

The questions on this sheet are to be discussed during the two tutorials and the practical sessions; some questions are to be done “by hand”, others require to use GAP. In the following,  $p$  always denotes a prime and  $n$  is a positive integer. [This sheet also contains some comments on the solutions.](#)

### Question 1 (tutorial)

Let  $G$  be a nontrivial finite  $p$ -group acting on a finite set  $\Omega$ . Recall that the  $G$ -orbit of  $\omega \in \Omega$  is defined as the subset  $\omega^G = \{\omega^g \mid g \in G\} \subseteq \Omega$ ; its stabiliser is the subgroup  $\text{Stab}_G(\omega) = \{g \in G \mid \omega^g = \omega\} \leq G$ .

- Prove the Orbit-Stabiliser-Theorem, that is, show that  $|G|/|\text{Stab}_G(\omega)| = |\omega^G|$ .
- Denote by  $\text{Fix}_\Omega(G) = \{\omega \in \Omega \mid \forall g \in G : \omega^g = \omega\}$  the set of  $G$ -fixed points in  $\Omega$ . Use a) to prove that  $|\Omega| \equiv |\text{Fix}_\Omega(G)| \pmod{p}$ ; in particular, if  $|\Omega|$  is a  $p$ -power, then  $|\text{Fix}_\Omega(G)|$  is divisible by  $p$ .
- Use b) to prove that the center  $Z(G) = \{h \in G \mid \forall g \in G : h^g = h\}$  of  $G$  is non-trivial.
- Let  $H < G$  be a proper subgroup. Consider an action of  $H$  and use b) to prove that  $N_G(H) > H$ .

**SOLUTION:** a) The map  $\psi: G \rightarrow \Omega$ ,  $g \mapsto \omega^g$  induces  $\hat{\psi}: \text{Stab}_G(\omega) \backslash G \rightarrow \omega^G$ ,  $\text{Stab}_G(\omega)g \rightarrow \omega^g$ . Clearly,  $\hat{\psi}$  is well-defined: if  $\text{Stab}_G(\omega)g = \text{Stab}_G(\omega)h$ , then  $h = sg$  for some  $s \in \text{Stab}_G(\omega)$ , and so  $\omega^h = \omega^{sg} = \omega^g$ . By construction,  $\hat{\psi}$  is surjective. If  $\text{Stab}_G(\omega)g$  and  $\text{Stab}_G(\omega)h$  are mapped to  $\omega^g = \omega^h$ , then  $gh^{-1} \in \text{Stab}_G(\omega)$ , and so  $\text{Stab}_G(\omega)g = \text{Stab}_G(\omega)h$ . This proves that  $\hat{\psi}$  is a bijection, and therefore  $|\omega^G| = |\text{Stab}_G(\omega) \backslash G|$ ; clearly,  $|\text{Stab}_G(\omega) \backslash G| = |G|/|\text{Stab}_G(\omega)|$ .

- Note that  $\omega \in \text{Fix}_\Omega(G)$  if and only if  $\omega^G = \{\omega\}$  is an orbit of size 1. Together with a), if  $\omega \in \Omega$ , then either  $\omega \in \text{Fix}_\Omega(G)$  or  $\omega^G$  has size divisible by  $p$ . Since the  $G$ -orbits in  $\Omega$  partition  $\Omega$ , it follows that  $|\Omega| \equiv |\text{Fix}_\Omega(G)| \pmod{p}$ .
- Let  $G$  act on itself via conjugation. Then  $h \in G$  lies in  $\text{Fix}_G(G)$  if and only if  $h^g = h$  for all  $g \in G$ , that is, if and only if  $h \in Z(G)$ ; this shows that  $\text{Fix}_G(G) = Z(G)$ . Clearly,  $1 \in Z(G)$ , hence  $|Z(G)| > 1$ . Now b) implies that  $|Z(G)| \equiv |G| \pmod{p}$ , so  $p \mid |Z(G)|$ ; together, it follows that  $Z(G)$  is nontrivial.
- The group  $H$  acts via left multiplication on the left cosets  $G/H = \{gH \mid g \in G\}$ . Since  $|G/H| = |G|/|H|$  is a  $p$ -power, it follows from b) that  $\text{Fix}_{G/H}(H)$  is divisible by  $p$ , and  $1H \in \text{Fix}_{G/H}(H)$  yields  $|\text{Fix}_{G/H}(H)| \geq p$ . Thus, there is  $g \in G \setminus H$  with  $gH \in \text{Fix}_{G/H}(H)$ , that is,  $hgH = gH$  for all  $h \in H$ . This implies  $g^{-1}hg \in H$  for all  $h \in H$ , that is,  $H^g = H$  and  $g \in N_G(H) \setminus H$ .

### Question 2 (tutorial)

Let  $G$  be a finite  $p$ -group.

- Prove that if  $N \trianglelefteq G$  and  $G/N$  is cyclic, then  $G' = [N, G]$ .
- Prove that if  $G/\gamma_2(G)$  is cyclic, then  $\gamma_2(G) = \{1\}$  and  $G$  is abelian.
- Prove that  $\Phi(G) = G'G^p$ ; here  $\Phi(G)$  is the Frattini subgroup of  $G$  (that is, the intersection of all maximal subgroups of  $G$ ) and  $G^p$  is the subgroup of  $G$  generated by all  $p$ -th powers.

**SOLUTION:** a) If  $G/N = \langle xN \rangle$  for some  $x \in G$ , then every  $g \in G$  can be written as  $g = x^i n$  for some  $n \in N$  and  $i \in \mathbb{Z}$ . Thus the generators of  $\gamma_2(G)$  are

$$\begin{aligned} [x^i n, x^j m] &= [x^i, x^j m]^n [n, x^j m] \\ &= [x^i, m]^n [x^i, x^j]^{mn} [n, m] [n, x^j]^m \\ &= [x^i, m]^n [n, m] [n, x^j]^m \in [N, G], \end{aligned}$$

which proves that  $G' \leq [N, G]$ . Clearly,  $[N, G] \leq G'$ , hence equality.

- b) It follows from a) that  $\gamma_2(G) = [G, G] = [G, \gamma_2(G)] = \gamma_3(G)$ , which forces  $\gamma_2(G) = 1$ .
- c) Every maximal subgroup  $M \leq G$  has index  $p$  and is normal in  $G$ , see also Q1d). Since  $G/M$  is elementary abelian, it follows that  $G'G^p \leq M$ ; thus  $G'G^p \leq \Phi(G)$ . Let  $I$  be the intersection of all maximal subgroups of  $G$  which contain  $G'G^p$ . Clearly,  $\Phi(G) \leq I$ . On the other hand,  $G/G'G^p$  is elementary abelian, so  $\Phi(G/G'G^p) = 1$ , which implies that  $I \leq G'G^p$ , so  $\Phi(G) \leq G'G^p$ . The claim follows.

### Question 3 (practical)

Use the SmallGroups Library of GAP to obtain a list of all  $p$ -groups  $G$  of size at most  $\max\{p^6, 1000\}$  with the property that  $G$  admits a subgroup  $A \leq G$  of size  $p^2$  with  $C_G(A) = A$ ; note that  $A$  necessarily contains the center of  $G$ . Do the following:

- a) Let  $G$  and  $A \leq G$  be as in the question and suppose  $G$  is nonabelian. Determine the orders of  $N_G(A)$  and  $Z(G)$ ; compute a few examples to see what these orders might be.
- b) For the groups in your list, compare their nilpotency class with their order; based on your observations, make a conjecture about the structure of the groups.
- c) *Challenge Question:* Prove your conjecture (for example, use a) and induction on the group order).

**SOLUTION:** If  $G = A$ , then  $G = A = Z(A) = N_G(A)$  is abelian of order  $p^2$  and nilpotency class 1. Thus, with the GAP code below we only construct those groups with  $G > A$ , and stores them in a list `res`: an entry in `res` has the form `L=[G, [A1, . . . , An] ]` where each  $A_i < G$  is self-centralising:

```
gap> getGoodGroups := function(G,p)
> local c,A,cl;
> c := Center(G);
> if Size(c)>p^2 then return []; fi;
> cl := List(ConjugacyClassesSubgroups(G),Representative);
> cl := Filtered(cl,x->Size(x)=p^2 and IsSubgroup(x,c) and Centraliser(G,x)=x);
> return cl;
> end;;
gap> myprimes := Filtered(Primes,x->x^2<1000);
gap> res := [];
gap> for p in myprimes do
> for n in Filtered([3..6],i->p^i<1000) do
> Print("start order ",p^n,"\n");
> for nr in [1..NumberSmallGroups(p^n)] do
> G := SmallGroup(p^n,nr);
> grps := getGoodGroups(G,p);
> if Size(grps)>0 then Add(res,[G,grps]); fi;
> od; od; od;
```

- a) For each pair  $G, A$  computed above, we inspect  $|G|$ ,  $|Z(G)|$ ,  $|N_G(A)|$ , and test whether  $N_G(A)$  is abelian:

```
gap> List(res, x -> [ Size(x[1]), Size(Center(x[1])),
                    List(x[2], A -> [Size(Normaliser(x[1],A)),
                                      IsAbelian(Normaliser(x[1],A))])]);
[output omitted]
```

The output indicates that  $Z(G)$  is cyclic of order  $p$  and  $N_G(A)$  is nonabelian of order  $p^3$ ; let us check this:

```
gap> ForAll(res, x -> ForAll(x[2],A->
    Order(Center(x[1]))=PrimePGroup(x[1])
    and Size(Normaliser(x[1],A))=PrimePGroup(x[1])^3
    and not IsAbelian(Normaliser(x[1],A))));
true
```

Now we prove this. Since  $G$  is nonabelian,  $G > A$ , and so  $N_G(A) > A$  by Q1d). Note that  $N_G(A)/C_G(A)$  embeds into  $\text{Aut}(A)$ . Since  $A$  is either cyclic or elementary abelian of order  $p^2$ , it follows that  $\text{Aut}(A)$  is either  $C_{p(p-1)}$  or  $\text{GL}_2(p)$ . In both cases,  $p$  divides  $|\text{Aut}(A)|$ , but  $p^2$  does not. Since  $C_G(A) = A$  has order  $p^2$ , it follows that  $N_G(A)$  is nonabelian of order  $p^3$ . Now clearly  $Z(G) \leq A \leq N_G(A)$ ; if  $Z(G) = A$ , then  $Z(N_G(A)) = A$  has index  $p$  in  $N_G(A)$ , which is not possible since  $N_G(A)$  is nonabelian; this proves that  $Z(G) < A$ , hence  $Z(G)$  is cyclic of order  $p$ .

b) Running through the examples, we see that every such group has maximal class, that is, order  $p^n$  and nilpotency class  $n - 1$ ; this can be seen, for example, with the command

```
List(res , x-> [ NilpotencyClassOfGroup(x[1]),
                  Collected(FactorsInt(Size(x[1]))) [1][2]]);
```

Thus we conjecture: if a  $p$ -group  $G$  has a subgroup  $A \leq G$  of order  $p^2$  with  $A = C_G(A)$ , then  $G$  has maximal class (that is, nilpotency class  $n - 1$ ).

c) Let  $G$  be a  $p$ -group with  $A \leq G$  of order  $p^2$  and  $C_G(A) = A$ . If  $G = A$ , then  $G$  has maximal class; now suppose  $G > A$ ; in particular,  $N_G(A) > A$ . We know from b) that  $N_G(A)$  is nonabelian of order  $p^3$  and that  $Z = Z(G)$  is cyclic of order  $p$ . Now consider  $\overline{G} = G/Z$  and  $\overline{N} = N_G(A)/Z$ . Note that  $C_{\overline{G}}(\overline{N}) \leq C_{\overline{G}}(A/Z) \leq N_G(A)/Z = \overline{N}$ , thus we can apply the induction hypothesis to  $\overline{G}$  and  $\overline{N}$  and obtain that  $\overline{G}$  has maximal class. Since  $\overline{G} = G/Z(G)$ , it follows that  $G$  has maximal class: consider the upper central series of  $G$ .

#### Question 4 (practical)

There are several ways to store and re-construct a pc-group in GAP; consult the manual at

<http://www.gap-system.org/Manuals/doc/ref/chap46.html>

for the following tasks.

- Read about and use the commands `CodePcGroup` and `PcGroupCode`.
- Prove that, indeed, every polycyclic presentation  $P$  can be encoded by a positive integer  $c = c(P)$ . Observe that if  $P$  has a generating set of cardinality  $n$ , then  $P$  has  $n$  power relations and  $n(n - 1)/2$  commutator relations. One (theoretical) way of encoding  $P$  as a number is to make use of the uniqueness of prime-power factorisations in  $\mathbb{Z}$ .
- Read about and use the command `GapInputPcGroup`.

**SOLUTION:** For a) and c), just play around with GAP. For b), recall that there are infinitely many primes, say  $p_0, p_1, \dots$ , where  $p_i$  denotes the  $i + 1$ -th prime. Now suppose  $P$  has a generating set of cardinality  $n$ , say  $\{g_1, \dots, g_n\}$ ; there are  $n$  power relations and  $n(n - 1)/2$  commutator relations; choose a canonical order of these relations (for example, if  $i \in \{1, \dots, n\}$ , then the  $i$ -th relation is the power relation of  $g_i$ , etc.) Each right hand side of a relation is a normalised word in  $\{g_1, \dots, g_n\}$ , and therefore uniquely determined by the corresponding list of exponents; to have a uniform description, let every list of exponents be of length  $n$ , that is, we included zeroes. Thus, the whole presentation is uniquely determined by the number  $e_0 = n$  and  $n + n(n - 1)/2$  lists of length  $n$ , that is, by a list of  $m = 1 + n(n + n(n - 1)/2)$  nonnegative integers, say  $(e_0, e_1, \dots, e_m)$ . Associate with this list the positive integer  $c(P) = \prod_{i=0}^m p_i^{e_i}$ . Given such a number  $c(P)$ , its prime power factorisation reveals the list  $(e_0, e_1, \dots, e_m)$ : for example, the exponent of the largest power of 2 dividing  $c(P)$  is  $e_0 = n$ , which determines the number of generators in the presentation. (*This encoding is not efficient; see the description of `CodePcGroup` in the GAP manual to see how this can be done more efficient.*)

#### Question 5 (tutorial/practical)

Let  $G = \text{Sym}(9)$  be the symmetric group of rank 9.

- By hand, determine a polycyclic series and a polycyclic presentation for the Sylow 3-subgroup of  $G$ .

- b) Now do the same calculation with GAP; compare with your results for a). The following commands might be useful:

```
gap> G := SymmetricGroup(9);;
gap> S := SylowSubgroup(G, 3);;
gap> iso := IsomorphismPcGroup(S);;
gap> Spc := Image(iso);;
gap> mypcgs := List(Pcgs(Spc), x->PreImagesRepresentative(iso, x));;
```

SOLUTION: a) Note that  $|G| = 9!$ , so every Sylow 3-subgroup of  $G$  has order  $3^4 = 81$ . Note that

$$S = \langle (1, 2, 3), (4, 5, 6), (7, 8, 9), (1, 4, 7)(2, 5, 8)(3, 6, 9) \rangle \leq G$$

has order 81, hence  $S$  is a Sylow 3-subgroup of  $G$ . Write  $g_1 = (1, 4, 7)(2, 5, 8)(3, 6, 9)$ ,  $g_2 = (1, 2, 3)$ ,  $g_3 = (4, 5, 6)$ , and  $g_4 = (7, 8, 9)$ ; note that each  $|g_i| = 3$  and  $g_j^{g_i} = g_j$  for  $i, j \geq 2$ . Also,  $g_2^{g_1} = g_3$ ,  $g_3^{g_1} = g_4$ , and  $g_4^{g_1} = g_2$ . Thus, if we define  $G_i = \langle g_i, \dots, g_4 \rangle$ , then  $G = G_1 \triangleright G_2 \triangleright G_3 \triangleright G_4 \triangleright = \{()\}$  is a polycyclic series and  $X = [g_1, g_2, g_3, g_4]$  is a PCGS with  $R(X) = [3, 3, 3, 3]$ . The corresponding pc presentation is determined as

$$H = \langle x_1, x_2, x_3, x_4 \mid x_1^3 = x_2^3 = x_3^3 = x_4^3 = 1, x_2^{x_1} = x_3, x_3^{x_1} = x_4, x_4^{x_1} = x_2, x_j^{x_i} = x_j \text{ for } i, j \geq 2 \rangle.$$

- b) (Just play around with GAP.)

### Question 6 (tutorial/practical)

Let  $G = \langle g_1, g_2, g_3 \mid g_1^4 = g_3, g_2^4 = g_3, g_3^4 = 1, g_2^{g_1} = g_2, g_3^{g_1} = g_3^2, g_3^{g_2} = g_3 \rangle$ .

- By hand, show that this polycyclic presentation is not consistent.
- By hand, find a consistent polycyclic presentation of  $G$ .
- Construct  $G$  in GAP using the following commands:

```
gap> F:=FreeGroup(["g1", "g2", "g3"]);;
gap> AssignGeneratorVariables(F);
#I Assigned the global variables [ g1, g2, g3 ]
gap> R:=[g1^4/g3, g2^4/g3, g3^4, Comm(g1,g2), g3^g1/g3^2, g3^g2/g3];;
gap> G:=F/R;
```

Do `IsomorphismPcGroup`, `StructureDescription`, and `PcGroupFpGroup`. The last command will yield an error message; re-define  $G$  by using the consistent pc-presentation you have obtained in b).

SOLUTION: a) The exponents of this presentation are  $(4, 4, 4)$  and the normalised words in the generators are  $\{g_1^{e_1} g_2^{e_2} g_3^{e_3} \mid 0 \leq e_1, e_2, e_3 \leq 3\}$ . Consistency checks show that the presentation is not consistent. For example, the collections of  $(g_3 g_1) g_1^3$  and  $g_3 (g_1^4)$  yield

$$(g_3 g_1) g_1^3 = g_1 g_3^2 g_1^3 = g_1 g_3 g_1 g_3^2 g_1^2 = g_1^2 g_3^4 g_1^2 = g_1^4 = g_3 \quad \text{and} \quad g_3 (g_1^4) = g_3^2$$

in  $G$ . In particular, this shows that  $g_3 = g_3^2$  in  $G$  and, thus,  $g_3 = 1$  in  $G$ .

- b) Using that  $g_3 = 1$ , we have that  $G = \langle g_1, g_2 \mid g_1^4, g_2^4, g_2^{g_1} = g_2 \rangle$ ; obviously, this presentation is consistent and describes a group isomorphic to  $C_4 \times C_4$ .

- c) (Just play around with GAP.)

### Question 7 (tutorial)

For a positive integer  $n$  let  $G(n) = \langle a, b \mid a^n, b^n, [a, b] = a \rangle$ .

- Prove by hand that if  $n = p$  is a prime, then  $G(p) \cong C_p$ .
- Does the same hold when  $n$  is not prime? (Maybe compute some examples with GAP.)

SOLUTION: a) Write  $G = G(p)$ . Note that  $G' = \langle [a, b] \rangle = \langle a \rangle$ , which implies that  $G/G' \cong C_p$  or  $G/G' \cong 1$ ; since  $a^p = 1$ , this also implies that  $G' \cong C_p$  or  $G' = 1$ . Thus,  $|G| \in \{1, p, p^2\}$ , so  $G$  is abelian. But this forces  $G' = 1$  and thus  $a = 1$ . Now  $G = \langle b \mid b^p \rangle \cong C_p$ .

b) The group  $G(n)$  is not always cyclic; here are some non-cyclic examples.

```
gap> Gn:=function(n)
> local F,R;
> F := FreeGroup(2);
> R := [F.1^n, F.2^n, Comm(F.1, F.2)/F.1];
> return F/R;
> end;
gap> for i in [1..50] do
> G:=Gn(i);
> if not IsCyclic(G) then
>   Display([i, StructureDescription(Image(IsomorphismPcGroup(G)))]);
> fi; od;
[ 6, "C3 x S3" ]
[ 12, "C3 x (C3 : C4)" ]
[ 18, "(C9 : C9) : C2" ]
[ 20, "C5 x (C5 : C4)" ]
[ 21, "C7 x (C7 : C3)" ]
[ 24, "C3 x (C3 : C8)" ]
[ 30, "C15 x S3" ]
[ 36, "(C9 : C9) : C4" ]
[ 40, "C5 x (C5 : C8)" ]
[ 42, "C7 x (S3 x (C7 : C3))" ]
[ 48, "C3 x (C3 : C16)" ]
```

In general,  $G = G(n)$  satisfies  $G' = \langle a \rangle \cong C_m$  for some  $m \mid n$ , and  $G/G' \cong C_n$  acts via  $a \mapsto a^2$  on  $G'$ . The latter requires that the order of  $a$  is odd. For example, if  $G = G(6)$ , then  $a^3 = 1$ , so  $G' = \langle a \rangle \cong C_3$ .

### Question 8 (tutorial/practical)

Consider the dihedral group  $G = \langle r, m \mid r^{2^{n-1}}, m^2, r^m = r^{2^{n-1}-1} \rangle$ .

- Find the normal form of the element  $w = rmr^2m^2r^3m^3$ .
- Find a polycyclic series of  $G$  whose associated PCGS has relative orders  $[2, \dots, 2]$ .
- Find a polycyclic presentation of  $G$ , associated to the PCGS you have found in b).
- Write a GAP function `getDn(n)` which constructs this group using the presentation you have found in c), via `PcGroupFpGroup`.

SOLUTION: a) First, we use that  $m^i = m^{i \bmod 2}$  for all  $i$ , and obtain  $w = rmr^5m$ . Second, we use that  $rm = mr^{2^{n-1}-1}$  and get

$$w = rmr^5m = rm^2(r^{2^{n-1}-1})^5 = r^{5(2^{n-1}-1)+1} = r^{-4} = r^{2^{n-1}-4},$$

note that  $r^{2^{n-1}-1} = r^{-1}$ .

b) Note that  $\langle r \rangle \trianglelefteq G$  is a normal subgroup of index 2; thus it suffices to find a PCGS of  $\langle r \rangle$  with relative orders 2. However, the latter group is cyclic of order  $2^{n-1}$ , so we can choose  $g_1 = m, g_2 = r, g_3 = r^2, \dots, g_n = r^{2^{n-1}}$ . If we define  $G_i = \langle g_i, \dots, g_n \rangle$ , then each  $G_{i+1}$  has index 2 in  $G_i$ , which yields a polycyclic series  $G_1 > \dots > G_n > 1$  with sections of order 2. Thus,  $X = [g_1, \dots, g_n]$  is a PCGS with relative orders  $R(X) = [2, \dots, 2]$ .

c) We use the notation of b). Note that all the elements  $g_2, \dots, g_n$  commute pairwise, and  $g_i^2 = g_{i+1}$  for  $i = 2, \dots, n-1$ . It remains to describe  $g_i^{g_1}$  for  $i = 2, \dots, n$ . Note that  $r^m = r^{-1}$ , hence

$$g_i^{g_1} = (r^{2^{i-1}})^m = r^{-2^{i-1}} = g_i^{-1}.$$

Observe also that  $g_i g_i g_{i+1} \cdots g_n = g_i^2 g_{i+1} \cdots g_n = g_{i+1}^2 g_{i+2} \cdots g_n = \dots = g_n^2 = 1$ , hence

$$g_i^{g_1} = g_i^{-1} = g_i g_{i+1} \cdots g_n$$

for every  $i = 2, \dots, n$ . This yields the following polycyclic presentation for  $G$ :

$$\langle g_1, \dots, g_n \mid \begin{array}{l} g_1^2 = 1, \\ g_i^2 = g_{i+1} \text{ for } i = 2, \dots, n-1, \\ g_n^2 = 1, \\ g_i^{g_j} = g_i \text{ for } 2 \leq j < i < n \\ g_i^{g_1} = g_i \cdots g_n \text{ for } i = 2, \dots, n \end{array} \rangle.$$

d) Here is some GAP code for that task:

```
getDn := function(n)
local gens, i, j, F, R;
  F := FreeGroup(n);
  gens := GeneratorsOfGroup(F);
  R := [gens[1]^2];
  for i in [2..n-1] do Add(R, gens[i]^2/gens[i+1]); od;
  Add(R, gens[n]^2);
  for j in [2..n-1] do for i in [j+1..n] do
    Add(R, Comm(gens[i], gens[j]));
  od; od;
  for i in [2..n-1] do
    Add(R, Comm(gens[i], gens[1]) / (Product(gens{[i+1..n]})));
  od;
  Add(R, Comm(gens[1], gens[n]));
  return PcGroupFpGroup(F/R);
end;
StructureDescription(getDn(10));
"D1024"
```

### Question 9 (tutorial)

By hand, compute a wpcp of the group

$$G = \langle a, b, c \mid a^9, b^9, c^9, [[b, a], a] = a^3, (aba)^9, (ba)^5 a = b, [a, c] \rangle;$$

you can use that  $G$  has order  $3^3$ .

SOLUTION: First compute  $H = G/P_1(G)$  as outlined in the lectures; abelianising the relations (and taking everything modulo 3) yields the  $1 \times 3$  matrix  $M = (0 \ 1 \ 0)$ . This tells us that  $H = \langle aP_1(G), cP_1(G) \rangle$  has rank 2, thus  $H \cong C_2^2$  and we can define  $H$  via the wpcp  $H = \langle a, c \mid a^3, c^3 \rangle$ . We define

$$\theta: G \rightarrow G/P_1(G), \quad (a, c) \mapsto (a, c).$$

The 3-covering of  $H$  is  $H^* = \text{Pc}\langle a, c, x_1, x_2, x_3 \mid a^3 = x_1, c^3 = x_2, [c, a] = x_3, x_1^3 = x_2^3 = x_3^3 = 1 \rangle$ , and consistency checks show that this presentation is consistent. Now use  $\theta$  to evaluate the relations of  $G$  in  $H^*$ :

$$\begin{array}{ll} a^9 = 1 \rightsquigarrow 1 = 1 & b^9 = 1 \rightsquigarrow 1 = 1 \\ c^9 = 1 \rightsquigarrow 1 = 1 & [[b, a], a] = a^3 \rightsquigarrow 1 = x_1 \\ (aba)^9 = 1 \rightsquigarrow 1 = 1 & (ba)^5 a = b \rightsquigarrow x_1^2 = 1 \\ [a, c] = 1 \rightsquigarrow x_3 = 1, & \end{array}$$

which tells us that  $G/P_2(G) \cong H^*/\langle x_1, x_3 \rangle = \text{Pc}\langle a, c, x \mid a^3 = 1, c^3 = x, x^3 = 1 \rangle \cong C_3 \times C_9$ . Since  $|G| = 27 = |G/P_2(G)|$ , we conclude that this is a wpcp for  $G$ .

**Question 10** (tutorial)

By hand, show that the nucleus of

$$Q_8 = \text{Pc}\langle a, b, c \mid a^2 = c, b^2 = c, c^2 = 1, [b, a] = c \rangle$$

in  $Q_8^*$  is trivial, and deduce that  $Q_8$  has no immediate 2-descendants.

**SOLUTION:** Write  $G = Q_8$  and note that  $G$  has  $p$ -class  $k = 2$ . Let  $G^*$  be the 2-cover with multiplier  $M$  and nucleus  $P_k(G^*)$ . We show that  $P_k(G^*) = 1$  so that  $UP_k(G^*) < M$  for every  $U < M$ , that is,  $M$  has no allowable subgroups, and therefore  $Q_8$  has no immediate descendants.

To prove that  $P_k(G^*)$  is trivial, we first write down a presentation for  $G^*$ . By considering  $G/P_1(G)$  as before, we find that  $G = \langle a, b \rangle$  has rank 2. Note that  $G$  is already given by a wpcp, and we can choose  $a^2 = c$  as the definition of  $c$ . Thus, a (inconsistent) presentation for  $G^*$  is

$$\text{Pc}\langle a, b, c, x_1, \dots, x_5 \mid a^2 = c, b^2 = cx_1, c^2 = x_2, [b, a] = cx_3, [c, b] = x_4, [c, a] = x_5, x_1^2 = \dots = x_5^2 = 1 \rangle;$$

note that  $M = \langle x_1, \dots, x_5 \rangle \leq G^*$  is the multiplier. Moreover,  $P_1(G^*)M/M \leq P_1(G) = \langle c \rangle$ , which shows that  $P_1(G^*)$  is contained in  $\langle c, x_1, \dots, x_5 \rangle$ . The latter is clearly abelian; if we show that  $c$  has order 2, then  $P_1(G)^*$  is elementary abelian, and  $P_k(G)^* = 1$  follows – which implies the claim.

To show that  $c$  has order 2, we need that  $x_2 = 1$ . We see this by doing a few consistency checks:  $aa^2 = ac$  and  $a^2a = ca = acx_5$  implies that  $x_5 = 1$ ; similarly,  $bb^2 = bcx_1$  and  $b^2b = cx_1b = bcx_1x_4$  implies  $x_4 = 1$ . Lastly,  $b^2a = cx_1a = acx_1x_5$  and  $b(ba) = babcx_3 = abcx_3bcx_3 = ab^2c^2x_3^2x_4 = acx_1x_2x_4$  force  $x_2 = 1$ ; recall that  $x_5 = x_4 = 1$ . Thus,  $c$  has order 2 in  $G^*$ , and the claim follows as described above.

**Question 11** (practical)

Make sure the GAP package Anupq is installed and running; you might have to do `./configure` and make in `pkg/anupq` before you can load it in gap with `LoadPackage('`anupq`')`. Look up the manual and use ...

- ... the command `Pq` to compute a wpcp of  $G$ ,
- ... the command `PqPCover` to compute the 2-covering group  $G^*$  of  $G$
- ... the command `PqDescendants` to compute all immediate descendants of  $G$ ,

for each group  $G$  in the questions above, and for

$$G = \langle x, y \mid [[y, x], x] = x^2, (xyx)^4, x^4, y^4, (yx)^3y = x \rangle \quad \text{with } p = 2.$$

**SOLUTION:** Here is some GAP code for the last group:

```
gap> LoadPackage("anupq");
gap> F:=FreeGroup(["x","y"]);
gap> AssignGeneratorVariables(F);
#I Assigned the global variables [ x, y ]
gap> R:=[Comm(Comm(y,x),x)/x^2, (x*y*x)^4, x^4, y^4, (y*x)^3*y/x];
gap> G:=F/R;
gap> Gwpcp:=Pq(G:Prime:=2);
gap> PrintPcpPresentation(PcGroupToPcpGroup(Gwpcp));
g1^2 = g5
g2^2 = g4
g3^2 = g5
g4^2 = id
g5^2 = id
g6^2 = id
g2 ^ g1 = g2 * g3
g3 ^ g1 = g3 * g5
g3 ^ g2 = g3 * g6
g4 ^ g1 = g4 * g5 * g6
```

```

gap> Size (Gwpcp);
64
gap> Gstar:=PqPCover (Gwpcp:Prime:=2);
gap> Size (Gstar);
512
gap> imdes:=PqDescendants (Gwpcp:Prime:=2);
[ <pc group of size 128 with 7 generators>,
  <pc group of size 128 with 7 generators>,
  <pc group of size 128 with 7 generators>,
  <pc group of size 128 with 7 generators> ]

```

**Question 12** (tutorial)

For  $n \in \mathbb{N}$  consider the cyclic group  $G = C_{p^n} = \text{Pc}\langle r \mid r^{(p^n)} \rangle$ ; compute  $G^*$  and show that  $G$  has immediate descendants.

SOLUTION: First, we write down a wpcp of  $G = C_{p^n}$ , namely

$$G = \text{Pc}\langle r_1, \dots, r_n \mid r_1^p = r_2, \dots, r_{n-1}^p = r_n, r_n^p = 1 \rangle.$$

The first  $n - 1$  relations are definitions; the non-defining relations are  $r_n^p = 1$  and the trivial commutator relations  $r_j^{r_i} = r_j$  for  $i < j$ . Thus we obtain

$$G^* = \text{Pc}\langle r_1, \dots, r_n, b, b_{i,j} (i < j) \mid r_1^p = r_2, \dots, r_{n-1}^p = r_n, r_n^p = b, r_j^{r_i} = r_j b_{i,j} (i < j), \text{ each } b_{i,j}^p = 1 \rangle.$$

Let's do a few consistency checks. First, if  $1 \leq i \leq j < n$ , then

$$r_j^p r_i = r_{j+1} r_i = r_i r_{j+1} b_{i,j+1} \quad \text{and} \quad r_j^{p-1} (r_j r_i) = r_i r_j^p b_{i,j}^p = r_i r_{j+1},$$

which shows that  $b_{i,k} = 1$  for all  $1 \leq i < k \leq n$ ; this yields a consistent wpcp, namely

$$G^* = \text{Pc}\langle r_1, \dots, r_n, b \mid r_1^p = r_2, \dots, r_{n-1}^p = r_n, r_n^p = b, b^p = 1 \rangle \cong C_{p^{n+1}}.$$

Note that  $G$  has  $p$ -class  $c = n$  since  $P_i(G) = \langle r_{i+1} \rangle$  for all  $i = 1, \dots, n - 1$ , and  $P_n(G) = 1$ . Now the multiplier of  $G$  is  $M = \langle b \rangle \leq G^*$  and the nucleus of  $G$  is  $P_c(G^*) = \langle b \rangle = M$ , that is,  $U = 1 \leq M$  is an allowable subgroup. This proves that  $G^*$  is indeed an immediate descendant of  $G$ .