

ON A MISCIBLE DISPLACEMENT MODEL IN POROUS MEDIA FLOW WITH MEASURE DATA

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ABSTRACT. We establish the existence of a solution to a non-linearly coupled elliptic-parabolic system of PDEs describing the single-phase, miscible displacement of one incompressible fluid by another in a porous medium. We consider a velocity-dependent diffusion-dispersion tensor and model the action of injection and production wells by measures on the domain. We obtain the solution by passing to the limit on problems with regularised well data.

1. INTRODUCTION

The single-phase, miscible displacement of one incompressible fluid by another in a porous medium occurs during the enhanced oil recovery process in petroleum engineering. Engineers displace the oil in a reservoir by the injection of a fluid, such as a polymeric solvent, into designated injection wells. A mixture of the oil and the invading fluid is then collected at production wells. Under the appropriate physical hypotheses, such a miscible displacement problem is described by a non-linearly coupled elliptic-parabolic system, occasionally referred to as the Peaceman model after it was first introduced by Peaceman and Rachford in [18]. We refer the reader to [3, 4, 17] for further details.

The reservoir in the physical problem is represented by a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) with Lipschitz continuous boundary $\partial\Omega$, and we denote by $(0, T)$ the time interval over which the process occurs.

In that which follows, Φ denotes the porosity of the reservoir and \mathbf{K} the absolute permeability of the reservoir. The Darcy velocity of the fluid mixture is denoted by \mathbf{u} , and the concentration-dependent quantities of viscosity and density by μ and ρ , respectively. We write \mathbf{g} for the constant, downward-pointing gravitational vector. The main unknowns of the problem are the pressure p of the fluid mixture and the concentration c of one of the components in the mixture. With this notation in place, the model is given by

$$(1.1) \quad \left. \begin{aligned} \mathbf{u}(x, t) &= -\frac{\mathbf{K}(x)}{\mu(c(x, t))} (\nabla p(x, t) - \rho(c(x, t))\mathbf{g}) \\ \operatorname{div} \mathbf{u}(x, t) &= (q^I - q^P)(x, t) \end{aligned} \right\}, \quad (x, t) \in \Omega \times (0, T),$$

$$(1.2) \quad \Phi(x)\partial_t c(x, t) - \operatorname{div}(\mathbf{D}(x, \mathbf{u})\nabla c - c\mathbf{u})(x, t) + (q^I c)(x, t) = (q^I \hat{c})(x, t), \\ (x, t) \in \Omega \times (0, T).$$

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The sums of injection well source terms and production well sink terms are given by q^I and q^P , respectively, and the concentration of the injected fluid is \hat{c} . After Peaceman [16], the diffusion-dispersion tensor \mathbf{D} frequently assumes the form

$$\mathbf{D}(x, \mathbf{u}) = \Phi(x) \left(d_m \mathbf{I} + |\mathbf{u}| \left(d_l E(\mathbf{u}) + d_t (\mathbf{I} - E(\mathbf{u})) \right) \right),$$

where \mathbf{I} is the identity matrix, $d_m > 0$ the molecular diffusion coefficient and $d_l > 0$ and $d_t > 0$ the longitudinal and transverse mechanical dispersion coefficients, respectively. In practice, d_m is very small relative to the mechanical dispersion coefficients, and d_l is much larger than d_t [7]. The matrix $E(\mathbf{u})$ is the projection along the direction of flow:

$$E(\mathbf{u}) = \left(\frac{\mathbf{u}_i \mathbf{u}_j}{|\mathbf{u}|^2} \right)_{1 \leq i, j \leq d}.$$

Following [13], the viscosity is usually given by

$$\mu(c) = \mu(0) \left(1 + (M^{1/4} - 1)c \right)^{-4} \quad \text{for } c \in [0, 1],$$

where $M = \mu(0)/\mu(1)$ is the mobility ratio.

We assume the reservoir boundary $\partial\Omega$ to be impermeable, which yields no-flow boundary conditions for (1.1)–(1.2):

$$(1.3) \quad \mathbf{u}(x, t) \cdot \mathbf{n} = 0, \quad (x, t) \in \partial\Omega \times (0, T), \text{ and}$$

$$(1.4) \quad \mathbf{D}(x, \mathbf{u}) \nabla c(x, t) \cdot \mathbf{n} = 0, \quad (x, t) \in \partial\Omega \times (0, T),$$

where \mathbf{n} is the exterior unit normal to $\partial\Omega$. The first of these enforces a compatibility condition upon the source terms:

$$\int_{\Omega} q^I(x, t) dx = \int_{\Omega} q^P(x, t) dx \quad \text{for all } t \in (0, T),$$

and we normalise the pressure to eliminate any arbitrary constants:

$$(1.5) \quad \int_{\Omega} p(x, t) dx = 0 \quad \text{for all } t \in (0, T).$$

Additionally, we specify an initial condition for the concentration:

$$(1.6) \quad c(x, 0) = c_0(x), \quad x \in \Omega.$$

The source terms q^I and sink terms q^P , henceforth collectively referred to as ‘source terms’, give the flow rates of the injected and produced fluids at the respective wells. The principle difficulty in their mathematical representation is that of scale. Typical wellbore diameters are fractions of a metre, whereas the diameter of a reservoir may be up to thousands of metres. At the former scale, a source term may be adequately modelled as a classical function supported on the wellbore. However, at the reservoir scale – that is, on the order of a typical mesh size of a discretisation used in reservoir simulation – the action of a well is effectively that of a (spatial) measure supported at a point in two-dimensional models, or on a line in three-dimensional models.

Studies addressing the well-posedness of the Peaceman model are few relative to numerical treatments of the problem; see Feng [12] for a survey of existence results. We note that Sammon [20] models the wells as mollified Dirac masses and assumes the diffusion coefficient \mathbf{D} to be independent of velocity. Subsequent

work by Mikelić [15] on the stationary problem with a slightly regularised diffusion-dispersion tensor models the wells as non-negative elements of $L^r(\Omega)$, $r \in (d, \infty)$. Both Feng [11] and Chen and Ewing [5] consider the diffusion-dispersion tensor as given above; the former takes $q^I, q^P \in L^\infty(0, T; L^2(\Omega))$, whereas the latter relaxes the regularity of the production well to $q^P \in L^\infty(0, T; H^{-1}(\Omega))$ and establishes existence for a variety of boundary conditions. Amirat and Ziani [1] assume $L^\infty(0, T; L^2(\Omega))$ regularity of the wells and prove the existence of a weak solution when $d_m = 0$, a condition motivated by the negligible contribution at high flow velocities of molecular diffusion relative to mechanical dispersion.

Under the assumption that \mathbf{D} is bounded, Fabrie and Gallouët [10] provide the first existence result for miscible displacement with wells modelled by measures. However, this assumption on \mathbf{D} is incompatible with the form of Peaceman's tensor above since the Darcy velocity \mathbf{u} is potentially unbounded. Choquet [6] considers this issue in the context of a compressible model of radionuclide transport with wells modelled by measures. The coupling of the equations in such compressible models is stronger than in the incompressible model we study in this paper, however it also provides estimates on the time derivative of the pressure which entail straightforward compactness results on this unknown. For the incompressible model (1.1)–(1.2), additional arguments are required to establish a convergence result on the pressure that permits passage to the limit in the coupled system.

In this paper we extend the work of [10] to establish the existence of a weak solution to (1.1)–(1.6) with a diffusion-dispersion tensor that generalises that of Peaceman and with wells modelled by measures. Section 2 lists the assumptions on the data and details the main result. In Section 3 we show that a weak solution exists for an approximate problem with regularised wells, which gives an alternative proof of a similar result of [5]. We prove our main result in Section 4 by passing to the limit on the approximate problem using some estimates and a stability result developed therein, the latter of which in particular allows us to dispense with an additional regularity assumption on the source terms required by [10]. The principal novelty of our work is perhaps in the treatment of the diffusion-dispersion term when passing to the limit, which is accomplished with reference to some technical lemmas presented in Appendix A.

2. ASSUMPTIONS AND MAIN RESULT

The assumptions on the data are as follows.

$$(2.1) \quad \begin{aligned} T \in \mathbb{R}_+^* \text{ and } \Omega \text{ is a bounded, open subset of } \mathbb{R}^d, d \geq 1, \\ \text{with a Lipschitz continuous boundary.} \end{aligned}$$

Denoting by $\mathcal{S}_d(\mathbb{R})$ the set of $d \times d$ symmetric matrices, the permeability satisfies

$$(2.2) \quad \begin{aligned} \mathbf{K} : \Omega \rightarrow \mathcal{S}_d(\mathbb{R}) \text{ is measurable and} \\ \exists k_* > 0 \text{ such that, for a.e. } x \in \Omega \text{ and } \forall \xi \in \mathbb{R}^d, \\ k_* |\xi|^2 \leq \mathbf{K}(x)\xi \cdot \xi \leq k_*^{-1} |\xi|^2. \end{aligned}$$

The assumptions on the porosity, density and viscosity are quite general:

$$(2.3) \quad \begin{aligned} \Phi \in L^\infty(\Omega) \text{ and there exists } \phi_* > 0 \text{ such that for a.e. } x \in \Omega, \\ \phi_* \leq \Phi(x) \leq \phi_*^{-1}, \end{aligned}$$

$$(2.4) \quad \rho \in C([0, 1], \mathbb{R}), \mu \in C([0, 1], (0, \infty)).$$

The diffusion-dispersion tensor satisfies

$$(2.5) \quad \begin{aligned} \mathbf{D} : \Omega \times \mathbb{R}^d &\rightarrow \mathcal{S}_d(\mathbb{R}) \text{ is a Carathéodory function such that} \\ \exists \alpha_{\mathbf{D}} > 0, \exists \Lambda_{\mathbf{D}} > 0 &\text{ such that, for a.e. } x \in \Omega \text{ and all } \zeta, \xi \in \mathbb{R}^d, \\ \mathbf{D}(x, \zeta) \xi \cdot \xi &\geq \alpha_{\mathbf{D}}(1 + |\zeta|)|\xi|^2 \text{ and } |\mathbf{D}(x, \zeta)| \leq \Lambda_{\mathbf{D}}(1 + |\zeta|). \end{aligned}$$

Finally, the injected and initial concentration are such that

$$(2.6) \quad \hat{c} \in L^\infty(0, T; C(\bar{\Omega})) \text{ satisfies } 0 \leq \hat{c}(x, t) \leq 1 \text{ for a.e. } (x, t) \in \Omega \times (0, T),$$

$$(2.7) \quad c_0 \in L^\infty(\Omega) \text{ satisfies } 0 \leq c_0(x) \leq 1 \text{ for a.e. } x \in \Omega,$$

and the source terms are such that

$$(2.8) \quad \begin{aligned} q^I &= a\nu \text{ and } q^P = b\nu, \text{ where} \\ \nu \in \mathcal{M}_+(\Omega) \text{ and } a, b &\in L^\infty(0, T; C(\bar{\Omega})) \text{ satisfy:} \\ a(x, t) &\geq 0, \quad b(x, t) \geq 0 \quad \forall (x, t) \in \Omega \times (0, T), \end{aligned}$$

where $\mathcal{M}_+(\Omega)$ is the set of bounded non-negative Radon measures on Ω . The compatibility condition imposed by (1.3) becomes

$$(2.9) \quad \int_{\Omega} a(x, t) d\nu(x) = \int_{\Omega} b(x, t) d\nu(x) \quad \forall t \in (0, T).$$

For a topological vector space $X(\Omega)$ of functions on Ω , we write $(X(\Omega))'$ for its topological dual and omit Ω when writing the duality pairing: $\langle \cdot, \cdot \rangle_{(X(\Omega))', X(\Omega)} = \langle \cdot, \cdot \rangle_{X', X}$. For $z \in [1, \infty)$, we write $z' = \frac{z}{z-1}$ for its conjugate exponent. Given $x \in \Omega$ and $\zeta \in \mathbb{R}^d$, we write $\mathbf{D}^{1/2}(x, \zeta)$ for the square root of $\mathbf{D}(x, \zeta)$, which is well-defined since $\mathbf{D}(x, \zeta)$ is a symmetric, positive-definite matrix. When a constant appears in an estimate we track only its relevant dependencies, which are always non-decreasing.

Remark 2.1. The particular forms of the viscosity and the diffusion-dispersion tensor given in Section 1 satisfy the assumptions (2.4), (2.5); for the latter, set $\alpha_{\mathbf{D}} = \phi_* \inf(d_m, d_l, d_t)$ and $\Lambda_{\mathbf{D}} = \phi_*^{-1} \sup(d_m, d_l, d_t)$.

Under these assumptions, we will consider the following notion of solutions to the Peaceman model.

Definition 2.1. Under the hypotheses (2.1)–(2.9), a weak solution of (1.1)–(1.6) is a triple (p, \mathbf{u}, c) satisfying

$$(2.10) \quad \begin{aligned} c &\in L^2(0, T; H^1(\Omega)), \quad 0 \leq c \leq 1, \\ c &\in L^\infty(0, T; L^1(\Omega, \nu)), \\ 0 &\leq c(x, t) \leq 1, \quad \text{for } \nu\text{-a.e. } x \in \Omega, \text{ for a.e. } t \in (0, T), \end{aligned}$$

$$(2.11) \quad \Phi \partial_t c \in L^2(0, T; (W^{1,s}(\Omega))'), \quad \forall s > 2d,$$

$$(2.12) \quad \Phi c \in C([0, T]; (W^{1,s}(\Omega))'), \quad \Phi c(\cdot, 0) = \Phi c_0 \text{ in } (W^{1,s}(\Omega))', \quad \forall s > 2d,$$

$$(2.13) \quad \mathbf{D}(\cdot, \mathbf{u}) \nabla c \in L^2(0, T; L^r(\Omega)^d), \quad \forall r < \frac{2d}{2d-1},$$

$$(2.14) \quad p \in L^\infty(0, T; W^{1,q}(\Omega)), \quad \mathbf{u} \in L^\infty(0, T; L^q(\Omega)^d), \quad \forall q < \frac{d}{d-1},$$

$$\begin{aligned}
 & \int_0^T \langle \Phi \partial_t c(\cdot, t), \varphi(\cdot, t) \rangle_{(W^{1,s})', W^{1,s}} dt \\
 & + \int_0^T \int_{\Omega} \mathbf{D}(x, \mathbf{u}(x, t)) \nabla c(x, t) \cdot \nabla \varphi(x, t) dx dt \\
 (2.15) \quad & - \int_0^T \int_{\Omega} \mathbf{u}(x, t) \cdot \nabla \varphi(x, t) c(x, t) dx dt \\
 & + \int_0^T \int_{\Omega} c(x, t) \varphi(x, t) b(x, t) d\nu(x) dt \\
 & = \int_0^T \int_{\Omega} \hat{c}(x, t) \varphi(x, t) a(x, t) d\nu(x) dt, \quad \forall \varphi \in L^2(0, T; W^{1,s}(\Omega)), \forall s > 2d,
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{u}(x, t) &= -\frac{\mathbf{K}(x)}{\mu(c(x, t))} (\nabla p(x, t) - \rho(c(x, t)) \mathbf{g}), \\
 (2.16) \quad & - \int_0^T \int_{\Omega} \mathbf{u}(x, t) \cdot \nabla \psi(x, t) dx dt = \int_0^T \int_{\Omega} \psi(x, t) a(x, t) d\nu(x) dt \\
 & - \int_0^T \int_{\Omega} \psi(x, t) b(x, t) d\nu(x) dt, \quad \forall \psi \in \bigcap_{q>d} L^1(0, T; W^{1,q}(\Omega)).
 \end{aligned}$$

The main result of this article is the following existence result.

Theorem 2.2. *Under the hypotheses (2.1)–(2.9), there exists a weak solution to (1.1)–(1.6) in the sense of Definition 2.1.*

Remark 2.2. The proof actually shows that (p, \mathbf{u}) is a solution to (1.1)–(1.3) in a stronger sense than what we state in (2.16); see the proof of Proposition 4.1.

We prove Theorem 2.2 by passing to the limit on problems with a regularised ν as defined in Proposition 2.3 below. Whilst the result of this proposition is already established in [5], we give an alternative proof based on the regularisation method we then use to prove Theorem 2.2. This serves to present the estimates we require and illustrate the techniques we use for the singular problem, in particular the technical lemmas of Appendix A.

Proposition 2.3. *Let (2.1)–(2.9) hold and suppose that $\nu \in L^2(\Omega)$. Then there exists a solution $(\tilde{p}, \tilde{\mathbf{u}}, \tilde{c})$ to (1.1)–(1.6) in the following sense:*

$$(2.17) \quad \tilde{c} \in L^2(0, T; H^1(\Omega)), \quad 0 \leq \tilde{c} \leq 1,$$

$$(2.18) \quad \Phi \partial_t \tilde{c} \in L^2(0, T; (W^{1,4}(\Omega))'),$$

$$(2.19) \quad \Phi \tilde{c} \in C([0, T]; (W^{1,4}(\Omega))'), \quad \Phi \tilde{c}(\cdot, 0) = \Phi c_0 \quad \text{in } (W^{1,4}(\Omega))',$$

$$(2.20) \quad \mathbf{D}(\cdot, \tilde{\mathbf{u}}) \nabla \tilde{c} \in L^2(0, T; L^{4/3}(\Omega)^d),$$

$$(2.21) \quad \tilde{p} \in L^\infty(0, T; H^1(\Omega)), \quad \tilde{\mathbf{u}} \in L^\infty(0, T; L^2(\Omega)^d),$$

$$\begin{aligned}
& \int_0^T \langle \Phi \partial_t \tilde{c}(\cdot, t), \varphi(\cdot, t) \rangle_{(W^{1,4})', W^{1,4}} dt \\
& + \int_0^T \int_{\Omega} \mathbf{D}(x, \tilde{\mathbf{u}}(x, t)) \nabla \tilde{c}(x, t) \cdot \nabla \varphi(x, t) dx dt \\
(2.22) \quad & - \int_0^T \int_{\Omega} \tilde{\mathbf{u}}(x, t) \cdot \nabla \varphi(x, t) \tilde{c}(x, t) dx dt \\
& + \int_0^T \int_{\Omega} \tilde{c}(x, t) \varphi(x, t) b(x, t) \nu(x) dx dt \\
& = \int_0^T \int_{\Omega} \hat{c}(x, t) \varphi(x, t) a(x, t) \nu(x) dx dt, \quad \forall \varphi \in L^2(0, T; W^{1,4}(\Omega)),
\end{aligned}$$

$$\begin{aligned}
& \tilde{\mathbf{u}}(x, t) = -\frac{\mathbf{K}(x)}{\mu(\tilde{c}(x, t))} (\nabla \tilde{p}(x, t) - \rho(\tilde{c}(x, t)) \mathbf{g}), \\
(2.23) \quad & - \int_0^T \int_{\Omega} \tilde{\mathbf{u}}(x, t) \cdot \nabla \psi(x, t) dx dt = \int_0^T \int_{\Omega} \psi(x, t) a(x, t) \nu(x) dx dt \\
& - \int_0^T \int_{\Omega} \psi(x, t) b(x, t) \nu(x) dx dt, \quad \forall \psi \in L^1(0, T; H^1(\Omega)).
\end{aligned}$$

Remark 2.3. It will be seen that in fact $\mathbf{D}^{1/2}(\cdot, \tilde{\mathbf{u}}) \nabla \tilde{c} \in L^2(0, T; L^2(\Omega)^d)$, which, together with (2.21), is stronger than (2.20).

One obtains the existence of a solution to this regularised problem by passing to the limit on problems with bounded diffusion-dispersion tensors, detailed in the next section.

3. EXISTENCE OF A SOLUTION TO THE REGULARISED PROBLEM

3.1. Approximating the diffusion-dispersion tensor. Assume that (2.1)–(2.9) hold. For $k \in \mathbb{R}_+$, consider the truncated diffusion-dispersion tensor

$$(3.1) \quad \mathbf{D}_k(x, \mathbf{u}) := \mathbf{D}\left(x, \frac{T_k(|\mathbf{u}|)}{|\mathbf{u}|} \mathbf{u}\right),$$

where $T_k : [0, \infty) \rightarrow [0, \infty)$ is the truncator

$$T_k(s) = \begin{cases} s & \text{if } s < k \\ k & \text{if } s \geq k. \end{cases}$$

It is simple to verify that for each $k \in \mathbb{R}_+$, \mathbf{D}_k is a bounded Carathéodory function that satisfies

$$(3.2) \quad \mathbf{D}_k(x, \mathbf{u}) \xi \cdot \xi \geq \alpha_{\mathbf{D}} |\xi|^2, \quad \text{for a.e. } x \in \Omega, \forall \xi \in \mathbb{R}^d.$$

Furthermore, \mathbf{D}_k can be estimated uniformly with respect to k :

$$(3.3) \quad |\mathbf{D}_k(x, \mathbf{u})| \leq \Lambda_{\mathbf{D}} (1 + |\mathbf{u}|), \quad \text{for a.e. } x \in \Omega,$$

and for a.e. $x \in \Omega$, $\mathbf{D}_k(x, \cdot) \rightarrow \mathbf{D}(x, \cdot)$ uniformly (as $k \rightarrow \infty$) on compact sets.

Now, in (1.1)–(1.4) replace \mathbf{D} by \mathbf{D}_k and assume that $\nu \in L^2(\Omega)$. Then [10] establishes the existence of a solution (p_k, \mathbf{u}_k, c_k) to (1.1)–(1.6) in the following sense:

$$(3.4) \quad c_k \in L^2(0, T; H^1(\Omega)) \cap C([0, T]; L^2(\Omega)), \quad 0 \leq c_k \leq 1,$$

$$(3.5) \quad \Phi \partial_t c_k \in L^2(0, T; (H^1(\Omega))'),$$

$$(3.6) \quad c_k(\cdot, 0) = c_0 \quad \text{in } L^2(\Omega),$$

$$(3.7) \quad p_k \in L^\infty(0, T; H^1(\Omega)), \quad \mathbf{u}_k \in L^\infty(0, T; L^2(\Omega)^d),$$

$$(3.8) \quad \begin{aligned} & \int_0^T \langle \Phi \partial_t c_k(\cdot, t), \varphi(\cdot, t) \rangle_{(H^1)', H^1} dt \\ & + \int_0^T \int_\Omega \mathbf{D}_k(x, \mathbf{u}_k(x, t)) \nabla c_k(x, t) \cdot \nabla \varphi(x, t) dx dt \\ & - \int_0^T \int_\Omega \mathbf{u}_k(x, t) \cdot \nabla \varphi(x, t) c_k(x, t) dx dt \\ & + \int_0^T \int_\Omega c_k(x, t) \varphi(x, t) b(x, t) \nu(x) dx dt \\ & = \int_0^T \int_\Omega \hat{c}(x, t) \varphi(x, t) a(x, t) \nu(x) dx dt, \quad \forall \varphi \in L^2(0, T; H^1(\Omega)), \end{aligned}$$

$$(3.9) \quad \begin{aligned} \mathbf{u}_k(x, t) &= -\frac{\mathbf{K}(x)}{\mu(c_k(x, t))} (\nabla p_k(x, t) - \rho(c_k(x, t)) \mathbf{g}), \\ & - \int_0^T \int_\Omega \mathbf{u}_k(x, t) \cdot \nabla \psi(x, t) dx dt = \int_0^T \int_\Omega \psi(x, t) a(x, t) \nu(x) dx dt \\ & - \int_0^T \int_\Omega \psi(x, t) b(x, t) \nu(x) dx dt, \quad \forall \psi \in L^1(0, T; H^1(\Omega)). \end{aligned}$$

In the course of proving this result, the authors show that there is a constant C_1 depending only on $\|a\|_{L^\infty(0, T; C(\bar{\Omega}))}$, $\|b\|_{L^\infty(0, T; C(\bar{\Omega}))}$ and $\|\nu\|_{L^2(\Omega)}$ such that

$$(3.10) \quad \|p_k\|_{L^\infty(0, T; H^1(\Omega))} \leq C_1, \quad \|\mathbf{u}_k\|_{L^\infty(0, T; L^2(\Omega)^d)} \leq C_1.$$

The next section details the additional estimates required to pass to the limit in (3.4)–(3.10) and hence prove Proposition 2.3.

Remark 3.1. Note that [10] actually proves the existence of a solution (p_k, \mathbf{u}_k, c_k) to (1.1)–(1.6) with the additional assumption of $\Phi \equiv 1$. However, it is straightforward to adapt this result to the case where the porosity satisfies (2.3). Examining the relevant estimates in [10], one observes that a coefficient of ϕ_* is introduced in the estimates that correspond to our estimates (3.11) herein, and as in our Proposition 3.3 below, one estimates $\Phi \partial_t c$ instead of $\partial_t c$. The compactness result of [2] can be applied in [10] in the same manner as in Section 3.3 below to account for the presence of Φ satisfying (2.3).

3.2. Estimates. We begin with some simple energy estimates, which in particular provide bounds on spatial derivatives of c_k .

Proposition 3.1. *Assume that (3.4)–(3.9) hold. There exists C_2 depending only on $\|a\|_{L^\infty(0, T; C(\bar{\Omega}))}$, $\|b\|_{L^\infty(0, T; C(\bar{\Omega}))}$ and $\|\nu\|_{L^1(\Omega)}$ such that for all $s \in [0, T]$,*

$$(3.11) \quad \phi_* \|c_k(s)\|_{L^2(\Omega)}^2 + 2 \int_0^s \int_\Omega \mathbf{D}_k(x, \mathbf{u}_k(x, t)) \nabla c_k(x, t) \cdot \nabla c_k(x, t) dx dt \leq C_2$$

Proof. Take $\varphi = \mathbf{1}_{[0,s]}c_k$ as a test function in (3.8) to obtain

$$\begin{aligned}
(3.12) \quad & \int_0^s \langle \Phi \partial_t c_k(\cdot, t), c_k(\cdot, t) \rangle_{(H^1)', H^1} dt \\
& + \int_0^s \int_{\Omega} \mathbf{D}_k(x, \mathbf{u}_k(x, t)) \nabla c_k(x, t) \cdot \nabla c_k(x, t) dx dt \\
& - \int_0^s \int_{\Omega} c_k(x, t) \mathbf{u}_k(x, t) \cdot \nabla c_k(x, t) dx dt \\
& + \int_0^s \int_{\Omega} c_k^2(x, t) b(x, t) \nu(x) dx dt \\
& = \int_0^s \int_{\Omega} \hat{c}(x, t) c_k(x, t) a(x, t) \nu(x) dx dt.
\end{aligned}$$

From (3.4), $c_k \in L^2(0, T; H^1(\Omega)) \cap L^\infty(\Omega \times (0, T))$. It follows that $\nabla \left(\frac{c_k^2}{2} \right) = c_k \nabla c_k \in L^2(0, T; L^2(\Omega))$ and so $\mathbf{1}_{[0,s]} \frac{c_k^2}{2}$ is a valid test function in (3.9). We therefore rewrite the convection term

$$\begin{aligned}
& - \int_0^s \int_{\Omega} c_k(x, t) \mathbf{u}_k(x, t) \cdot \nabla c_k(x, t) dx dt \\
& = - \int_0^s \int_{\Omega} \mathbf{u}_k(x, t) \cdot \nabla \left(\frac{c_k^2(x, t)}{2} \right) dx dt \\
& = \int_0^s \int_{\Omega} \left(\frac{c_k^2(x, t)}{2} \right) (a(x, t) - b(x, t)) \nu(x) dx dt.
\end{aligned}$$

Integration by parts on the first term of (3.12) yields

$$\int_0^s \langle \Phi \partial_t c_k(\cdot, t), c_k(\cdot, t) \rangle_{(H^1)', H^1} dt = \frac{1}{2} (\|\sqrt{\Phi} c_k(\cdot, s)\|_{L^2(\Omega)}^2 - \|\sqrt{\Phi} c_0\|_{L^2(\Omega)}^2).$$

Substituting back in (3.12),

$$\begin{aligned}
& \frac{1}{2} (\|\sqrt{\Phi} c_k(\cdot, s)\|_{L^2(\Omega)}^2 - \|\sqrt{\Phi} c_0\|_{L^2(\Omega)}^2) \\
& + \int_0^s \int_{\Omega} \mathbf{D}_k(x, \mathbf{u}_k(x, t)) \nabla c_k(x, t) \cdot \nabla c_k(x, t) dx dt \\
& = \int_0^s \int_{\Omega} \left(\hat{c}(x, t) c_k(x, t) - \frac{c_k^2(x, t)}{2} \right) a(x, t) \nu(x) dx dt \\
& \quad - \int_0^s \int_{\Omega} \frac{c_k^2(x, t)}{2} b(x, t) \nu(x) dx dt \\
& \leq (\|a\|_{L^\infty(0, T; C(\bar{\Omega}))} + \|b\|_{L^\infty(0, T; C(\bar{\Omega}))}) \|\nu\|_{L^1(\Omega)} s,
\end{aligned}$$

where the last inequality follows from the uniform bound (3.4) and the hypotheses (2.3) and (2.7) on the porosity and the initial concentration. Estimate (3.11) is then a straightforward consequence. \square

Corollary 3.2. *There exist C_3 and C_4 , both depending only on $\|a\|_{L^\infty(0, T; C(\bar{\Omega}))}$, $\|b\|_{L^\infty(0, T; C(\bar{\Omega}))}$ and $\|\nu\|_{L^1(\Omega)}$, such that*

$$(3.13) \quad \|\mathbf{D}_k^{1/2}(\cdot, \mathbf{u}_k) \nabla c_k\|_{L^2(0, T; L^2(\Omega)^d)} \leq C_3$$

$$(3.14) \quad \|\mathbf{D}_k(\cdot, \mathbf{u}_k) \nabla c_k\|_{L^2(0, T; L^{4/3}(\Omega)^d)} \leq C_4.$$

Proof. Writing $\mathbf{D}_k(x, \mathbf{u}_k) \nabla c_k \cdot \nabla c_k = |\mathbf{D}_k^{1/2}(x, \mathbf{u}_k) \nabla c_k|^2$ and invoking Proposition 3.1 gives (3.13). For (3.14), let $\zeta, \xi \in \mathbb{R}^d$ and observe that

$$|\mathbf{D}_k^{1/2}(x, \zeta) \xi|^2 = \mathbf{D}_k(x, \zeta) \xi \cdot \xi \leq |\mathbf{D}_k(x, \zeta)| |\xi|^2 \leq \Lambda_{\mathbf{D}} (1 + |\zeta|) |\xi|^2.$$

Taking square roots of both sides establishes that for a.e. $x \in \Omega$,

$$(3.15) \quad |\mathbf{D}_k^{1/2}(x, \zeta)| \leq \Lambda_{\mathbf{D}}^{1/2} (1 + |\zeta|^{1/2}) \quad \text{for all } \zeta \in \mathbb{R}^d.$$

Hence, Estimate (3.10) shows that $\|\mathbf{D}_k^{1/2}(\cdot, \mathbf{u}_k)\|_{L^\infty(0, T; L^4(\Omega)^{d \times d})}$ is bounded uniformly with respect to k . Using Hölder's inequality to write

$$\begin{aligned} \|\mathbf{D}_k(\cdot, \mathbf{u}_k) \nabla c_k\|_{L^2(0, T; L^{4/3}(\Omega)^d)} \\ \leq \|\mathbf{D}_k^{1/2}(\cdot, \mathbf{u}_k)\|_{L^\infty(0, T; L^4(\Omega)^{d \times d})} \|\mathbf{D}_k^{1/2}(\cdot, \mathbf{u}_k) \nabla c_k\|_{L^2(0, T; L^2(\Omega)^d)} \end{aligned}$$

thus gives (3.14). \square

The problem of interest is dependent on both space and time and so it is standard to complete these spatial estimates with time-derivative estimates on c_k , in order to obtain the strong compactness of this sequence of functions.

Proposition 3.3. *Let (3.4)–(3.9) hold. Then there exists C_5 depending only on $\|a\|_{L^\infty(0, T; C(\bar{\Omega}))}$, $\|b\|_{L^\infty(0, T; C(\bar{\Omega}))}$ and $\|\nu\|_{L^2(\Omega)}$ such that*

$$(3.16) \quad \|\Phi \partial_t c_k\|_{L^2(0, T; (W^{1,4}(\Omega))')} \leq C_5.$$

Proof. Let $\varphi \in L^2(0, T; W^{1,4}(\Omega))$. From (3.4) and (3.8) we have

$$\begin{aligned} & \left| \int_0^T \langle \Phi \partial_t c_k(\cdot, t), \varphi(\cdot, t) \rangle_{(W^{1,4})', W^{1,4}} dt \right| \\ & \leq \int_0^T \int_\Omega |\mathbf{D}_k(x, \mathbf{u}_k(x, t)) \nabla c_k(x, t) \cdot \nabla \varphi(x, t)| dx dt \\ & \quad + \int_0^T \int_\Omega |\mathbf{u}_k(x, t) \cdot \nabla \varphi(x, t) c_k(x, t)| dx dt \\ & \quad + \int_0^T \int_\Omega |c_k(x, t) \varphi(x, t) b(x, t) \nu(x)| dx dt \\ & \quad + \int_0^T \int_\Omega |\hat{c}(x, t) \varphi(x, t) a(x, t) \nu(x)| dx dt \\ & \leq \|\mathbf{D}_k(\cdot, \mathbf{u}_k) \nabla c_k\|_{L^2(0, T; L^{4/3}(\Omega)^d)} \|\nabla \varphi\|_{L^2(0, T; L^4(\Omega)^d)} \\ & \quad + \|\mathbf{u}_k\|_{L^\infty(0, T; L^2(\Omega)^d)} \|\nabla \varphi\|_{L^1(0, T; L^2(\Omega)^d)} \\ & \quad + \|\varphi\|_{L^1(0, T; L^2(\Omega))} \|\nu\|_{L^2(\Omega)} (\|a\|_{L^\infty(0, T; C(\bar{\Omega}))} + \|b\|_{L^\infty(0, T; C(\bar{\Omega}))}). \end{aligned}$$

We then use Estimates (3.10) and (3.14) to obtain

$$\left| \int_0^T \langle \Phi \partial_t c_k(\cdot, t), \varphi(\cdot, t) \rangle_{(W^{1,4})', W^{1,4}} dt \right| \leq C_5 \|\varphi\|_{L^2(0, T; W^{1,4}(\Omega))},$$

which completes the proof. \square

3.3. Passing to the limit. In order to apply classical compactness results we introduce the Hilbert space

$$\Phi H^1(\Omega) := \{\Phi v \mid v \in H^1(\Omega)\},$$

with norm

$$\|w\|_{\Phi H^1(\Omega)} := \left\| \frac{w}{\Phi} \right\|_{H^1(\Omega)} \quad \text{for } w \in \Phi H^1(\Omega).$$

Observe that since the porosity is independent of time, the previous estimate (3.16) gives a bound in $L^2(0, T; (W^{1,4}(\Omega))')$ of the sequence $(\partial_t(\Phi c_k))_{k \in \mathbb{N}}$. Furthermore, applying (3.2) to the left-hand side of (3.11) shows that the sequence $(c_k)_{k \in \mathbb{N}}$ is bounded in $L^2(0, T; H^1(\Omega))$, and therefore that the sequence $(\Phi c_k)_{k \in \mathbb{N}}$ is bounded in $L^2(0, T; \Phi H^1(\Omega))$. One may verify that $\Phi H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, so that the invocation of a compactness result of Aubin [2] is valid. Combining these with the bound on the sequence (c_k) in $L^\infty(\Omega \times (0, T))$ from (3.4), we have that, up to a subsequence,

$$(3.17) \quad \begin{aligned} c_k &\rightarrow \tilde{c} \quad \text{in } L^\infty(\Omega \times (0, T)) \text{ weak-*,} \quad 0 \leq \tilde{c} \leq 1 \text{ a.e. in } \Omega \times (0, T), \\ c_k &\rightarrow \tilde{c} \quad \text{weakly in } L^2(0, T; H^1(\Omega)), \text{ proving (2.17),} \end{aligned}$$

$$(3.18) \quad \begin{aligned} \Phi \partial_t c_k &\rightarrow \Phi \partial_t \tilde{c} \quad \text{weakly in } L^2(0, T; (W^{1,4}(\Omega))'), \text{ proving (2.18),} \\ \Phi c_k &\rightarrow \Phi \tilde{c} \quad \text{in } L^2(0, T; L^2(\Omega)) \text{ and} \\ \Phi c_k &\rightarrow \Phi \tilde{c} \quad \text{in } C([0, T]; (W^{1,4}(\Omega))'), \text{ which proves (2.19).} \end{aligned}$$

From (3.18) and the uniform positivity of Φ we deduce that, up to a subsequence,

$$c_k \rightarrow \tilde{c} \quad \text{in } L^2(0, T; L^2(\Omega)) \text{ and a.e. in } \Omega \times (0, T).$$

Estimates (3.10) and the first step in the proof of [10, Proposition 4.5] demonstrates that there exists $(\tilde{p}, \tilde{\mathbf{u}})$ satisfying (2.21) and (2.23) and that, up to a subsequence,

$$\nabla p_k \rightarrow \nabla \tilde{p} \quad \text{in } L^2(0, T; L^2(\Omega)^d) \quad \text{and} \quad \tilde{\mathbf{u}}_k \rightarrow \tilde{\mathbf{u}} \quad \text{in } L^2(0, T; L^2(\Omega)^d).$$

To prove Proposition 2.3, it remains to establish (2.20), and pass to the limit on (3.8) to demonstrate that \tilde{c} satisfies (2.22). To this end, we apply Corollary A.2 to obtain $\mathbf{D}_k(\cdot, \mathbf{u}_k) \rightarrow \mathbf{D}(\cdot, \tilde{\mathbf{u}})$ in $L^2(0, T; L^2(\Omega)^{d \times d})$. This, together with (3.14) and (3.17) verifies the hypotheses of Lemma A.3 with $v_k = \nabla c_k$, $w_k = \mathbf{D}_k(\cdot, \mathbf{u}_k)$, a, r_1, r_2, s_1, s_2 all 2 and $b = 4/3$ and hence

$$\mathbf{D}_k(\cdot, \mathbf{u}_k) \nabla c_k \rightarrow \mathbf{D}(\cdot, \tilde{\mathbf{u}}) \nabla \tilde{c} \quad \text{weakly in } L^2(0, T; L^{4/3}(\Omega)^d),$$

which proves (2.20). Taking $\varphi \in L^2(0, T; W^{1,4}(\Omega))$, we pass to the limit in (3.8) using weak-strong convergence. This concludes the proof of Proposition 2.3.

Remark 3.2. Following Remark 2.3, we show that $\mathbf{D}^{1/2}(\cdot, \tilde{\mathbf{u}}) \nabla \tilde{c} \in L^2(0, T; L^2(\Omega)^d)$ using the same technique as above. Apply Corollary A.2 to obtain $\mathbf{D}_k^{1/2}(\cdot, \mathbf{u}_k) \rightarrow \mathbf{D}^{1/2}(\cdot, \tilde{\mathbf{u}})$ in $L^2(0, T; L^2(\Omega)^{d \times d})$. As before, the uniform bound (3.13) on the sequence $(\mathbf{D}_k^{1/2}(\cdot, \mathbf{u}_k) \nabla c_k)_{k \in \mathbb{N}}$ and (3.17) allow us to apply Lemma A.3 and hence

$$(3.19) \quad \mathbf{D}_k^{1/2}(\cdot, \mathbf{u}_k) \nabla c_k \rightarrow \mathbf{D}^{1/2}(\cdot, \tilde{\mathbf{u}}) \nabla \tilde{c} \quad \text{weakly in } L^2(0, T; L^2(\Omega)^d).$$

We employ this result in the next section, where we establish the main result of this paper.

4. EXISTENCE OF A SOLUTION WITH SINGULAR WELL DATA

As stated in the introduction, we obtain the existence of a solution to (1.1)–(1.6) with measure source terms by passing to the limit on the regularised problem.

4.1. Stability result for the elliptic equation. The following proposition is a stronger version of [10, Proposition 3.3]. In this reference, the stability result is proved under additional assumptions on the measure source term and its approximation $(f_n)_{n \in \mathbb{N}}$, stated in order to get uniqueness of the limit via a regularity result of Meyers [14]. We prove here that, by invoking Stampacchia’s notion of solution for linear elliptic equations with measures [8, 19, 21], we can omit these additional assumptions and prove nevertheless the convergence of the whole sequence of approximate solutions.

Proposition 4.1. *Let $(A_n)_{n \in \mathbb{N}} \subset L^\infty(\Omega; M_d(\mathbb{R}))$ be a sequence of bounded measurable matrix-valued functions, $(F_n)_{n \in \mathbb{N}} \subset L^2(\Omega)^d$, $(f_n)_{n \in \mathbb{N}} \subset L^2(\Omega)$. We assume that, for some $A : \Omega \rightarrow M_d(\mathbb{R})$, $F \in L^2(\Omega)^d$ and $f \in (C(\bar{\Omega}))'$,*

- (1) $A_n \rightarrow A$ pointwise a.e. on Ω , $(A_n)_{n \in \mathbb{N}}$ is bounded in $L^\infty(\Omega; M_d(\mathbb{R}))$ and uniformly elliptic,
- (2) $F_n \rightarrow F$ in $L^2(\Omega)^d$,
- (3) $f_n \rightarrow f$ for the weak-* topology of $(C(\bar{\Omega}))'$ and, for all $n \in \mathbb{N}$, $\int_\Omega f_n(x) dx = 0$.

We let v_n be the weak solution to $-\operatorname{div}(A_n \nabla v_n - F_n) = f_n$ with homogeneous Neumann boundary conditions. That is,

$$(4.1) \quad \begin{cases} v_n \in H^1(\Omega) \text{ satisfies } \int_\Omega v_n(x) dx = 0 \text{ and, } \forall \varphi \in H^1(\Omega), \\ \int_\Omega A_n(x) \nabla v_n(x) \cdot \nabla \varphi(x) dx = \int_\Omega f_n(x) \varphi(x) dx + \int_\Omega F_n(x) \cdot \nabla \varphi(x) dx. \end{cases}$$

Then there exists a weak solution v to $-\operatorname{div}(A \nabla v - F) = f$ with homogeneous Neumann boundary conditions:

$$(4.2) \quad \begin{cases} v \in \bigcap_{q < \frac{d}{d-1}} W^{1,q}(\Omega) \text{ satisfies } \int_\Omega v(x) dx = 0 \text{ and, } \forall \varphi \in \bigcup_{r > d} W^{1,r}(\Omega), \\ \int_\Omega A(x) \nabla v(x) \cdot \nabla \varphi(x) dx = \int_\Omega \varphi(x) df(x) + \int_\Omega F(x) \cdot \nabla \varphi(x) dx, \end{cases}$$

such that, as $n \rightarrow \infty$, $v_n \rightarrow v$ in $W^{1,q}(\Omega)$ for all $q < \frac{d}{d-1}$.

Remark 4.1. Note that the *whole* sequence v_n converges strongly to a solution of (4.2). The proof also shows that $(v_n)_{n \in \mathbb{N}}$ actually converges to a solution of $-\operatorname{div}(A \nabla v - F) = f$ in a stronger sense than (4.2), namely Stampacchia’s duality formulation, which ensures the uniqueness of the solution to this problem (this is actually the reason why the stability result does not hold “up to a subsequence”).

Proof. Let us first recall Stampacchia’s setting for solving elliptic equations with measure data. For a given bounded and coercive matrix-valued function A , we define $T_A : (H^1(\Omega))' \rightarrow H^1(\Omega)$ as the inverse of $-\operatorname{div}(A^T \nabla)$ with homogeneous

Neuman boundary conditions, i.e. for $g \in (H^1(\Omega))'$, $w = T_A(g)$ is the solution to

$$(4.3) \quad \begin{cases} w \in H^1(\Omega), & \int_{\Omega} w(x) \, dx = 0 \text{ and} \\ \forall \psi \in H^1(\Omega), & \int_{\Omega} A(x)^T \nabla w(x) \cdot \nabla \psi(x) \, dx = \langle g, \psi \rangle_{(H^1)'} \end{cases}$$

It has been proved (see e.g. [21] for Dirichlet boundary conditions, [8] for other boundary conditions) that for any $p > d$, $T_A^p = (T_A)|_{(W^{1,p'}(\Omega))'}$ takes values in $H^1(\Omega) \cap C(\bar{\Omega})$. Hence $(T_A^p)^* : \mathcal{M}(\bar{\Omega}) + (H^1(\Omega))' \rightarrow W^{1,p'}(\Omega)$ is well defined. Stampacchia then defines, for $h \in \mathcal{M}(\bar{\Omega}) + (H^1(\Omega))'$, the solution to

$$(4.4) \quad \begin{cases} -\operatorname{div}(A\nabla z) = h & \text{in } \Omega, \\ A\nabla z \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \end{cases}$$

as $z = (T_A^p)^*(h)$. It can easily be seen that z does not depend on p (because $T_A^p = T_A^s$ on $(W^{1,\max(p',s')}(\Omega))'$), and we therefore denote $z = (T_A)^*(h)$ without the p .

Several equivalent formulations exist that make clear why this z can be considered a re-formulation of (4.4), see [8, 19, 21]. It can in particular be seen that if $h \in (H^1(\Omega))'$, then $z = (T_A)^*(h)$ is the classical weak solution of (4.4) with mean value zero.

In particular, defining $h_n \in (H^1(\Omega))'$ by $\langle h_n, \psi \rangle_{(H^1)'} = \int_{\Omega} f_n(x)\psi(x) \, dx + \int_{\Omega} F_n(x) \cdot \nabla \psi(x) \, dx$, we see that $v_n = (T_{A_n})^*(h_n)$. Notice that h_n is the sum of two terms, one converging to f in $(C(\bar{\Omega}))'$ weak-* and the other strongly converging to $\psi \mapsto \int_{\Omega} F(x) \cdot \nabla \psi(x) \, dx$ in $(H^1(\Omega))'$. We can therefore use the stability results of [8, 9] (established for Dirichlet, mixed or Robin boundary conditions, but the proof is identical for Neumann boundary conditions) which prove that, under the assumptions in the proposition, $v_n \rightarrow v := (T_A)^*(h)$ strongly in $W^{1,q}(\Omega)$ for all $q < \frac{d}{d-1}$, where $h \in \mathcal{M}(\bar{\Omega}) + (H^1(\Omega))'$ is defined by

$$\forall \psi \in C(\bar{\Omega}) \cap H^1(\Omega), \quad \langle h, \psi \rangle_{\mathcal{M} + (H^1)', C \cap H^1} = \int_{\Omega} \psi(x) \, df(x) + \int_{\Omega} F(x) \cdot \nabla \psi(x) \, dx.$$

The conclusion then follows immediately, as it is known [8, 19, 21] that $v = (T_A)^*(h)$ satisfies (4.2). \square

4.2. Approximation and estimates. The first two steps of the proof of [10, Theorem 2.1] detail the approximations necessary to regularise the singular problem while retaining the no-flow compatibility conditions: a sequence $(a_n, \nu_n)_{n \in \mathbb{N}}$ is constructed such that

$$(4.5) \quad \nu_n \geq 0, \quad \|\nu_n\|_{L^1(\Omega)} \leq \nu(\Omega),$$

$$(4.6) \quad a_n(x, t) \rightarrow a(x, t), \text{ as } n \rightarrow \infty, \text{ for a.e. } (x, t) \in \Omega \times (0, T),$$

$$(4.7) \quad a_n \geq 0 \text{ and } (a_n) \text{ is bounded in } L^\infty(0, T; C(\bar{\Omega})), \text{ and}$$

$$(4.8) \quad \int_{\Omega} a_n(x, t)\nu_n(x) \, dx = \int_{\Omega} b(x, t)\nu_n(x) \, dx, \quad \text{for a.e. } t \in (0, T).$$

This last property ensures that the compatibility condition (2.9) still holds when ν is replaced by ν_n in (1.1)–(1.4). We refer the reader to [10] for further details.

Remark 4.2. The construction of $(\nu_n)_{n \in \mathbb{N}}$ necessary for the present work is simpler than in [10], since we do not require the sequence to be bounded in $(W^{1,q}(\Omega))'$ for all $q > 2$ (see Section 4.1).

Replacing a by a_n and ν by ν_n in (1.1)–(1.6), Proposition 2.3 shows that there is a solution $(\tilde{p}_n, \tilde{\mathbf{u}}_n, \tilde{c}_n)$ to (1.1)–(1.6) in the sense of (2.17)–(2.23). Furthermore, [10, Proposition 4.3] shows that for any $1 \leq q < d/(d-1)$, there exists C_6 not depending upon n such that

$$(4.9) \quad \|\tilde{p}_n\|_{L^\infty(0,T;W^{1,q}(\Omega))} \leq C_6, \quad \|\tilde{\mathbf{u}}_n\|_{L^\infty(0,T;L^q(\Omega)^d)} \leq C_6.$$

It remains to re-examine the estimates of Section 3.2 in light of (4.9).

From Estimate (3.11) and Properties (4.5), (4.7), one deduces that there exists C_7 not depending on n such that

$$(4.10) \quad \|\tilde{c}_n\|_{L^2(0,T;H^1(\Omega))} \leq C_7.$$

The elliptic estimates (4.9) yield an analogous result to Corollary 3.2.

Corollary 4.2. *Let (2.17)–(2.23) hold. Then*

- *There exists C_8 , not depending on n , such that*

$$(4.11) \quad \|\mathbf{D}^{1/2}(\cdot, \tilde{\mathbf{u}}_n) \nabla \tilde{c}_n\|_{L^2(0,T;L^2(\Omega)^d)} \leq C_8.$$

- *For any $1 \leq r < 2d/(2d-1)$ there exists C_9 , not depending upon n , such that*

$$(4.12) \quad \|\mathbf{D}(\cdot, \tilde{\mathbf{u}}_n) \nabla \tilde{c}_n\|_{L^2(0,T;L^r(\Omega)^d)} \leq C_9.$$

Proof. Estimate (4.11) follows from (3.13), (3.19), (4.5) and (4.7).

To prove (4.12), note that for any $1 \leq q < d/(d-1)$, Estimate (4.9) gives C_{10} not depending upon n such that

$$\|\tilde{\mathbf{u}}_n\|_{L^\infty(0,T;L^{2q}(\Omega))} \leq C_{10}.$$

It follows from the growth estimate (3.15) that $\|\mathbf{D}^{1/2}(\cdot, \tilde{\mathbf{u}}_n)\|_{L^\infty(0,T;L^{2q}(\Omega)^{d \times d})}$ is bounded uniformly with respect to n . The Hölder inequality then gives

$$\begin{aligned} & \|\mathbf{D}(\cdot, \tilde{\mathbf{u}}_n) \nabla \tilde{c}_n\|_{L^2(0,T;L^r(\Omega)^d)} \\ & \leq \|\mathbf{D}^{1/2}(\cdot, \tilde{\mathbf{u}}_n)\|_{L^\infty(0,T;L^{2q}(\Omega)^{d \times d})} \|\mathbf{D}^{1/2}(\cdot, \tilde{\mathbf{u}}_n) \nabla \tilde{c}_n\|_{L^2(0,T;L^2(\Omega)^d)}, \end{aligned}$$

where r is chosen so that $\frac{1}{r} = \frac{1}{2q} + \frac{1}{2}$. Since q can be any number in $[1, \frac{d}{d-1})$, r can be arbitrarily chosen in $[1, \frac{2d}{2d-1})$ and the proof is complete. \square

Finally, analogous to Proposition 3.3, the following estimate is necessary to utilise compactness results detailed in the next section.

Proposition 4.3. *Let (2.17)–(2.23) hold. Then for any $s > 2d$ there exists C_{11} not depending upon n such that*

$$(4.13) \quad \|\Phi \partial_t \tilde{c}_n\|_{L^2(0,T;(W^{1,s}(\Omega))')} \leq C_{11}.$$

Proof. Let $s > 2d$ and $\varphi \in L^2(0, T; W^{1,s}(\Omega))$. From (2.22),

$$\begin{aligned} & \left| \int_0^T \langle \Phi \partial_t \tilde{c}_n(\cdot, t), \varphi(\cdot, t) \rangle_{(W^{1,s})', W^{1,s}} dt \right| \\ & \leq \| \mathbf{D}(\cdot, \tilde{\mathbf{u}}_n) \nabla \tilde{c}_n \|_{L^2(0, T; L^r(\Omega)^d)} \| \nabla \varphi \|_{L^2(0, T; L^{r'}(\Omega)^d)} \\ & \quad + \| \tilde{\mathbf{u}}_n \|_{L^\infty(0, T; L^r(\Omega)^d)} \| \nabla \varphi \|_{L^1(0, T; L^{r'}(\Omega)^d)} \\ & \quad + \| \varphi \|_{L^1(0, T; L^\infty(\Omega))} \| \nu_n \|_{L^1(\Omega)} \left(\| a_n \|_{L^\infty(0, T; C(\bar{\Omega}))} + \| b \|_{L^\infty(0, T; C(\bar{\Omega}))} \right), \end{aligned}$$

where r is chosen such that $r' = s$, which imposes $r = s' < \frac{2d}{2d-1} < \frac{d}{d-1}$. Using (4.5), (4.7), (4.9), (4.12) and the embedding of $W^{1,s}(\Omega)$ into $L^\infty(\Omega)$, we infer that

$$\left| \int_0^T \langle \Phi \partial_t \tilde{c}_n(\cdot, t), \varphi(\cdot, t) \rangle_{(W^{1,s})', W^{1,s}} dt \right| \leq C_{12} \| \varphi \|_{L^2(0, T; W^{1,s}(\Omega))},$$

where C_{12} does not depend on n , hence the result. \square

4.3. Passing to the limit. Analogously to Section 3.3, from (2.17), (4.10), (4.13) and once again using the Aubin compactness theorem, one may assume that, up to a subsequence,

$$(4.14) \quad \tilde{c}_n \rightarrow c \quad \text{weakly in } L^2(0, T; H^1(\Omega)),$$

$$(4.15) \quad \tilde{c}_n \rightarrow c \quad \text{in } L^\infty(\Omega \times (0, T)) \text{ weak-}^*, \quad 0 \leq c \leq 1 \text{ a.e. in } \Omega \times (0, T),$$

$$(4.16) \quad \Phi \partial_t \tilde{c}_n \rightarrow \Phi \partial_t c \quad \text{weakly in } L^2(0, T; (W^{1,s}(\Omega))') \quad \forall s > 2d,$$

$$(4.17) \quad \Phi \tilde{c}_n \rightarrow \Phi c \quad \text{in } L^2(0, T; L^2(\Omega)), \text{ and}$$

$$(4.18) \quad \Phi \tilde{c}_n \rightarrow \Phi c \quad \text{in } C([0, T]; (W^{1,s}(\Omega))') \quad \forall s > 2d.$$

From (4.16) we deduce (2.11). Properties (4.18) and (2.19) give (2.12). As before, from (4.17) we infer that, up to another subsequence,

$$(4.19) \quad \tilde{c}_n \rightarrow c \quad \text{in } L^2(0, T; L^2(\Omega)) \text{ and a.e. on } \Omega \times (0, T).$$

For the elliptic equation, from Estimates (4.9) we have, up to a subsequence,

$$(4.20) \quad \tilde{p}_n \rightarrow p \quad \text{in } L^\infty(0, T; W^{1,q}(\Omega)) \text{ weak-}^* \quad \forall 1 \leq q < d/(d-1), \text{ and}$$

$$(4.21) \quad \tilde{\mathbf{u}}_n \rightarrow \mathbf{u} \quad \text{in } L^\infty(0, T; L^q(\Omega)^d) \text{ weak-}^* \quad \forall 1 \leq q < d/(d-1).$$

Taking $\psi \in L^1(0, T; W^{1,q}(\Omega))$ for some $q > d$, it is straightforward to pass to the limit on (2.23) using (4.19), (4.20) and (4.21). For the right-hand side of (2.23), note that for $q > d$, $W^{1,q}(\Omega) \subset C(\bar{\Omega})$ and $(a_n(\cdot, t) - b(\cdot, t))\nu_n \rightarrow (a(\cdot, t) - b(\cdot, t))\nu$ in $(C(\bar{\Omega}))'$ weak- $*$ (see below) and use the dominated convergence theorem in time. This proves (2.14) and (2.16).

We now invoke the stability result Proposition 4.1 to obtain stronger convergence of the pressure and Darcy velocity, which is necessary to pass to the limit on the parabolic equation (2.22).

Let $t \in (0, T)$ such that $\tilde{c}_n(\cdot, t) \rightarrow c(\cdot, t)$ a.e. on Ω (almost every t satisfies this by (4.19)). We set

$$\begin{aligned} A_n(x) &:= \frac{\mathbf{K}(x)}{\mu(\tilde{c}_n(x, t))}, & F_n(x) &:= \rho(\tilde{c}_n(x, t))\mathbf{g}, \\ A(x) &:= \frac{\mathbf{K}(x)}{\mu(c(x, t))}, & F(x) &:= \rho(c(x, t))\mathbf{g}. \end{aligned}$$

Then from (2.2) and (2.4) it is clear that A_n is uniformly bounded and elliptic. By (4.19) and the continuity of μ and ρ on the unit interval, $A_n(\cdot) \rightarrow A(\cdot)$ a.e. on Ω and $F_n(\cdot) \rightarrow F(\cdot)$ in $L^2(\Omega)$ as $n \rightarrow \infty$, so that the first two hypotheses of Proposition 4.1 are satisfied. One also requires that

$$(a_n(\cdot, t) - b(\cdot, t))\nu_n \rightarrow (a(\cdot, t) - b(\cdot, t))\nu \text{ in } (C(\bar{\Omega}))' \text{ weak-},$$

which is established in [10]. Then Proposition 4.1 shows that $\tilde{p}_n(\cdot, t) \rightarrow p(\cdot, t)$ in $W^{1,q}(\Omega)$ for all $q < \frac{d}{d-1}$. In particular, for a.e. $t \in (0, T)$,

$$(4.22) \quad \nabla \tilde{p}_n(\cdot, t) \rightarrow \nabla p(\cdot, t) \text{ and } \tilde{\mathbf{u}}_n(\cdot, t) \rightarrow \mathbf{u}(\cdot, t) \text{ in } L^q(\Omega)^d \text{ for all } 1 \leq q < \frac{d}{d-1},$$

without the extraction of subsequences after (4.20), (4.21). From (4.9) we have, for a.e. $t \in (0, T)$, $\|\tilde{\mathbf{u}}_n(\cdot, t)\|_{L^q(\Omega)} \leq C_6 \in L^p(0, T)$ for any $p < \infty$. Dominated convergence in $L^p(0, T; L^q(\Omega)^d)$ therefore gives

$$(4.23) \quad \tilde{\mathbf{u}}_n \rightarrow \mathbf{u} \text{ in } L^p(0, T; L^q(\Omega)^d), \quad \forall p < \infty, \quad \forall q < \frac{d}{d-1}.$$

This strong convergence of the Darcy velocity is critical to addressing in particular the convergence of the diffusion-dispersion term. To this end, Corollary A.2 with $H_n = \mathbf{D}^{1/2}$ gives

$$(4.24) \quad \mathbf{D}^{1/2}(\cdot, \tilde{\mathbf{u}}_n) \rightarrow \mathbf{D}^{1/2}(\cdot, \mathbf{u}) \text{ in } L^p(0, T; L^{2q}(\Omega)^{d \times d}), \quad \forall p < \infty, \quad \forall q < \frac{d}{d-1},$$

which, together with (4.14) and Estimate (4.11), allow us to apply Lemma A.3 with $v_n = \nabla \tilde{c}_n$, $w_n = \mathbf{D}^{1/2}(\cdot, \tilde{\mathbf{u}}_n)$ and $a = b = r_1 = r_2 = s_2 = 2$, $s_1 = 2q \geq 2$. We obtain

$$(4.25) \quad \mathbf{D}^{1/2}(\cdot, \tilde{\mathbf{u}}_n) \nabla \tilde{c}_n \rightarrow \mathbf{D}^{1/2}(\cdot, \mathbf{u}) \nabla c \text{ weakly in } L^2(0, T; L^2(\Omega)^d).$$

Using (4.24), (4.25) and Estimate (4.12) we apply Lemma A.3 again with $w_n = \mathbf{D}^{1/2}(\cdot, \tilde{\mathbf{u}}_n)$, $v_n = \mathbf{D}^{1/2}(\cdot, \tilde{\mathbf{u}}_n) \nabla \tilde{c}_n$ and $a = r_1 = r_2 = s_2 = 2$, $s_1 = 2q \geq 2$ and $b = r$ which yields

$$(4.26) \quad \mathbf{D}(\cdot, \tilde{\mathbf{u}}_n) \nabla \tilde{c}_n \rightarrow \mathbf{D}(\cdot, \mathbf{u}) \nabla c \text{ weakly in } L^2(0, T; L^r(\Omega)^d), \quad \forall r < \frac{2d}{2d-1},$$

and proves in particular (2.13).

Remark 4.3. This splitting technique for obtaining convergence of the diffusion-dispersion term fills an apparent gap in the proof of [6, Lemma 8] or in the passing to the limit in [6, Eq. (4.6)]. Using our notation, it seems to us that the weak convergence of $\mathbf{D}(\cdot, \tilde{\mathbf{u}}_n) \nabla \tilde{c}_n$ and $\tilde{\mathbf{u}}_n \cdot \nabla \tilde{c}_n$ are justified in this reference by simply invoking the weak convergence of $\nabla \tilde{c}_n$ in $L^2(\Omega \times (0, T))$ and the strong convergence of $\tilde{\mathbf{u}}_n$ in $L^q(\Omega \times (0, T))$ for $q < \frac{d+2}{d+1}$. However, since $\frac{d+2}{d+1} < 2$, a direct weak-strong convergence argument cannot be applied and so more sophisticated reasoning is necessary. The argument we propose above (the splitting of $\mathbf{D}(\cdot, \tilde{\mathbf{u}}_n)$ into $\mathbf{D}(\cdot, \tilde{\mathbf{u}}_n)^{1/2} \mathbf{D}(\cdot, \tilde{\mathbf{u}}_n)^{1/2}$ and the double application of Lemma A.3) can be adapted to the proof of the convergence of $\tilde{\mathbf{u}}_n \cdot \nabla \tilde{c}_n$ by writing

$$\tilde{\mathbf{u}}_n \cdot \nabla \tilde{c}_n = (1 + |\tilde{\mathbf{u}}_n|^{1/2}) \frac{\tilde{\mathbf{u}}_n}{1 + |\tilde{\mathbf{u}}_n|^{1/2}} \cdot \nabla \tilde{c}_n$$

and by using Lemma A.3 twice along with the fact that $(\frac{\tilde{\mathbf{u}}_n}{1 + |\tilde{\mathbf{u}}_n|^{1/2}} \cdot \nabla \tilde{c}_n)_{n \in \mathbb{N}}$ is bounded in $L^2(\Omega \times (0, T))$.

Suppose now that $\varphi \in L^2(0, T; W^{1,s}(\Omega))$ for some $s > 2d$. From (2.22),

$$\begin{aligned} & \int_0^T \langle \Phi \partial_t \tilde{c}_n(\cdot, t), \varphi(\cdot, t) \rangle_{(W^{1,4})', W^{1,4}} dt \\ & + \int_0^T \int_{\Omega} \mathbf{D}(x, \tilde{\mathbf{u}}_n(x, t)) \nabla \tilde{c}_n(x, t) \cdot \nabla \varphi(x, t) dx dt \\ & - \int_0^T \int_{\Omega} \tilde{\mathbf{u}}_n(x, t) \cdot \nabla \varphi(x, t) \tilde{c}_n(x, t) dx dt \\ & + \int_0^T \int_{\Omega} \tilde{c}_n(x, t) \varphi(x, t) b(x, t) \nu_n(x) dx dt \\ & = \int_0^T \int_{\Omega} \hat{c}(x, t) \varphi(x, t) a_n(x, t) \nu_n(x) dx dt, \end{aligned}$$

which we write as

$$T_1 + T_2 + T_3 + T_4 = T_5.$$

We pass to the limit on T_1 using (4.16) and on T_2 using (4.26). For T_3 , we use (4.15) and $\tilde{\mathbf{u}}_n \cdot \nabla \varphi \rightarrow \mathbf{u} \cdot \nabla \varphi$ in $L^1(0, T; L^1(\Omega))$, which follows from (4.23) and the choice of test function. We handle T_5 in a similar manner to the source terms in the elliptic equation using dominated convergence in time.

It remains to pass to the limit on T_4 . Note that by the preceding arguments, T_4 has a limit as $n \rightarrow \infty$. To identify this limit, we refer the reader to [10, Lemma 5.1], where the authors demonstrate that we can specify c on null sets with respect to the Lebesgue measure in such a way that the last two lines of (2.10) are satisfied. This proves (2.15) and thus concludes the proof of Theorem 2.2.

APPENDIX A. SOME USEFUL LEMMAS

We present here the technical lemmas that we use to pass to the limit on the diffusion-dispersion term.

Lemma A.1. *Let Ω be a bounded subset of \mathbb{R}^N , $N \in \mathbb{N}$ and for each $n \in \mathbb{N}$, let $H_n : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ be a Carathéodory function such that*

- *there exist positive constants C_{13}, γ such that for a.e. $x \in \Omega$,*

$$(A.1) \quad |H_n(x, \xi)| \leq C_{13}(1 + |\xi|^\gamma), \quad \forall \xi \in \mathbb{R}^N, \forall n \in \mathbb{N},$$

- *there is a Carathéodory function $H : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ such that for a.e. $x \in \Omega$,*

$$(A.2) \quad H_n(x, \cdot) \rightarrow H(x, \cdot) \quad \text{uniformly on compact sets as } n \rightarrow \infty.$$

If $p \in [\gamma, \infty)$ and $(u_n)_{n \in \mathbb{N}} \subset L^p(\Omega)^N$ is a sequence with $u_n \rightarrow u$ in $L^p(\Omega)^N$ as $n \rightarrow \infty$, then

$$H_n(\cdot, u_n) \rightarrow H(\cdot, u) \quad \text{in } L^{p/\gamma}(\Omega) \text{ as } n \rightarrow \infty.$$

Proof. First note that if $v \in L^p(\Omega)^N$, then $H_n(\cdot, v) \in L^{p/\gamma}(\Omega)$, since by (A.1) there is C_{14} such that

$$\|H_n(\cdot, v)\|_{L^{p/\gamma}(\Omega)} \leq C_{14}(\text{meas}(\Omega)^{\gamma/p} + \|v\|_{L^p(\Omega)^N}^\gamma).$$

Suppose that $H_n(\cdot, u_n)$ does not converge to $H(\cdot, u)$ in $L^{p/\gamma}(\Omega)$. Then there is $\varepsilon > 0$ and a subsequence of $(H_n(\cdot, u_n))_{n \in \mathbb{N}}$, still denoted by $(H_n(\cdot, u_n))_{n \in \mathbb{N}}$, with

$$(A.3) \quad \|H_n(\cdot, u_n) - H(\cdot, u)\|_{L^{p/\gamma}(\Omega)} \geq \varepsilon.$$

Since $u_n \rightarrow u$ in $L^p(\Omega)^N$ as $n \rightarrow \infty$, the partial converse to the dominated convergence theorem establishes the existence of a subsequence of $(u_n)_{n \in \mathbb{N}}$, again still

denoted by $(u_n)_{n \in \mathbb{N}}$, and a function $v \in L^p(\Omega)^N$ with $u_n(x) \rightarrow u(x)$ for a.e. $x \in \Omega$, and $|u_n(x)| \leq v(x)$ for a.e. $x \in \Omega$. Now, H_n is Carathéodory, so together with (A.2) it follows that $H_n(x, u_n(x)) \rightarrow H(x, u(x))$ for a.e. $x \in \Omega$. Furthermore,

$$|H_n(x, u_n(x))| \leq C_{13}(1 + (v(x))^\gamma) \quad \text{for a.e. } x \in \Omega,$$

so the dominated convergence theorem gives

$$H_n(\cdot, u_n) \rightarrow H(\cdot, u) \quad \text{in } L^{p/\gamma}(\Omega),$$

contrary to (A.3). \square

Corollary A.2. *Let Ω , H_n and H satisfy the hypotheses of Lemma A.1. If $p, q \in [\gamma, \infty)$ and $(u_n)_{n \in \mathbb{N}} \subset L^p(0, T; L^q(\Omega)^N)$ is a sequence with*

$$u_n \rightarrow u \text{ in } L^p(0, T; L^q(\Omega)^N) \text{ as } n \rightarrow \infty, \text{ then}$$

$$H_n(\cdot, u_n) \rightarrow H(\cdot, u) \quad \text{in } L^{p/\gamma}(0, T; L^{q/\gamma}(\Omega)) \text{ as } n \rightarrow \infty.$$

Proof. Since $u_n \rightarrow u$ in $L^p(0, T; L^q(\Omega)^N)$, the partial converse to the dominated convergence theorem in $L^p(0, T; L^q(\Omega)^N)$ establishes the existence of a subsequence $(u_n)_{n \in \mathbb{N}}$, still denoted by $(u_n)_{n \in \mathbb{N}}$, and a function $v \in L^p(0, T)$ such that for a.e. $t \in (0, T)$,

$$u_n(\cdot, t) \rightarrow u(\cdot, t) \quad \text{in } L^q(\Omega)^N, \text{ and } \|u_n(\cdot, t)\|_{L^q(\Omega)^N} \leq v(t).$$

Now, by Lemma A.1, for a.e. $t \in (0, T)$,

$$H_n(\cdot, u_n(\cdot, t)) \rightarrow H(\cdot, u(\cdot, t)) \quad \text{in } L^{q/\gamma}(\Omega).$$

Furthermore,

$$\|H_n(\cdot, u_n(\cdot, t))\|_{L^{q/\gamma}(\Omega)} \leq C_{13}(1 + (v(t))^\gamma) \quad \text{for a.e. } t \in (0, T),$$

so the dominated convergence theorem gives

$$H_n(\cdot, u_n) \rightarrow H(\cdot, u) \quad \text{in } L^{p/\gamma}(0, T; L^{q/\gamma}(\Omega)),$$

which concludes the proof. \square

Lemma A.3. *Let Ω be a bounded, open subset of \mathbb{R}^N and for each $n \in \mathbb{N}$, let $w_n : \Omega \times (0, T) \rightarrow \mathbb{R}$ and $v_n : \Omega \times (0, T) \rightarrow \mathbb{R}$ be such that*

$$w_n \rightarrow w \quad \text{in } L^{r_1}(0, T; L^{s_1}(\Omega)), \text{ and}$$

$$v_n \rightarrow v \quad \text{weakly in } L^{r_2}(0, T; L^{s_2}(\Omega)),$$

where $r_1, r_2, s_1, s_2 \geq 1$ are such that $1/r_1 + 1/r_2 \leq 1$ and $1/s_1 + 1/s_2 \leq 1$. Suppose also that the sequence $(w_n v_n)_{n \in \mathbb{N}}$ is bounded in $L^a(0, T; L^b(\Omega))$, where $a, b \in (1, \infty)$. Then

$$w_n v_n \rightarrow wv \quad \text{weakly in } L^a(0, T; L^b(\Omega)).$$

Proof. The assumption that $(w_n v_n)_{n \in \mathbb{N}}$ is bounded in $L^a(0, T; L^b(\Omega))$ yields the existence of $\chi \in L^a(0, T; L^b(\Omega))$ such that, up to a subsequence,

$$w_n v_n \rightarrow \chi \quad \text{weakly in } L^a(0, T; L^b(\Omega)),$$

which implies that $w_n v_n \rightarrow \chi$ in the sense of distributions on $\Omega \times (0, T)$. From the other hypotheses, $L^{r_1}(0, T; L^{s_1}(\Omega)) \hookrightarrow L^{r_2}(0, T; L^{s_2}(\Omega))'$, so using weak-strong convergence we have $w_n v_n \rightarrow wv$ in the sense of distributions on $\Omega \times (0, T)$, hence $\chi = wv$. \square

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