CHAPTER 1

Decomposition of the figure-8 knot

This book is an introduction to knots, links, and their geometry. Before we begin, we need to define carefully exactly what we mean by knots and links, and that is done in this chapter in the first section. More importantly, we also need to develop tools to work with knots and links and the 3-manifolds they define, namely knot and link complements. After covering basic definitions, this chapter gives a geometric method, explained carefully by example, to decompose a knot or link complement into simple pieces. The methods here are an introduction to topological techniques in 3-manifold geometry and topology, and an introduction to some of the tools used in the field.

1.1. Basic terminology

DEFINITION 1.1. A *knot* $K \subset S^3$ is a subset of points homeomorphic to a circle S^1 under a piecewise linear (PL) homeomorphism. We may also think of a knot as a PL embedding $K\colon S^1 \to S^3$. We will use the same symbol K to refer to the map and its image $K(S^1)$.

More generally, a *link* is a subset of S^3 PL homeomorphic to a disjoint union of copies of S^1. Alternately, we may think of a link as a PL embedding of a disjoint union of copies of S^1 into S^3.

A piecewise linear homeomorphism of S^1 is one that takes S^1 to a finite number of linear segments. Restricting to such homeomorphisms allows us to assume that a knot $K \subset S^3$ has a regular tubular neighborhood, that is there is an embedding of a solid torus $S^1 \times D^2$ into S^3 such that $S^1 \times \{0\}$ maps to K. An embedding of S^1 into S^3 that cannot be made piecewise linear defines an object called a *wild knot*. Wild knots may have very interesting geometry, but we will only be concerned with the classical knots of definition 1.1 here.

DEFINITION 1.2. We will say that two knots (or links) K_1 and K_2 are equivalent if they are *ambient isotopic*, that is, if there is a (PL) homotopy $h\colon S^3 \times [0,1] \to S^3$ such that $h(*,t) = h_t\colon S^3 \to S^3$ is a homeomorphism for each t, and $h(K_1, 0) = h_0(K_1) = K_1$ and $h(K_1, 1) = h_1(K_1) = K_2$.

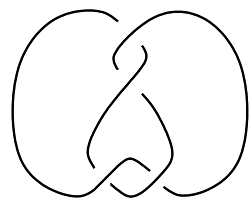

FIGURE 1.1. A diagram of the figure-8 knot

A PL embedding of S^1 into S^3 defines a 3-manifold, the *knot complement* (or link complement).

DEFINITION 1.3. For a knot K, let $N(K)$ denote an open regular neighborhood of K in S^3. The *knot complement* is the manifold $S^3 - N(K)$. Notice that it is a compact 3-manifold with boundary homeomorphic to a torus.

For our applications, we will typically be interested in the interior of the manifold $S^3 - N(K)$. We will denote this manifold by $S^3 - K$, and usually (somewhat sloppily) just refer to this manifold as the knot complement.

DEFINITION 1.4. A *knot invariant* (or link invariant) is a function from the set of links to some other set whose value depends only on the equivalence class of the link.

Knot and link invariants are used to prove that two knots or links are distinct, or to measure the complexity of the link in various ways.

Eventually, we will put a hyperbolic structure on (the interior of) our knot complements. We will define a hyperbolic structure on a manifold more carefully in a later chapter. For now, if a manifold has a hyperbolic structure, then we can measure geometric information about the manifold, including lengths of geodesics, volume, minimal surfaces, etc.

Classically, a knot is described by a diagram.

DEFINITION 1.5. A *knot diagram* is a 4-valent graph with over/under crossing information at each vertex. The diagram is embedded in a plane $S^2 \subset S^3$ called the *projection plane*, or plane of projection. Figure 1.1 shows a diagram of the figure-8 knot.

1.2. Polyhedra

Sometimes it is easier to study manifolds, including knot complements, if we split them into smaller, simpler pieces, for example 3-balls. We are going to decompose the figure-8 knot complement into two carefully marked 3-balls, namely ideal polyhedra. This decomposition appears in the notes [**Thurston, 1979**], and with a little more explanation in [**Thurston, 1997**]. The procedure was generalized to all link complements in [**Menasco, 1983**]. This work is essentially what we present below in the text and in exercises.

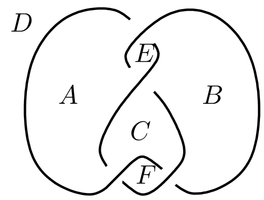

FIGURE 1.2. Faces for the figure-8 knot complement.

DEFINITION 1.6. A *polyhedron* is a closed 3-ball whose boundary is labeled with a finite graph, containing a finite number vertices and edges, so that complementary regions, i.e. faces, are simply connected faces.

An *ideal polyhedron* is a polyhedron with all vertices removed. That is, to form an ideal polyhedron, start with a regular polyhedron and remove the points corresponding to vertices.

We will cut $S^3 - K$ into two ideal polyhedra. We will then have a description of $S^3 - K$ as a gluing of two ideal polyhedra. That is, given a description of the polyhedra, and gluing information on the faces of the polyhedra, we may reconstruct the knot complement $S^3 - K$.

The example we will walk through is that of the figure-8 knot complement. We will see that this particular knot complement has many nice properties.

In the exercises, you will be asked to walk through the techniques below to determine decompositions of other knot complements into ideal polyhedra.

1.2.1. Overview. Start with a diagram of the knot. There will be two polyhedra in our decomposition. These can be visualized as two balloons: One balloon expands above the diagram, and one balloon expands below the diagram. As the balloons continue expanding, they will bump into each other in the regions cut out by the graph of the diagram. Label these regions. In figure 1.2, the regions are labeled A, B, C, D, E, and F. These will correspond to faces of the polyhedra.

The faces meet up in edges. There is one edge for each crossing. It runs vertically from the knot at the top of the crossing to the knot at the bottom (or the other way around). The balloon expands until faces meet at edges. Figure 1.3 shows how the top balloon would expand at a crossing. The edge is drawn as an arrow from the top of the crossing to the bottom. Faces labeled T and U meet across the edge. Rotating the picture 180° about the edge, we would see an identical picture with S meeting V.

It may be helpful to examine the meeting of faces at an edge by 3-dimensional model. Henry Segerman has come up with a paper model to illustrate of the phenomenon of figure 1.3. Start with a sheet of paper labeled as in figure 1.4. Cut out the shaded square in the middle. Now fold

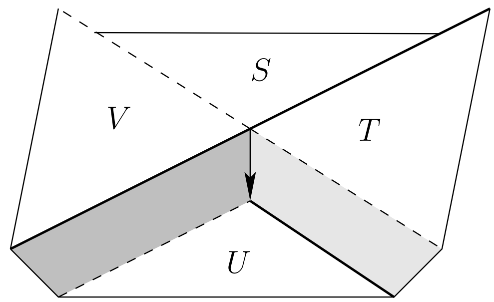

FIGURE 1.3. The knot runs along diagonals. Faces labeled U and T meet at the edge shown, marked by an arrow

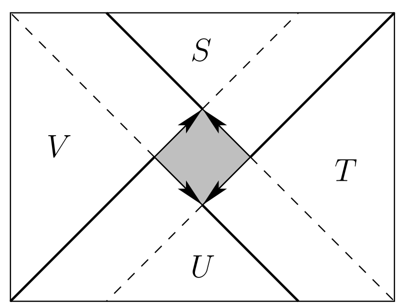

FIGURE 1.4. Cut out the shaded square. Start with a pair of parallel lines. Fold the thick part of the line in a direction opposite that of the dashed part of the line. Fold parallel thick and dashed lines in opposite directions. Correct folding results in a model that looks like figure 1.3.

the paper until it looks like that in figure 1.3. By rotating the paper model, we can see how faces meet up.

Stringing crossings such as this one together, we obtain the complete polyhedral decomposition of the knot. This is the geometric intuition behind the polyhedral expansion. We now explain a combinatorial method to describe the polyhedra.

1.2.2. Step 1. Sketch faces and edges into the diagram.

Recall a diagram is a 4-valent graph lying on a plane, the plane of projection. The regions on the plane of projection that are cut out by the graph will be the faces, including the outermost unbounded region of the plane of projection. We start by labeling those, in figure 1.2.

Edges come from arcs that connect the two strands of the diagram at a crossing. These are called *crossing arcs*. For ease of explanation, we are going to draw each edge four times, as follows. Shown on the left of figure 1.5 is a single edge corresponding to a crossing arc. Note that the

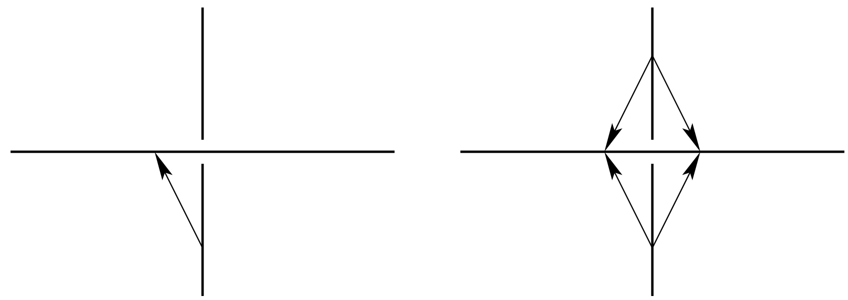

FIGURE 1.5. A single edge.

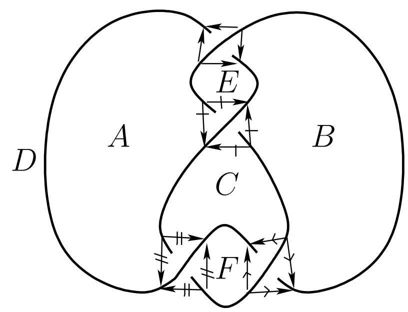

FIGURE 1.6. Edges of the figure-8 knot

edge is ambient isotopic in S^3 to the three additional edges shown on the right in figure 1.5.

The reason for sketching each edge four times is that it allows us to visualize easily which edges bound the faces we have already labeled. In figure 1.6, we have drawn four copies of each of the four edges we get from crossing arcs of the diagram. Note that the face labeled A, for example, will be bordered by three edges, one with two tick marks, one with a single tick mark, and one with no tick marks.

REMARK 1.7. The orientations on the edges can be chosen to run in either direction; that is, arrows on the edges can run from overcrossing to undercrossing or vice versa, as long as we are consistent with orientations corresponding to the same edge. We have chosen the orientations in figure 1.6 to simplify a later step, and to match a figure in chapter 4. The opposite choice for any edge is also fine.

1.2.3. Step 2. Shrink the knot to ideal vertices on the top polyhedron.

Now we come to the reason for using *ideal* polyhedra, rather than regular polyhedra. Notice that the edges stretch from a part of the knot to a part of the knot. However, the manifold we are trying to model is the knot complement, $S^3 - K$. Therefore, the knot K does not exist in the manifold.

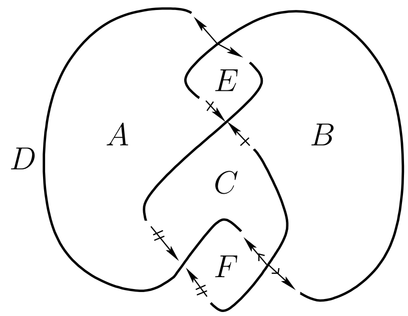

FIGURE 1.7. Isotopic edges in top polyhedron identified.

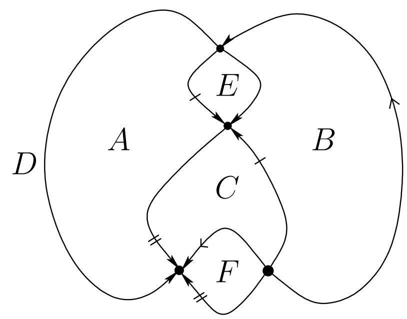

FIGURE 1.8. Top polyhedron, viewed from the inside.

An edge with its two vertices on K must necessarily be an ideal edge, i.e. its vertices are not contained in the manifold $S^3 - K$.

Since the knot is *not* part of the manifold, we will shrink strands of the knot to ideal vertices. Focus first on the polyhedron on top. Each component of the knot we "see" from inside the top polyhedron will be shrunk to a single ideal vertex. These visible knot components correspond to sequences of overcrossings of the diagram. Compare to figure 1.3 — note that at an undercrossing, the component of the knot ends in an edge, but at an overcrossing the knot continues on. Moreover, note that at an undercrossing, the knot runs into just one edge, but at an overcrossing the knot passes the same edge twice, once on each side.

In terms of the four copies of the edge in figure 1.5, when we consider the polyhedron on top, we may identify the two edges which are isotopic along an overstrand, but not those isotopic along understrands. See figure 1.7.

Shrink each overstrand to a single ideal vertex. The result is pattern of faces, edges, and ideal vertices for the top polyhedron, shown in figure 1.8. Notice that the face D is a disk, containing the point at infinity.

1.2.4. Step 3. Shrink the knot to ideal vertices for the bottom polyhedron.

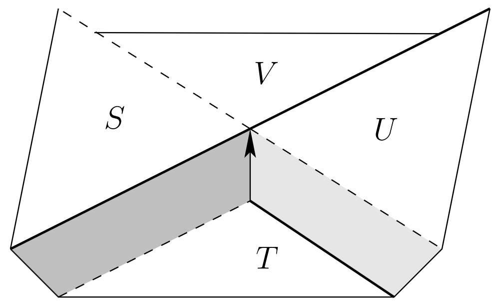

FIGURE 1.9. 3-dimensional model, opposite side as in figure 1.3.

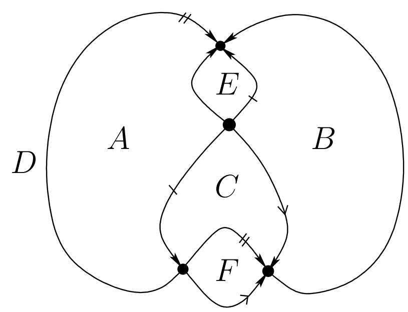

FIGURE 1.10. Bottom polyhedron, viewed from the outside.

Notice that underneath the knot, the picture of faces, edges, and vertices will be slightly different. In particular, when finding the top polyhedron, we collapsed overstrands to a single ideal vertex. When you put your head underneath the knot, what appear as overstrands from below will appear as understrands on the usual knot diagram.

The easiest way to see this difference is to take the 3-dimensional model constructed in figure 1.4. Figure 1.3 shows the view of the faces meeting at an edge from the top. If you turn the model over to the opposite side, you will see how the faces meet underneath. Figure 1.9 illustrates this. Note U now meets T, and S meets V.

In terms of the combinatorics, edges of figure 1.5 which are isotopic by sliding an endpoint along an understrand are identified to each other on the bottom polyhedron, but edges only isotopic by sliding an endpoint along an overstrand are not identified.

As above, collapse each knot strand corresponding to an understrand to a single ideal vertex. The result is figure 1.10.

One thing to notice: we sketched the top polyhedron with our heads inside the ball on top, looking out. If we move the face D away from the point at infinity, then it wraps *above* the other faces shown in figure 1.8.

(a.) Trefoil. (b.) The 5_2 knot. (c.) The 6_3 knot.

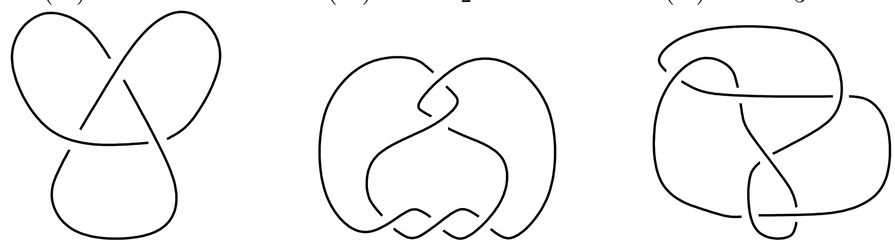

FIGURE 1.11. Three examples of knots.

On the other hand, we sketched the bottom polyhedron with our heads outside the ball on the bottom. If we move the face D away from the point at infinity, it wraps *below* the other faces shown in figure 1.10.

1.2.5. Rebuilding the knot complement from the polyhedra. Figures 1.8 and 1.10 show two ideal polyhedra that we obtained by studying the figure-8 knot complement. We claim that they glue to give the figure-8 knot complement. That is, attach face A on the bottom polyhedron to the face labeled A on the top polyhedron, ensuring that the edges bordering face A match up. Similarly for the other faces.

This process of gluing faces and edges gives exactly the complement of the knot. By construction, faces glue to give the faces illustrated in figure 1.6, and edges glue to give the edges there, except when we have finished, all four edges in an isotopy shown in that figure have been glued together.

1.3. Generalizing: Exercises

This polyhedral decomposition works for any knot or link diagram, to give a polyhedral decomposition of its complement.

EXERCISE 1.1. As a warmup exercise, determine the polyhedral decomposition for one (or more) of the knots shown in figure 1.11. Sketch both top and bottom polyhedra.

Your solution should consist of two ideal polyhedra, i.e. marked graphs on the surface of a ball, with faces and edges marked according to the gluing pattern. For example, the complete diagrams in Figures 1.8 and 1.10 form the solution for the figure-8 knot.

DEFINITION 1.8. An *alternating* diagram is one in which crossings alternate between over and under as we travel along the diagram in a fixed direction.

All the examples of knot diagrams we have encountered so far are alternating. The diagram of the knot 8_{19} in figure 1.12 is not alternating. (In fact, the knot 8_{19} has no alternating diagram.)

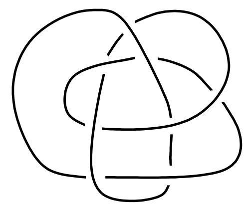

FIGURE 1.12. The knot 8_{19}, which has no alternating diagram.

EXERCISE 1.2. Determine the polyhedral decomposition for the given diagram of the knot 8_{19}. Note: as above, many ideal vertices are obtained by shrinking overstrands to a point. However, you will have to use, for example, figure 1.3 to determine what happens between two understrands.

Recall that the *valence* of a vertex in a graph is the number of edges that meet that vertex. The valence of an ideal vertex is defined similarly.

EXERCISE 1.3. (a) If a knot diagram is alternating, we obtain a very special ideal polyhedron. In particular, all ideal vertices will have the same valence. What is it? Show that the ideal vertices for an alternating knot all have this valence.

(b) What are the possible valences of ideal vertices in general, i.e. for non-alternating knots)? For which $n \geq 0 \in \mathbb{Z}$ is there a knot diagram whose polyhedral decomposition yields an ideal vertex of valence n? Explain your answer, with (portions of) knot diagrams.

EXERCISE 1.4. Note that in the polyhedral decomposition for alternating knots, the polyhedra are given by simply labeling each ball with the projection graph of the knot and declaring each vertex to be ideal. Prove this statement for any alternating knot. Show the result is false for non-alternating knots.

EXERCISE 1.5. The decomposition admits a checkerboard coloring: faces are either white or shaded, and white faces meet shaded faces across an edge. Moreover, faces are identified from top to bottom by a “gear rotation”: white faces on the top are rotated once counter-clockwise and then glued to the identical face on the bottom; shaded faces are rotated once clockwise and then glued to the identical face on the bottom. This is shown for the figure-8 knot in figure 1.13. Prove the above statement for any alternating knot.

The diagrams we have encountered so far are all reduced. We can follow the above procedure for non-reduced diagrams. For example, we can obtain a polyhedral decomposition for diagrams which contain a *simple nugatory crossing*, which we define to be as shown in figure 1.14.

EXERCISE 1.6. Show that the polyhedral decomposition will contain a monogon, i.e. a face whose boundary is a single edge and a single vertex, if and only if the diagram has a simple nugatory crossing.

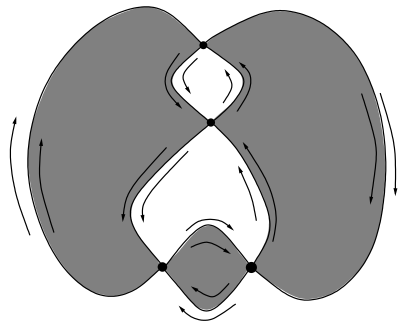

FIGURE 1.13. Checkerboard coloring and "gear rotation" for the figure-8 knot.

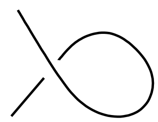

FIGURE 1.14. A simple nugatory crossing.

Let bigons be bygone. — William Menasco

DEFINITION 1.9. A *bigon* is a face of the polyhedral decomposition that has just two edges (and two ideal vertices).

Note that the two edges of a bigon face must be isotopic to each other. Hence, we sometimes will remove bigon faces from the polyhedral decomposition, identifying their two edges.

EXERCISE 1.7. For the figure-8 knot, sketch the two polyhedra we get when bigon faces are removed. How many edges are there in this new, bigon-free decomposition? The resulting polyhedra are well known solids in this case. What are they?

For each of the polyhedra obtained in exercise 1.1, sketch the resulting polyhedra with bigons removed.

EXERCISE 1.8. Suppose we start with an alternating knot diagram with at least two crossings, and do the polyhedral decomposition above, collapsing bigons at the last step. What are possible valences of vertices? Sketch the diagram of a single alternating knot that has all possible valences of ideal vertices in its polyhedral decomposition.

What valences of vertices can you get if you don't require the diagram to be alternating but collapse bigons? Can you find 1-valent vertices? For any $n > 4 \in \mathbb{Z}$, can you find n-valent vertices?