Hyperbolic knot theory

Jessica S. Purcell

Preface

This book is meant to be a hands-on introduction to major areas of research in hyperbolic 3-manifolds, motivated and grounded by way of examples of knots in S^3. I find knots to be a wonderful playground in which to explore hyperbolic geometry and its invariants. Moreover, there are still many open questions about the relationship of hyperbolic geometry and knots and knot invariants. For example, it is unknown in general how hyperbolic geometry affects things like knot homologies, polynomials and quantum invariants, and many diagrammatic invariants, and there are several important open questions in these areas. This book serves as an introduction to the hyperbolic geometry side of such questions.

Introduction to the geometry of knots

Knots have been studied mathematically since approximately 1833, when Gauss developed the linking number of two knots. Peter Tait was one of the first to try to classify knots up to equivalence. He created the first knot tables, listing knots up to seven crossings. See, for example, the article by Silver [**Silver, 2006**].

The study of the geometry of knots, particularly hyperbolic geometry, began with work of Robert Riley in the 1970s [**Riley, 1975**], and developed further in the late 1970s and early 1980s, with work of William Thurston.

In the 1980s, Thurston conjectured that every 3-manifold decomposes along spheres and incompressible tori into pieces that admit uniquely one of eight 3-dimensional geometries (geometric structures) [**Thurston, 1982**]. This was his geometrization conjecture, and the proof of the full conjecture was announced in 2003 by Perelman. However, Thurston proved the conjecture for certain classes of manifolds, including manifolds with boundary, and knot complements, in the early 1980s. Thus it has been known for many years that knot complements in S^3 decompose into geometric pieces.

What are the geometric pieces? Peter Scott has written an excellent introduction to the eight 3-dimensional geometries [**Scott, 1983**]. Seven of the eight 3-dimensional geometries will not concern us here, but they

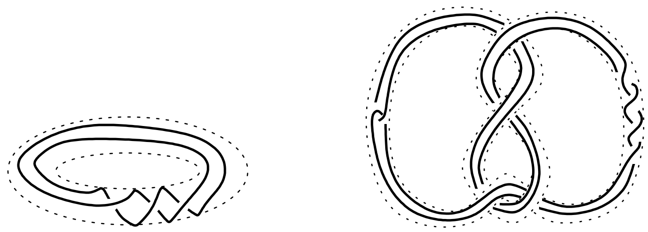

FIGURE 0.1. Left: a torus knot. Right: a satellite knot.

include the so-called Seifert fibered geometries. The last and most important geometry is hyperbolic.

Thurston showed that a knot complement will either be Seifert fibered, hyperbolic, or toroidal, meaning it contains an embedded torus that is essential (roughly, it cannot be pushed to the boundary or replaced with a simplified surface). If the knot is toroidal, to obtain its geometric pieces one must cut along the incompressible torus and consider each resulting piece separately. We know exactly which knot complements are Seifert fibered, toroidal, or hyperbolic, again due to work of Thurston in the 1980s: The knots whose complement can be Seifert fibered consist of *torus knots*: knots which can be drawn on the surface of a torus, as in figure 0.1. Toroidal knot complements are exactly the *satellite knots*: knots which can be drawn inside the complement of a (possibly knotted) solid torus, as on the right in figure 0.1. All other knots are hyperbolic.

Hyperbolic knots form the largest and least understood class of knots. Of all prime knots up to 16 crossings, classified by Hoste, Thistlethwaite, and Weeks [**Hoste et al., 1998**], 13 are torus knots, 20 are satellite knots, and the remaining 1,701,903 are hyperbolic.

By the rigidity theorem of Mostow and Prasad, if a knot complement admits a hyperbolic structure, then that structure is unique [**Mostow, 1973, Prasad, 1973**]. More carefully, Mostow showed that if there was an isomorphism between the fundamental groups of two closed hyperbolic 3-manifolds, then there was an isometry taking one to the other. Prasad extended this work to 3-manifolds with torus boundary, including knot complements. Thus if two hyperbolic knot complements have isomorphic fundamental group, then they have exactly the same hyperbolic structure. Finally, Gordon and Luecke showed that two knot complements with the same fundamental group are equivalent [**Gordon and Luecke, 1989**] (up to mirror reflection).

Thus a hyperbolic structure on a knot complement is a complete invariant of the knot. If we could completely understand hyperbolic structures on knot complements, we could completely classify hyperbolic knots. That is the main motivation for this book.

Prerequisites and notes to students

I have tried to keep prerequisites to a minimum. A basic course in topology is required, as well as some knowledge of basic algebraic topology, particularly fundamental group and covering spaces. Occasionally, experience with Riemannian geometry will also be helpful, but not required. We also occasionally assume basic results in differential topology, such as the fact that smooth manifolds admit tubular neighborhoods, and that submanifolds can be isotoped to meet transversely. If you are happy to accept these results you have the required prerequisites from differential topology.

Also, this book is meant to be interactive — that is why I have included examples and exercises. I hope you will work through the examples as they are presented, and generalize them in exercises. Many important results are saved for the exercises.