n-1] do
  Add(R,Comm(gens[i],gens[1])/(Product(gens[i+1..n])));
  od;
Add(R,Comm(gens[1],gens[n]));
return PcGroupFpGroup(F/R);
end;
StructureDescription(getDn(10));
"D1024"
```

**Question 9** (tutorial)

By hand, compute a wpcp of the group

\[
G = \langle a, b, c \mid a^9, b^9, c^9, [b, a], a = a^3, (aba)^9, (ba)^5a = b, [a, c] \rangle;
\]

you can use that \( |G| = 3^3 \).

**SOLUTION:** First compute \( H = G/P_1(G) \) as outlined in the lectures; abelianising the relations (and taking everything modulo 3) yields the \( 1 \times 3 \) matrix \( M = (0 \ 1 \ 0) \). This tells us that \( H = (aP_1(G), cP_1(G)) \) has rank 2, thus \( H \cong C_2^2 \) and we can define \( H \) via the wpcp \( H = \langle a, c \mid a^3, c^3 \rangle \). We define

\[
\theta: G \to G/P_1(G), \quad (a, c) \mapsto (a, c).
\]

The 3-covering of \( H \) is \( H^* = \text{Pc}\langle a, c, x_1, x_2, x_3 \mid a^3 = x_1, c^3 = x_2, [c, a] = x_3, x_1^3 = x_2^3 = x_3^3 = 1 \rangle \), and consistency checks show that this presentation is consistent. Now use \( \theta \) to evaluate the relations of \( G \) in \( H^* \):

\[
\begin{align*}
a^9 &= 1 \leadsto 1 = 1 & b^9 &= 1 \leadsto 1 = 1 \\
c^9 &= 1 \leadsto 1 = 1 & [b, a] &= a^3 \leadsto 1 = x_1 \\
(aba)^9 &= 1 \leadsto 1 = 1 & (ba)^5a &= b \leadsto x_1^3 = 1 \\
[a, c] &= 1 \leadsto x_3 = 1,
\end{align*}
\]

which tells us that \( G/P_2(G) \cong H^*/\langle x_1, x_3 \rangle = \text{Pc}\langle a, c, x \mid a^3 = 1, c^3 = x, x^3 = 1 \rangle \cong C_3 \times C_9 \). Since \( |G| = 27 = |G/P_2(G)| \), we conclude that this is a wpcp for \( G \).
By hand, show that the nucleus of 

\[ Q_8 = \text{Pc}(a, b, c \mid a^2 = c, b^2 = c, c^2 = 1, [b, a] = c) \]

in \( Q_8^* \) is trivial, and deduce that \( Q_8 \) has no immediate 2-descendants.

**SOLUTION:** Write \( G = Q_8 \) and note that \( G \) has \( p \)-class \( k = 2 \). Let \( M^* \) be the 2-cover with multiplicator \( M \) and nucleus \( P_k(G^*) \). We show that \( P_k(G^*) = 1 \) so that \( U P_k(G^*) < M \) for every \( U < M \), that is, \( M \) has no allowable subgroups, and therefore \( Q_8 \) has no immediate descendants.

To prove that \( P_k(G^*) \) is trivial, we first write down a presentation for \( G^* \). By considering \( G/P_1(G) \) as before, we find that \( G = \langle a, b \rangle \) has rank 2. Note that \( G \) is already given by a wpcp, and we can choose \( a^2 = c \) as the definition of \( c \). Thus, a (inconsistent) presentation for \( G^* \) is

\[ \text{Pc}(a, b, c, x_1, \ldots, x_5 \mid a^2 = c, b^2 = cx_1, c^2 = x_2, [b, a] = cx_3, [c, b] = x_4, [c, a] = x_5, x_1^2 = \ldots = x_5^2 = 1) \]

note that \( M = \langle x_1, \ldots, x_5 \rangle \leq G^* \) is the multiplicator. Moreover, \( P_1(G^*) M/M \leq P_1(G) = \langle c \rangle \), which shows that \( P_1(G^*) \) is contained in \( \langle c, x_1, \ldots, x_5 \rangle \). The latter is clearly abelian; if we show that \( c \) has order 2, then \( P_1(G^*) \) is elementary abelian, and \( P_k(G)^* = 1 \) follows – which implies the claim.

To show that \( c \) has order 2, we need that \( x_2 = 1 \). We see this by doing a few consistency checks: \( a a^2 = ac \) and \( a^2 a = ca = ac x_5 \) implies that \( x_5 = 1 \); similarly, \( b b^2 = b c x_1 \) and \( b^2 b = c x_1 b = b c x_1 x_4 \) implies \( x_4 = 1 \). Lastly, \( b^2 a = c x_1 a = a c x_1 x_5 \) and \( b(ba) = bab c x_3 = a b c x_3 b c x_5 = a b^2 c^2 x_3 x_4 = a c x_1 x_2 x_4 \) force \( x_2 = 1 \); recall that \( x_5 = x_4 = 1 \). Thus, \( c \) has order 2 in \( G^* \), and the claim follows as described above.

**Question 11** (practical)

Make sure the GAP package Anupq is installed and running; you might have to do ./configure and make in pkg/anupq before you can load it in gap with LoadPackage('`anupq`'). Look up the manual and use ...

a) ... the command \texttt{Pq} to compute a wpcp of \( G \),

b) ... the command \texttt{PqPcover} to compute the 2-covering group \( G^* \) of \( G \)

c) ... the command \texttt{PqDescendants} to compute all immediate descendants of \( G \),

for each group \( G \) in the questions above, and for

\[ G = \langle x, y \mid [y, x], x = x^2, (xy) x^4, x^4, y^4, (yx)^3 y = x \rangle \quad \text{with} \quad p = 2. \]

**SOLUTION:** Here is some GAP code for the last group:

```gap
> LoadPackage("anupq");
> F:=FreeGroup(["x","y"]);;
> AssignGeneratorVariables(F);
# I Assigned the global variables [ x, y ]
> R:=[Comm(Comm(y,x),x)/x^2, (x*y*x)^4, x^4, y^4, (y*x)^3*y/x];;
> G:=F/R;;
> Gwpcp:=Pq(G:Prime:=2);;
> PrintPcpPresentation(PcGroupToPcpGroup(Gwpcp));
> g1:=g2^2 = g5
> g2^2 = g4
> g3^2 = g5
> g4^2 = id
> g5^2 = id
> g6^2 = id
> g2 * g1 = g2 * g3
> g3 * g1 = g3 * g5
> g2 * g2 = g3 * g6
> g4 * g1 = g4 * g5 * g6
```
Question 12 (tutorial)
For $n \in \mathbb{N}$ consider the cyclic group $G = C_{p^n} = \langle r \mid r^{p^n} \rangle$; compute $G^*$ and show that $G$ has immediate descendants.

SOLUTION: First, we write down a wpcp of $G = C_{p^n}$, namely

$$G = \langle r_1, \ldots, r_n \mid r_1^p = r_2, \ldots, r_{n-1}^p = r_n, r_n^p = 1 \rangle.$$ 

The first $n-1$ relations are definitions; the non-defining relations are $r_i^p = 1$ and the trivial commutator relations $r_j^i = r_j$ for $i < j$. Thus we obtain

$$G^* = \langle r_1, \ldots, r_n, b, b_{i,j} (i < j) \mid r_1^p = r_2, \ldots, r_{n-1}^p = r_n, r_n^p = b, r_j^i = r_j b_{i,j} (i < j), \text{each } b_{i,j}^p = 1 \rangle.$$ 

Let’s do a few consistency checks. First, if $1 \leq i \leq j < n$, then

$$r_j^i r_i = r_{i+1} r_i = r_i r_{i+1} b_{i,j+1} \quad \text{and} \quad r_j^{p-1}(r_j r_i) = r_j r_i b_{i,j} = r_i r_{j+1},$$

which shows that $b_{i,k} = 1$ for all $1 \leq i < k \leq n$; this yields a consistent wpcp, namely

$$G^* = \langle r_1, \ldots, r_n, b \mid r_1^p = r_2, \ldots, r_{n-1}^p = r_n, r_n^p = b, b^p = 1 \rangle \cong C_{p^n+1}.$$ 

Note that $G$ has $p$-class $c = n$ since $P_i(G) = \langle r_{i+1} \rangle$ for all $i = 1, \ldots, n-1$, and $P_n(G) = 1$. Now the multiplicator of $G$ is $M = \langle b \rangle \leq G^*$ and the nucleus of $G$ is $P_*(G^*) = \langle b \rangle = M$, that is, $U = 1 \leq M$ is an allowable subgroup. This proves that $G^*$ is indeed an immediate descendant of $G$. 

```plaintext
gap> Size(Gwpcp);
64
gap> Gstar:=PqPCover(Gwpcp:Prime:=2);
gap> Size(Gstar);
512
gap> imdes:=PqDescendants(Gwpcp:Prime:=2);
[ <pc group of size 128 with 7 generators>,
  <pc group of size 128 with 7 generators>,
  <pc group of size 128 with 7 generators>,
  <pc group of size 128 with 7 generators> ]
```