CHAPTER 2

# Calculating in hyperbolic space

We will need to manipulate objects in 2 and 3-dimensional hyperbolic space. This chapter provides a very brief introduction to the tools that will be needed in the future, the objects that will be studied (lines, triangles, tetrahedra, metric properties), and examples of calculations that will appear.

We will use terminology and calculations from standard elementary Riemannian geometry. The reader who is not as comfortable with Riemannian geometry might find it helpful to follow along in the first few chapters of an introductory Riemannian geometry text, such as do Carmo [**do Carmo, 1992**, Chapter 1]. We will not provide all the details to all the statements given. The idea is that we want to begin calculating on knot complements and other 3-manifolds immediately, without getting lost early in details. Thus our aim is to provide just enough information here to start calculating in future chapters. Many more details and results can be found in other books, including full books on hyperbolic geometry. Anderson gives a very nice introduction to 2-dimensional hyperbolic geometry [**Anderson, 2005**]. More details in all dimensions appear in Ratcliffe [**Ratcliffe, 2006**]. The book by Marden includes more on groups of isometries of hyperbolic space, including many consequences to infinite volume hyperbolic 3-manifolds [**Marden, 2007**]. An introduction to hyperbolic geometry that includes a discussion of its visualization is also given by Thurston [**Thurston, 1997**].

## 2.1. Hyperbolic geometry in dimension two

We start with hyperbolic 2-space, $\mathbb{H}^2$.

There are several models of hyperbolic space. Here, we will work with the upper half plane model. In this model, hyperbolic 2-space $\mathbb{H}^2$ is defined to be the set of points in the upper half plane:

$$\mathbb{H}^2 = \{x + i\,y \in \mathbb{C} \mid y > 0\},$$

equipped with the metric whose first fundamental form is given by

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$


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That is, start with the usual Euclidean metric on $\mathbb{R}^2$, whose first fundamental form is $dx^2 + dy^2$. To obtain the metric on the hyperbolic plane, rescale the usual Euclidean metric by $1/y$, where $y$ is height in the plane.

Note that a point in $\mathbb{H}^2$ can either be thought of as a complex number $x + i\,y \in \mathbb{C}$ or as a point $(x, y) \in \mathbb{R}^2$. Both perspectives are useful: $\mathbb{R}^2$ leads more easily to coordinates and calculations, and $\mathbb{C}$ works seamlessly with our definition of isometries below. Changing perspectives does not affect our results, so we will regularly switch between the two without comment.

Our first task is to explore the meaning of the hyperbolic metric, and how it affects measurements.

**2.1.1. Hyperbolic 2-space and Riemannian geometry.** In this subsection, we briefly review how the metric and the space $\mathbb{H}^2$ described above fits into a more general picture of Riemannian geometry, and the tools we will use from that subject to calculate. If you are not yet familiar with Riemannian geometry, feel free to skim this section, noting equations (2.1), (2.2), and (2.3). This section was primarily written for a student who has seen some Riemannian geometry, but may have difficulty applying abstract concepts of that field to the specific metric of hyperbolic geometry. In my experience, a few key equations will be enough to get started.

In Riemannian geometry, a *Riemannian metric* on a manifold $M$ is defined to be a correspondence associating to each point $p \in M$ an inner product $\langle \cdot, \cdot \rangle_p$ on the tangent space $T_pM$. This inner product gives us a way of measuring the lengths of vectors tangent to $M$ at $p$, as well as computing areas, angles between curves, etc. Recall that the *first fundamental form* is defined by $\langle v, v \rangle_p$ for $v \in T_pM$.

In our case, the Riemannian manifold we consider is $\mathbb{H}^2$, and we have natural local coordinates on the manifold given by $x + i\,y \in \mathbb{C}$, or $(x, y) \in \mathbb{R}^2$, for $y > 0$. We may use these coordinates to describe the Riemannian metric. In particular, at the point $(x, y) \in \mathbb{H}^2$, a tangent vector $v \in T_{(x,y)}\mathbb{H}^2$ can also be described by coordinates $v = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y}$, and we write it as a vector

$$v = \begin{pmatrix} v_x \\ v_y \end{pmatrix}.$$

Then the metric on $\mathbb{H}^2$ is given by a matrix

$$\langle v, w \rangle = (v_x, v_y) \begin{pmatrix} 1/y^2 & 0 \\ 0 & 1/y^2 \end{pmatrix} \begin{pmatrix} w_x \\ w_y \end{pmatrix}.$$

One of the simplest geometric measurements we can compute using the definition of the metric is the arc length of a curve. If $\gamma(t)$ is a (differentiable) curve in $\mathbb{H}^2$, for $t \in [a, b]$, then we obtain a tangent vector $\gamma'(t)$ at each point of $\gamma(t)$ in $\mathbb{H}^2$, called the velocity vector. Recall that the *arc length* of $\gamma$ for $t \in [a, b]$ is defined to be

$$|\gamma| = \int_a^b \sqrt{\langle \gamma'(s), \gamma'(s) \rangle}\, ds.$$

In the case of $\mathbb{H}^2$, we will have coordinates $\gamma(t) = (\gamma_x(t), \gamma_y(t))$, and $\gamma'(t) = (\gamma_x'(t), \gamma_y'(t))^T$. Thus the arc length will be

$$|\gamma| = \int_a^b \sqrt{(\gamma_x'(s))^2 + (\gamma_y'(s))^2}\, \frac{1}{\gamma_y(s)}\, ds. \tag{2.1}$$

We will use this formula to compute examples in the next subsection.

Another piece of geometric information we can compute with a metric is the volume of a region, which we typically call "area" in two dimensions. In the most general setting, if $R \subset M$ is contained in a coordinate neighborhood of the Riemannian manifold $M$, with coordinates $(x_1, \dots, x_n)$ and metric given by the matrix $g_{ij}$ in these coordinates, then we can compute the volume of $R$ to be

$$\operatorname{vol}(R) = \int_R d\operatorname{vol} = \int_R \sqrt{\det(g_{ij})}\, dx_1 \dots dx_n. \tag{2.2}$$

The form $d\operatorname{vol}$ is the *volume form.* Thus in our setting, with $M = \mathbb{H}^2$ and metric as above,

$$\operatorname{area}(R) = \int_R \frac{1}{y^2}\, dx\, dy. \tag{2.3}$$

**2.1.2. Computing arc lengths and areas.** Now we will use the formulas obtained above to make calculations, to better understand the hyperbolic space $\mathbb{H}^2$.

EXAMPLE 2.1. Fix a height $h > 0$, and consider first a horizontal line segment between points $(0, h) = i\, h$ and $(1, h) = 1 + i\, h$ in $\mathbb{H}^2$. We may parametrize the line segment by $\gamma(t) = (t, h)$, for $t \in [0, 1]$. Note that because $h$ is fixed, the arc length of $\gamma$ is just its usual Euclidean length rescaled by $1/h$. That is, the arc length of $\gamma$ is $|\gamma| = 1/h$. Thus when $h = 1$, the length of $\gamma$ is 1. When $h$ becomes large, the arc length becomes very small. In other words, points with the same height become very close together as their heights increase. On the other hand, as $h$ approaches 0, the length of $\gamma$ approaches infinity. In fact, points near the real line $\mathbb{R} = \{(x, 0) \in \mathbb{R}^2\}$ can be very far apart.

EXAMPLE 2.2. Consider now a vertical line between points $(x, a)$ and $(x, b)$, for $x, a, b$ fixed in $\mathbb{R}$, $0 < a < b$. Such a line can be parametrized by $\zeta(t) = (x, t)$ for $t \in [a, b]$. So $\zeta'(t) = (0, 1)$. Thus its arc length is given by

$$|\zeta| = \int_a^b \sqrt{0+1}\, \frac{1}{s}\, ds = \log\left(\frac{b}{a}\right).$$

If we set $b = 1$ and let $a$ approach 0, note that the arc length of $\zeta$ gets arbitrarily large, approaching infinity. Similarly setting $a = 1$ and letting $b$ approach infinity gives arbitrarily long lengths.

The real line $\mathbb{R} = \{(x, 0) \in \mathbb{R}^2\}$ along with the point at infinity $\infty$ play an important role in the geometry of $\mathbb{H}^2$, although these points are not contained in $\mathbb{H}^2$.

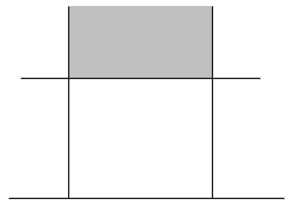

FIGURE 2.1. The region of example example 2.4

DEFINITION 2.3. We call $\mathbb{R} \cup \{\infty\}$ the *boundary at infinity* for $\mathbb{H}^2$. Note it is homeomorphic to a circle $S^1$, and hence is sometimes called the *circle at infinity*. It is denoted by $S^1_\infty$, $\partial\mathbb{H}^2$, and sometimes $\partial_\infty\mathbb{H}^2$.

Areas behave quite differently in hyperbolic space than in Euclidean space.

EXAMPLE 2.4. In this example, we will compute the area of the region $R$ of $\mathbb{H}^2$ bounded by the lines $x = 0$, $x = 1$, and $y = 1$. The region is shown in figure 2.1.

Using equation equation (2.3), we see that the area of the region is given by

$$\begin{aligned} \operatorname{area}(R) &= \int_R \frac{1}{y^2}\, dx\, dy \\ &= \int_0^1 \int_1^\infty \frac{1}{y^2}\, dy\, dx \\ &= \int_0^1 1\, dx = 1 \end{aligned}$$

This example shows that regions with infinite Euclidean area may have finite hyperbolic area.

**2.1.3. Geodesics and isometries.** Recall that a *geodesic* between points $p$ and $q$ is a length minimizing curve between those points.

THEOREM 2.5. *The geodesics in $\mathbb{H}^2$ consist of vertical straight lines and semi-circles with center on the real line.* □

Note these are exactly the circles and lines in the upper half plane that meet $S^1_\infty$ at right angles. See figure 2.2.

The proof of theorem 2.5 is left as an exercise in Riemannian geometry. The simplest way to prove the theorem uses coordinates and a bit more Riemannian geometry than we have reviewed so far. The interested reader can work through the details. The fact that these are the geodesics of $\mathbb{H}^2$ is all we will need going forward.

An *isometry* between Riemannian manifolds $M$ and $N$ is a diffeomorphism $f\colon M \to N$ such that

$$\langle v, w\rangle_p = \langle df_p(v), df_p(w)\rangle_{f(p)} \quad \text{for all } p \in M, v, w \in T_pM.$$

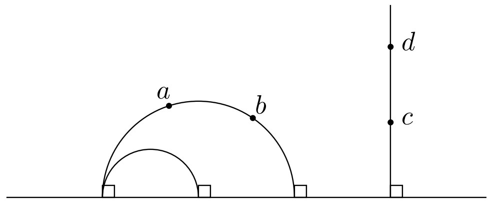


FIGURE 2.2. Some geodesics and points in $\mathbb{H}^2$.

Isometries preserve lengths, angles, and other geometric information. We are most interested in orientation preserving isometries from hyperbolic space to itself, i.e. diffeomorphisms $\phi\colon \mathbb{H}^2 \to \mathbb{H}^2$ that preserve the metric and orientation on $\mathbb{H}^2$. All such isometries form a group acting on $\mathbb{H}^2$. We will assume the following theorem.

THEOREM 2.6. *The full group of isometries of $\mathbb{H}^2$ is generated by inversions in geodesics in $\mathbb{H}^2$.*

*The group of orientation preserving isometries of $\mathbb{H}^2$ is the group of* linear fractional transformations

$$z \mapsto \frac{az+b}{cz+d},$$

*where $a, b, c, d \in \mathbb{R}$, and $ad - bc > 0$.* □

By taking the quotient of $a$, $b$, $c$, and $d$ by $\sqrt{ad-bc}$, the linear fractional transformation is equivalent to an element of $\mathrm{PSL}(2,\mathbb{R})$, the group of projective 2 by 2 matrices with real coefficients and determinant 1. That is, we may view $A \in \mathrm{PSL}(2,\mathbb{R})$ as given by a matrix

$$A = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a$, $b$, $c$, $d \in \mathbb{R}$ and $ad - bc = 1$. The sign in front reflects the fact that it is *projective*; it is well-defined only up to multiplication by $\pm\mathrm{Id}$. On the other hand, $A$ acts on $\mathbb{H}^2$ via

$$Az = \frac{az+b}{cz+d}.$$

Note the action is unaffected when we multiply $a$, $b$, $c$, and $d$ by the same real constant, thus it is necessary to take projective matrices.

Recall that linear fractional transformations take circles and lines to circles and lines, so they map geodesics to geodesics. For more information on these transformations, see for example [**Ahlfors, 1978**, pp 76–89].

The following lemma is very useful.

LEMMA 2.7. *Given any three distinct points $z_1$, $z_2$, and $z_3$ in $\partial\mathbb{H}^2$, there exists an orientation preserving isometry of $\mathbb{H}^2$ taking $z_3$ to $\infty$, and taking $\{z_1, z_2\}$ to $\{0, 1\}$. It follows that there exists an isometry of $\mathbb{H}^2$ taking any* *three distinct points on $\partial\mathbb{H}^2$ to any other three distinct points, with appropriate orientation.*

PROOF. This is a standard fact of linear fractional transformations. We need to take some care to preserve orientation. If necessary, switch $z_1$ and $z_2$ so that the sequence $z_1, z_2, z_3$ runs in clockwise order around $\partial\mathbb{H}^2$.

If none of $z_1$, $z_2$, and $z_3$ are infinity, then a linear fractional transformation sending $z_1$ to 1, $z_2$ to 0, and $z_3$ to $\infty$ is given by

$$z \mapsto \frac{z - z_2}{z - z_3}\frac{z_1 - z_3}{z_1 - z_2}.$$

Because the sequence $z_1, z_2, z_3$ is in clockwise order, the determinant

$$(z_1 - z_3)(z_1 - z_2)(z_2 - z_3)$$

of this transformation is positive. Thus it gives the desired orientation preserving isometry.

If $z_1 = \infty$, $z_2 = \infty$, or $z_3 = \infty$, then the isometry is given by

$$z \mapsto \frac{z - z_2}{z - z_3}, \quad z \mapsto \frac{z_1 - z_3}{z - z_3}, \quad z \mapsto \frac{z - z_2}{z_1 - z_2}$$

respectively. One can check that again, because we ensured the sequence $z_1, z_2, z_3$ is in clockwise order, the determinant of each transformation is positive. □

Many metric calculations in $\mathbb{H}^2$ can be simplified greatly by applying an appropriate isometry, including the use of lemma 2.7. For example, the following lemma is easily proved using an isometry.

LEMMA 2.8. *Two distinct geodesics $\ell_1$ and $\ell_2$ in $\mathbb{H}^2$ either*

*(1) intersect in a single point in the interior of $\mathbb{H}^2$,*
*(2) intersect in a single point on the boundary $\partial\mathbb{H}^2$, or*
*(3) are completely disjoint in $\mathbb{H}^2 \cup \partial\mathbb{H}^2$.*

*In the third case, there is a unique geodesic $\ell_3$ that is perpendicular to both $\ell_1$ and $\ell_2$.*

PROOF. We may apply an isometry $g$ of $\mathbb{H}^2$, taking endpoints of $\ell_1$ to 0 and $\infty$, and taking one of the endpoints of $\ell_2$ to 1. The image of the second endpoint of $\ell_2$ under $g$ is then some point $w$ in $\partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$. Note that $g(\ell_1)$ is the vertical line from 0 to $\infty$ in $\mathbb{H}^2$. The point $w$ determines the image of $g(\ell_2)$.

If $w = 0$ or if $w = \infty$, then we are in the second case, and $g(\ell_2)$ is a semi-circle with endpoints 0 and 1, or a vertical line from 1 to $\infty$.

If $w \in \mathbb{R}$ is less than zero, then we are in the first case. The two endpoints of $g(\ell_2)$ are separated by the line $g(\ell_1)$, so the geodesics must meet.

Finally, if $w \in \mathbb{R}$ is greater than zero, then we are in the third case, and the geodesics are disjoint. One way to see that there is a unique geodesic perpendicular to both is to apply another isometry $h$, taking $\sqrt{w}$ to 0 and $-\sqrt{w}$ to $\infty$. That is, let $h\colon \mathbb{H}^2 \to \mathbb{H}^2$ be given by

$$h(z) = \frac{z - \sqrt{w}}{z + \sqrt{w}}.$$

Note that $h(0) = -1$, $h(\infty) = 1$, so $h(g(\ell_1))$ is the geodesic that is a semi-circle with endpoints at $-1$ and 1. Also,

$$h(1) = \frac{1 - \sqrt{w}}{1 + \sqrt{w}} \quad \text{and} \quad h(w) = \frac{w - \sqrt{w}}{w + \sqrt{w}} = -\frac{1 - \sqrt{w}}{1 + \sqrt{w}}.$$

So $h(g(\ell_2))$ is the geodesic that is a semi-circle with endpoints $h(1)$ and $-h(1)$. Thus images of both geodesics are semi-circles with center at 0. The geodesic from 0 to $\infty$ is therefore perpendicular to both, and it is the unique such geodesic. Set $\ell_3$ to be the image of the line from 0 to $\infty$ under the composition $g^{-1} \circ h^{-1}$. □

In the previous proof, knowing which isometry $h$ to apply in the last step required a calculation. However, once that isometry was applied, the existence and uniqueness of the geodesic $\ell_3$ was clear.

Computing lengths of geodesics is also simplified by applying isometries.

Example 2.9. Length computation.

Suppose you wish to compute the length of a segment, or the distance between two points in $\mathbb{H}^2$. One strategy for computing is to apply an isometry taking the two points to a simpler picture. For example, in figure 2.2, we may find an isometry taking the geodesic containing points $a$ and $b$ to the vertical geodesic from 0 to $\infty$. Then under this isometry, the points $a$ and $b$ map to points of the form $(0, t_1)$ and $(0, t_2)$.

In example 2.2, we already computed the length of the vertical segment between $(0, t_1)$ and $(0, t_2)$; its length is $\log(t_1/t_2)$ (assuming here that $t_2 < t_1$, otherwise take the negative of the log). This gives the distance between $a$ and $b$.

### 2.1.4. Triangles and horocycles.

Definition 2.10. An *ideal triangle* in $\mathbb{H}^2$ is a triangle with three geodesic edges, with all three vertices on $\partial\mathbb{H}^2$.

There is an isometry of $\mathbb{H}^2$ taking any ideal triangle to the ideal triangle with vertices 0, 1, and $\infty$, by lemma 2.7. Hence all ideal triangles in $\mathbb{H}^2$ are isometric. In fact, we will see that they all have finite area. Because they are isometric, any ideal triangle has the same area.

Definition 2.11. A *horocycle* at an ideal point $p \in \partial\mathbb{H}^2$ is defined as a curve perpendicular to all geodesics through $p$. When $p$ is a point on $\mathbb{R} \subset \partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$, a horocycle is a Euclidean circle tangent to $p$, as in figure 2.3. When $p$ is the point $\infty$, a horocycle at $p$ is a line parallel to $\mathbb{R}$. That is, in this case the horocycle consists of points of the form $\{(x, y) \mid y = c\}$ where $c > 0$ is constant.

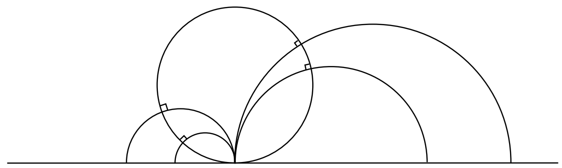

FIGURE 2.3. A horocycle

DEFINITION 2.12. A *horoball* is the region of $\mathbb{H}^2$ interior to a horocycle.

Note a horoball will either be a Euclidean disk tangent to $\mathbb{R} \subset \partial\mathbb{H}^2$ or a region consisting of points of the form $\{(x, y) \mid y > c\}$.

In example 2.4, we computed the area of a portion of a horoball, and we observed it was finite. Using this, we can show that the area of an ideal triangle is finite.

LEMMA 2.13. *The area of an ideal triangle is finite.*

PROOF. Given any ideal triangle in $\mathbb{H}^2$, we may apply an isometry taking its vertices to 0, 1, and $\infty$. Let $T$ denote this ideal triangle. Consider the intersection of $T$ with the horoball about infinity of height 1. This is the region $R$ of example 2.4.

Note that the isometries

$$z \mapsto \frac{z-1}{z} \quad \text{and} \quad z \mapsto \frac{-1}{z-1}$$

take the horoball about infinity to horoballs of Euclidean diameter 1 centered at 1 and at 0, respectively, and takes $T$ to $T$. Thus the intersections of $T$ with these horoballs also have areas 1.

Finally, note that the complement of these horoballs in $T$ is a compact region in $\mathbb{H}^2$, hence it has finite area. Thus the area of $T$ is 3 plus the area of the compact region outside of the three horoballs. $\square$

From the lemma, we see that the area of an ideal triangle is larger than 3. In fact, the exercises lead you through a calculation showing that the area of an ideal triangle is $\pi$.

## 2.2. Hyperbolic geometry in dimension three

Hyperbolic 3-space is defined as follows:

$$\mathbb{H}^3 = \{(x + iy, t) \in \mathbb{C} \times \mathbb{R} \mid t > 0\},$$

under the metric with first fundamental form

$$ds^2 = \frac{dx^2 + dy^2 + dt^2}{t^2}. \tag{2.4}$$

We have the following theorems, which we will assume. Their proofs can be found in texts on hyperbolic geometry.

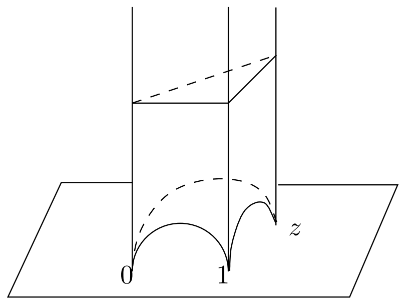


FIGURE 2.4. Ideal tetrahedron

THEOREM 2.14. *The geodesics in* $\mathbb{H}^3$ *consist of vertical lines and semi-circles orthogonal to the boundary* $\partial\mathbb{H}^3 = \mathbb{C} \cup \{\infty\}$. *Totally geodesic planes are vertical planes and hemispheres centered on* $\mathbb{C}$. □

THEOREM 2.15. *The full group of isometries of* $\mathbb{H}^3$ *is generated by inversions in geodesic planes.*

*The group of orientation preserving isometries of* $\mathbb{H}^3$ *is* $\mathrm{PSL}(2,\mathbb{C})$. *Its action on the boundary* $\partial\mathbb{H}^3 = \mathbb{C} \cup \{\infty\}$ *is the usual action of* $\mathrm{PSL}(2,\mathbb{C})$ *on* $\mathbb{C} \cup \{\infty\}$, *via Möbius transformation.* □

That is, an element $A \in \mathrm{PSL}(2,\mathbb{C})$ can be represented by a matrix, up to multiplication by $\pm\mathrm{Id}$. If

$$A = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{C}), \text{ then } A(z) = \frac{az+b}{cz+d}, \text{ for } z \in \partial\mathbb{H}^3.$$

We have not described how to extend the action of an element of $\mathrm{PSL}(2,\mathbb{C})$ to the interior of hyperbolic 3-space. There is a unique way to do so; Marden works through it carefully in [**Marden, 2007**, Chapter 1]. However, we will not need the formula, and it is complicated, so we omit it here.

THEOREM 2.16. *Apart from the identity, any element of* $\mathrm{PSL}(2,\mathbb{C})$ *is exactly one of three types:*

*(1)* elliptic, *which has two fixed points on* $\partial\mathbb{H}^3$ *and rotates about the axis between them in* $\mathbb{H}^3$, *fixing the axis pointwise,*
*(2)* parabolic, *which has a single fixed point on* $\partial\mathbb{H}^3$,
*(3)* loxodromic, *which has two fixed points on* $\partial\mathbb{H}^3$, *and translates and rotates about the axis between them.*

DEFINITION 2.17. An *ideal tetrahedron* is a tetrahedron in $\mathbb{H}^3$ with all four vertices on $\partial\mathbb{H}^3$.

Since there exists a Möbius transformation taking any three points to 1, 0, and $\infty$ in $\mathbb{C} \cup \{\infty\}$, we may assume our tetrahedron has vertices at 0, 1 and $\infty$, and at some point $z \in \mathbb{C} \setminus \{0,1\}$. So any ideal tetrahedron is parameterized by $z$. See figure 2.4.

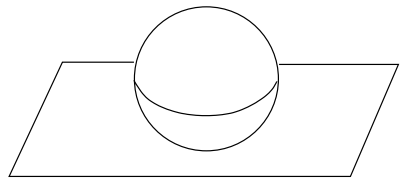

FIGURE 2.5. Horosphere

The value of $z$ has geometric meaning. For example, the argument of $z$ is the dihedral angle between the vertical planes through $0, 1, \infty$ and through $0, z, \infty$.

The modulus of $z$ also has geometric meaning. Consider the hyperbolic geodesic through $z \in \mathbb{C}$ that meets the vertical line from 0 to $\infty$ in a right angle at a point $p_1$. Consider also the geodesic through $1 \in \mathbb{C}$ that meets the vertical line from 0 to $\infty$ at a right angle at point $p_2$. The hyperbolic distance between $p_1$ and $p_2$ is exactly $|\ln |z||$ (exercise). Hence

$$\ln z = (\text{signed dist between altitudes}) + i(\text{dihedral angle}).$$

DEFINITION 2.18. A *horosphere* about $\infty$ in $\partial\mathbb{H}^3$ is a plane parallel to $\mathbb{C}$, consisting of points $\{(x + iy, c) \in \mathbb{C} \times \mathbb{R}\}$ where $c > 0$ is constant. Note for any $c > 0$, it is perpendicular to all geodesics through $\infty$. When we apply an isometry that takes $\infty$ to some $p \in \mathbb{C}$, a horosphere is taken to a Euclidean sphere tangent to $p$. By definition, this is a horosphere about $p$. A *horoball* is the region interior to a horosphere.

The metric on $\mathbb{H}^3$ induces a metric on a horosphere. This induced metric will always be Euclidean. When we intersect horospheres about 0, 1, $\infty$ and $z$ with an ideal tetrahedron through those points, we obtain four Euclidean triangles. These four triangles are similar (exercise).

## 2.3. Exercises

EXERCISE 2.1. Prove theorem 2.5, that is that vertical lines and semi-circles are geodesics, without using isometries of $\mathbb{H}^2$. One way to solve this problem is to use Riemannian geometry, such as calculations in coordinates on $\mathbb{H}^2$. Break the problem into two steps.

(1) Prove that vertical lines $L(t) = (x, t)$, $t > 0$, are geodesics in $\mathbb{H}^2$.
(2) Prove that semi-circles $C(t) = (x + r\cos(t), r\sin(t))$, $t \in (0, \pi)$ are geodesics in $\mathbb{H}^2$.

EXERCISE 2.2. Prove that an inversion of $\mathbb{H}^2$ in a hyperbolic geodesic is an isometry of $\mathbb{H}^2$. This is an orientation reversing isometry.

EXERCISE 2.3. Prove that any isometry of $\mathbb{H}^2$ is the product of inversions in hyperbolic geodesics.

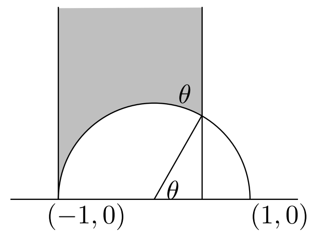


FIGURE 2.6. 2/3-ideal triangle.

EXERCISE 2.4. Work through the classification of isometries of $\mathbb{H}^2$ as elliptic, parabolic, or hyperbolic. (E.g. Thurston [**Thurston, 1979**, page 67]).

EXERCISE 2.5. Lemma 2.7 shows there exists an orientation preserving isometry of $\mathbb{H}^2$ taking any three points of $\partial\mathbb{H}^2$ to any other three points, provided we are careful with orientation. Prove a similar statement for $\mathbb{H}^3$: Given distinct $b$, $c$ and $d$ in $\mathbb{C} \cup \{\infty\}$, prove there exists an orientation preserving isometry of $\mathbb{H}^3$ taking $b$ to 1, $c$ to 0, and $d$ to $\infty$. Write it down as a matrix in $\mathrm{PSL}(2, \mathbb{C})$. Note in $\mathbb{H}^3$ we no longer have to worry about orientation.

EXERCISE 2.6. Prove an analogue of lemma 2.8 in $\mathbb{H}^3$: Two distinct geodesics $\ell_1$ and $\ell_2$ either intersect in a single point in the interior of $\mathbb{H}^3$, intersect in a single point on $\partial\mathbb{H}^3$, or are completely disjoint in $\mathbb{H}^3 \cup \partial\mathbb{H}^3$. In the third case, show there exists a unique geodesic that is perpendicular to both $\ell_1$ and $\ell_2$.

EXERCISE 2.7. (Cross ratios.) Given $a \in \mathbb{C}$, the image of $a$ under the isometry of exercise 2.5 is said to be the *cross ratio* of $a, b, c, d$, and is denoted $\lambda(a, b; c, d)$.

Let $x$ be the point on the geodesic in $\mathbb{H}^3$ between $c$ and $d$ such that the geodesic from $a$ to $x$ is perpendicular to that between $c$ and $d$. Let $y$ be the point on the geodesic between $c$ and $d$ such that the geodesic from $b$ to $y$ is perpendicular to that between $c$ and $d$. Prove the hyperbolic distance between $x$ and $y$ is equal to $|\ln|\lambda(a, b; c, d)||$.

EXERCISE 2.8. (Areas of ideal triangles.) Prove that the area of an ideal hyperbolic triangle is $\pi$. (E.g. use calculus.)

EXERCISE 2.9. (Areas of 2/3-ideal triangles.)

(a) A 2/3-ideal triangle is a triangle with two vertices on the boundary at infinity $\partial\mathbb{H}^2$, and the third in the interior of $\mathbb{H}^2$ such that the interior angle at the third vertex is $\theta$. Show that all 2/3-ideal triangles of angle $\theta$ are congruent to the triangle shown in figure 2.6.
(b) Define a function $A\colon (0, \pi) \to \mathbb{R}$ by: $A(\theta)$ is the area of the 2/3-ideal triangle with interior angle $\pi - \theta$. Show that $A(\theta_1 + \theta_2) = A(\theta_1) + A(\theta_2)$, when this is defined. (Hint: Figure 2.7 may be useful.)

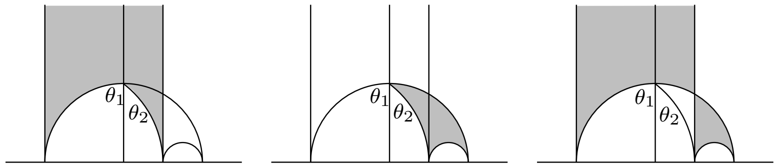


FIGURE 2.7. Areas of triangles.

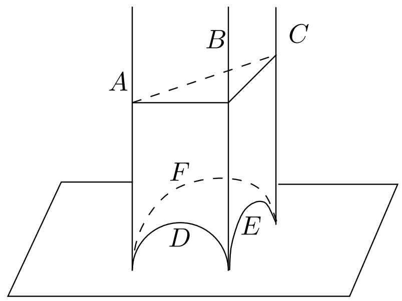


FIGURE 2.8.

(c) It follows that $A$ is $\mathbb{Q}$-linear. Since $A$ is continuous, it must be $\mathbb{R}$-linear. Show $A(\theta) = \theta$.

EXERCISE 2.10. (Areas of general triangles.) Using the previous two problems, show that the area of a triangle with interior angles $\alpha$, $\beta$, and $\gamma$ is equal to $\pi - \alpha - \beta - \gamma$. Note an ideal vertex has interior angle 0.

EXERCISE 2.11. (Ideal tetrahedra and dihedral angles.) The dihedral angles on a tetrahedron are labeled $A$, $B$, $C$, $D$, $E$, and $F$ in figure 2.8. Using linear algebra, prove that opposite dihedral angles agree. That is, show $A = E$, $B = F$, and $C = D$.

EXERCISE 2.12. (Ideal tetrahedra and cross ratios.) Orient an ideal tetrahedron with vertices $a, b, c, d$. When we apply a Möbius transformation taking $b, c, d$ to $1, 0, \infty$, respectively, the point $a$ goes to the cross ratio $\lambda(a, b; c, d)$. Label the edge from $c$ to $d$ by the complex number $\lambda = \lambda(a, b; c, d)$. We may do this for each edge of the tetrahedron, labeling by a different cross ratio. (Notice you need to keep track of orientation.) Find all labels on the edges of the tetrahedra in terms of $\lambda$.

EXERCISE 2.13. (Volume of a region in a horoball) Let $R$ be the region in $\mathbb{H}^3$ given by $A \times [1, \infty)$, where $A$ is some parallelogram contained in the horosphere about $\infty$ of height 1, i.e. $A \subset \{(x + i\,y, 1)\}$. Prove that $\mathrm{vol}(R) = \mathrm{area}(A)/2$.