CHAPTER 3

Geometric structures on manifolds

In this chapter, we give our first examples of hyperbolic manifolds, combining ideas from the previous two chapters.

3.1. Geometric structures

3.1.1. Introductory example: The torus. A geometric structure you are likely familiar with is a 2-dimensional Euclidean structure on a torus. Given any parallelogram, we obtain a torus by gluing the top and bottom sides of the parallelogram, and the right and left sides, as shown in figure 3.1.

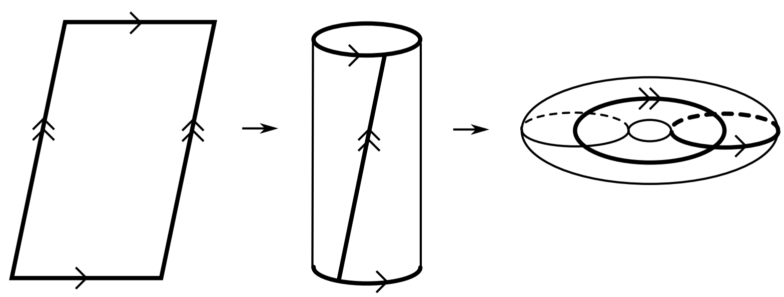

FIGURE 3.1. A parallelogram glued to a torus

The universal cover of the torus is obtained by gluing copies of the parallelogram to itself in $\mathbb{R}^2$. We may glue infinitely many copies in two directions, and we obtain a tiling of the plane $\mathbb{R}^2$ by isometric parallelograms, as in figure 3.2. These parallelograms define a lattice in $\mathbb{R}^2$, and deck transformations of the universal cover $\mathbb{R}^2$ of the torus are given by Euclidean translations by points of the lattice. That is, if the parallelogram is determined by vectors $\overrightarrow{v}$ and $\overrightarrow{w}$ along its sides, then any deck transformation is of the form $a\overrightarrow{v} + b\overrightarrow{w}$ for $a, b \in \mathbb{Z}$. This construction works for any choice of parallelogram.

Now modify this construction by choosing a more general quadrilateral instead of a parallelogram. We can still identify opposite sides in an orientation preserving manner, so when we glue we still get an object homeomorphic

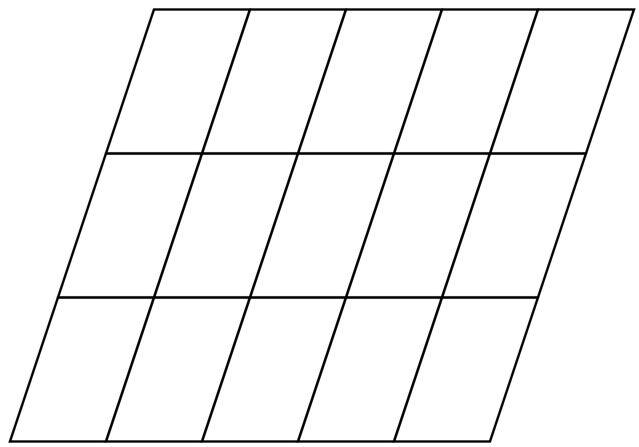

FIGURE 3.2. The universal cover of a Euclidean torus.

to a torus. However, the quadrilateral no longer determines a tiling of $\mathbb{R}^2$, nor a lattice. Indeed, when we glue copies of the quadrilateral to itself, as we did when constructing the universal cover above, we have to shrink, expand, and rotate the quadrilateral to glue copies, and the result is not a tiling of the plane. See figure 3.3.

These examples of the torus can be generalized to different surfaces and manifolds. The torus was created by gluing quadrilaterals. More generally, we will glue different types of polygons, including ideal polygons, and in 3-dimensions, polyhedra.

DEFINITION 3.1. Let M be a 2-manifold. A *topological polygonal decomposition* of M is a combinatorial way of gluing polygons so that the result is homeomorphic to M.

We allow ideal polygons, i.e. those with one or more ideal vertex. Additionally, by *gluing* we mean something that takes faces to faces, edges to edges, and vertices to vertices.

Both constructions of the torus above give examples of topological polygonal decompositions of the torus.

DEFINITION 3.2. A *geometric polygonal decomposition* of M is a topological polygonal decomposition along with a metric on each polygon such that gluing is by isometry and the result of the gluing is a smooth manifold with a complete metric.

Recall that a metric space is *complete* if every Cauchy sequence converges; and recall that a *Cauchy sequence* is a sequence $\{x_i\}_{i=1}^{\infty}$ such that for each $\epsilon > 0$, there exists a positive integer N such that $d(x_i, x_j) < \epsilon$ if $i, j \geq N$.

The first construction of the torus gives a complete Euclidean metric on the torus, by pulling back the Euclidean metric on the parallelogram.

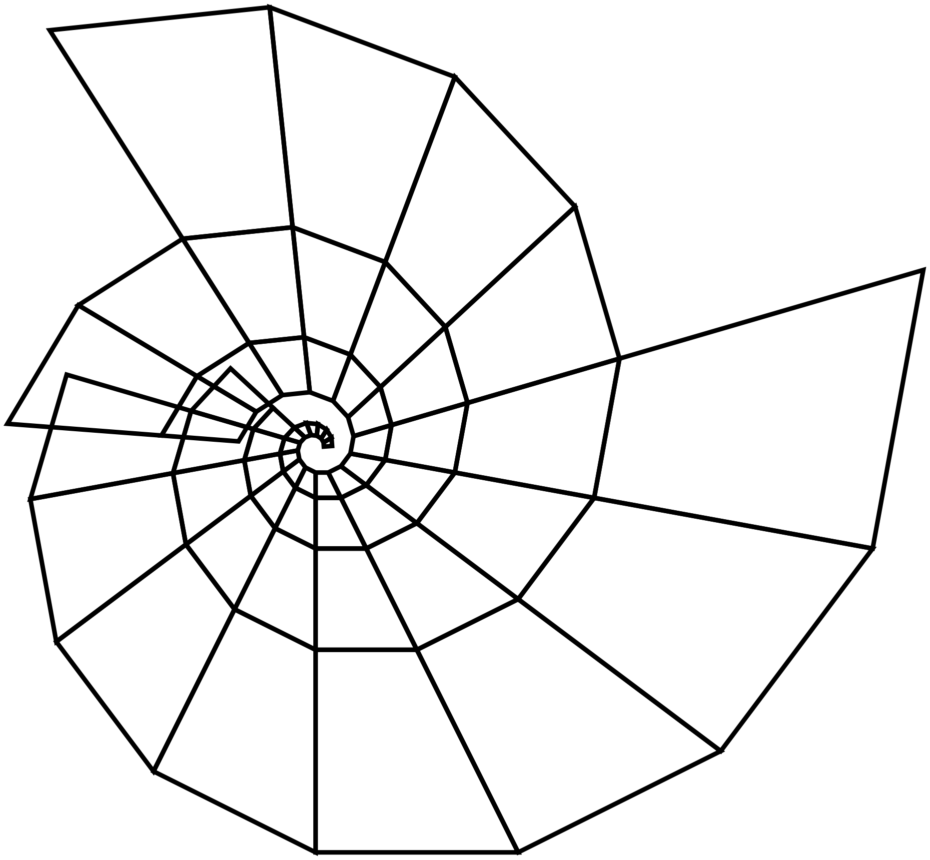

FIGURE 3.3. When we construct a torus from a quadrilateral, generally a single point is omitted from the plane.

Because gluings of the sides of the parallelogram are by Euclidean isometries, this will be well-defined. The second construction of the torus does not give a complete Euclidean metric, or any Euclidean metric: gluings of the quadrilaterals are by affine transformations (rotation, translation, scale), not isometries of the Euclidean plane, so we cannot pull back a well-defined metric. Note also that toward the center of figure 3.3, the quadrilaterals are becoming arbitrarily small. In fact, all copies of the quadrilaterals will miss a point in the figure (see exercise 3.8).

We will also be studying polygonal decompositions of manifolds and their generalization to three dimensions: polyhedral decompositions. More generally, we can discuss geometric structures on manifolds.

3.1.2. Geometric structures on manifolds.

DEFINITION 3.3. Let X be a manifold, and G a group acting on X. We say a manifold M has a (G, X)-structure if for every point $x \in M$, there exists a *chart* (U, ϕ), that is, a neighborhood $U \subset M$ of x and a homeomorphism $\phi\colon U \to \phi(U) \subset X$. Charts satisfy the following: If two charts (U, ϕ) and (V, ψ) overlap, then the *transition map* or *coordinate change map*

$$\gamma = \phi \circ \psi^{-1}\colon\ \psi(U \cap V) \to \phi(U \cap V)$$

is an element of G.

In the examples we encounter here, X will be simply connected, and G a group of real analytic diffeomorphisms acting transitively on X. Recall that real analytic diffeomorphisms are uniquely determined by their restriction to any open set. This is true, for example, of isometries of Euclidean space, and isometries of hyperbolic space.

In addition, typically our manifold X will admit a known metric, and G will be the group of isometries of X. It will follow that M inherits a metric from X (exercise). We will say that M has a *geometric structure*.

Example 3.4 (Euclidean torus). Let X be 2-dimensional Euclidean space, $\mathbb{E}^2$. Let G be isometries of Euclidean space $\mathrm{Isom}(\mathbb{E}^2)$. The torus admits a $(\mathrm{Isom}(\mathbb{E}^2), \mathbb{E}^2)$ structure, also called a *Euclidean structure*.

To help us understand the definition, let's look at some charts and transition maps for this example.

We know the universal cover of the torus is given by tiling the plane $\mathbb{R}^2$ with parallelograms. For simplicity, we will work with the example in which each parallelogram is a square, and one square has vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$ in $\mathbb{R}^2$. Call this square the *basic* square.

Now pick any point p on the torus. This will lift to a collection of points on $\mathbb{R}^2$, one for each copy of the unit square. Take a disk of radius $1/4$, say, around each lift. These all project under the covering map to an open neighborhood U of p in the torus. Therefore we have the following charts: (U, ϕ) is a chart, where ϕ maps U into the disk of radius $1/4$ centered around the lift $\widehat{p}_0$ of p in the basic square. Another chart is (U, ψ), where ψ maps U into the disk of radius $1/4$ about the lift $\widehat{p}_1$ of p in some other square. Such a lift is given by a translation of $\widehat{p}_0$ by a vector $(m, n) \in \mathbb{Z} \times \mathbb{Z}$, in the lattice determined by the basic square. Thus $\phi \circ \psi^{-1}$ will be a Euclidean translation by integral values in the x and y direction. These are Euclidean isometries.

More generally, let q be a point such that a lift $\widehat{q}_0$ of q has distance less than $1/2$ to $\widehat{p}_0$ in the basic square. Thus a disk of radius $1/4$ about $\widehat{p}_0$ overlaps with a disk of radius $1/4$ about $\widehat{q}_0$. These disks project to give open neighborhoods U and V of p and q respectively in the torus. Since these neighborhoods overlap, we need to ensure that any corresponding charts differ by a Euclidean isometry in the region of overlap. Obtain charts by mapping U to your favorite disk of radius $1/4$ about a lift of p. Map V to your favorite disk of radius $1/4$ about a lift of q in $\mathbb{R}^2$; see figure 3.4 for an example. Again, regardless of the choice of ϕ and ψ, the overlap

$$\phi \circ \psi^{-1}\colon\ \psi(U \cap V) \to \phi(U \cap V)$$

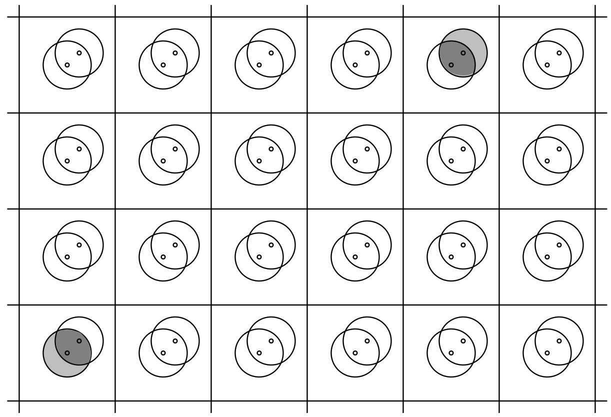

FIGURE 3.4. Euclidean structure on a torus: Transition maps are Euclidean translations.

will be a Euclidean translation of the intersection of the two disks by some $(n, m) \in \mathbb{Z} \times \mathbb{Z}$ corresponding to the choice of lifts. See figure 3.4.

The above discussion extends to arbitrary neighborhoods U and V: transition maps will always be translations by $(n, m) \in \mathbb{Z} \times \mathbb{Z}$. Therefore, we conclude that the torus obtained by gluing sides of the square with vertices $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$ admits a $(\mathrm{Isom}(\mathbb{E}^2), \mathbb{E}^2)$-structure, where $\mathbb{E}^2$ denotes $\mathbb{R}^2$ with the standard Euclidean metric. This is a *Euclidean structure*, for short.

EXAMPLE 3.5 (The affine torus). Again let $X = \mathbb{R}^2$, but this time let G be the affine group acting on $\mathbb{R}^2$. That is, G consists of invertible affine transformations, where recall an affine transformation is a linear transformation followed by a translation:

$$x \mapsto Ax + b.$$

The torus of figure 3.3 admits a $(G, \mathbb{R}^2)$ structure. This can be seen in a manner similar to that in the previous example. Charts will differ by a scaling, rotation, then translation.

In practice, we rarely use charts to show manifolds have a particular (G, X)-structure. Instead, as in the two previous examples, we build manifolds by starting with an existing manifold X and quotienting out by the action of a group, or by gluing together polygons.

3.1.3. Hyperbolic surfaces. Let $X = \mathbb{H}^2$, and let $G = \mathrm{Isom}(\mathbb{H}^2)$, the group of isometries of $\mathbb{H}^2$. When a 2-manifold admits an $(\mathrm{Isom}(\mathbb{H}^2), \mathbb{H}^2)$-structure, we say the manifold admits a *hyperbolic structure*, or is hyperbolic.

We will look at some examples of hyperbolic 2-manifolds obtained from geometric polygonal decompositions. To do so, we start with a collection of hyperbolic polygons in $\mathbb{H}^2$, for example, a collection of triangles. We allow vertices to either be finite, i.e. in the interior of $\mathbb{H}^2$, or ideal, on $\partial_\infty \mathbb{H}^2$. In any case, we will always assume each polygon is convex, and edges are portions of geodesics in $\mathbb{H}^2$. Now, to each edge, associate exactly one other edge. Just as in the case of the torus, glue polygons along associated edges by an isometry of $\mathbb{H}^2$.

When does the result of this gluing give a manifold that admits a hyperbolic structure? We obtain a hyperbolic structure exactly when each point in the result has a neighborhood U and a homeomorphism into $\mathbb{H}^2$ so that transition maps are in $\mathrm{Isom}(\mathbb{H}^2)$. The following lemma gives a condition that will guarantee it.

LEMMA 3.6. *A gluing of hyperbolic polygons yields a 2-manifold with a hyperbolic structure if and only if each point in the gluing has a neighborhood (in the quotient topology) isometric to a disk in* $\mathbb{H}^2$.

More generally, a gluing of n-dimensional hyperbolic polyhedra yields a hyperbolic n-manifold if and only if each point has a neighborhood (in the quotient topology) isometric to a ball in $\mathbb{H}^n$.

PROOF. We will prove the more general statement. Suppose first that a gluing of hyperbolic polyhedra M yields an n-manifold with hyperbolic structure. Then every point x in M has a neighborhood U and a chart $\phi\colon U \to \phi(U) \subset \mathbb{H}^n$ such that transition maps are isometries of $\mathbb{H}^n$. By restricting, we may assume $\phi(U)$ is a ball in $\mathbb{H}^n$. On the other hand, each neighborhood U comes from the quotient topology on the gluing. Since gluings are via isometries, U is made up of portions of balls on the polyhedra in $\mathbb{H}^n$, attached by gluing isometries. It follows that U is isometric to a ball, via the isometry ϕ.

Now suppose that under the quotient topology, every point of M has a neighborhood isometric to a ball of $\mathbb{H}^n$. Then this isometry gives a chart $\phi\colon U \to \phi(U) \subset \mathbb{H}^n$. If (U, ϕ) and (V, ψ) are charts and $U \cap V \neq \emptyset$, then $\phi \circ \psi^{-1}\colon \psi(U \cap V) \to \phi(U \cap V)$ is the composition of isometries, hence an isometry, so M has an $(\mathrm{Isom}(\mathbb{H}^n), \mathbb{H}^n)$ structure. □

When does each point in a gluing M of hyperbolic polygons have a neighborhood isometric to a disk in the hyperbolic plane? Let x be a point in the gluing, and consider its lifts to the polygons. There are three cases.

(1) If x lifts to a point $\widehat{x}$ in the interior of one of the polygons, then that lift is unique. In this case, for small enough $\epsilon > 0$, there is a disk about $\widehat{x}$ of radius ϵ embedded in the interior of the polygon in $\mathbb{H}^2$. This projects under the quotient map to a disk about x in M isometric to a disk in $\mathbb{H}^2$.
(2) If x lifts to a point on an edge of a polygon, then it has two lifts, $\widehat{x}_0$ and $\widehat{x}_1$, on two different edges that are glued to each other by the gluing map. A neighborhood of x in the quotient topology lifts to give a "half-neighborhood" of $\widehat{x}_0$ glued to a corresponding "half-neighborhood" of $\widehat{x}_1$. Each contains a half-disk in $\mathbb{H}^2$, and we may

scale the disks so that they glue to a disk under the gluing map. Thus in this case as well, x has a neighborhood isometric to a disk in $\mathbb{H}^2$.

(3) If x lifts to a finite vertex of a polygon, then it may have several lifts, possibly including several vertices of the collection of polygons. In this case, we need to be more careful. The following lemma gives a condition that will guarantee we have an isometry to a hyperbolic disk in this case as well.

LEMMA 3.7. *A gluing of hyperbolic polygons gives a 2-manifold with a hyperbolic structure if and only if for each finte vertex v of the polygons, the sum of interior angles at each vertex glued to v is 2π.*

PROOF. This is an immediate consequence of lemma 3.6 and the observation that around a vertex, portions of the polygons meet in a cycle, with total angle around the finite vertex equal to the sum of interior angles of the polyhedron at that vertex. We need to check that each finite vertex has a neighborhood isometric to a neighborhood in $\mathbb{H}^2$. This will hold if and only if the sum of interior angles is 2π. □

3.2. Complete structures

Given a gluing of hyperbolic polygons, suppose the angle sum at each finite vertex is 2π, so that we have a hyperbolic structure by lemma 3.7. Does it necessarily follow that we have a geometric polygonal decomposition, as in definition 3.2?

Recall that for a geometric polygonal decomposition, we needed a geometric structure on each polygon so that the result of the gluing is a smooth manifold with a complete metric. Our hyperbolic structure gives a smooth manifold with a metric. However, in the presence of ideal vertices, the metric may not be complete.

It will be easier to discuss criteria for completeness using the language of *developing maps* and *holonomy*. Our exposition of these terms is based on that of Thurston [**Thurston, 1997**].

3.2.1. Developing map and holonomy. The developing map, which we define in this subsection, encodes information on the (G, X)-structure of a manifold. It is a local homeomorphism into X. When a manifold has a polygonal decomposition, say by polygons in $X = \mathbb{R}^2$ or $\mathbb{H}^2$, the developing map "develops" the gluing information on the polygons by attaching copies of the polygons along edges in the space X, as we did for the torus in Figures 3.2 and 3.3.

More generally, a developing map can be defined for any manifold M with a (G, X)-structure, assuming as before that X is a manifold and G is a group of real analytic diffeomorphisms acting transitively on X. Any chart (U, ϕ) gives a homeomorphism of U onto $\phi(U) \subset X$. To define the developing map, we wish to extend this map.

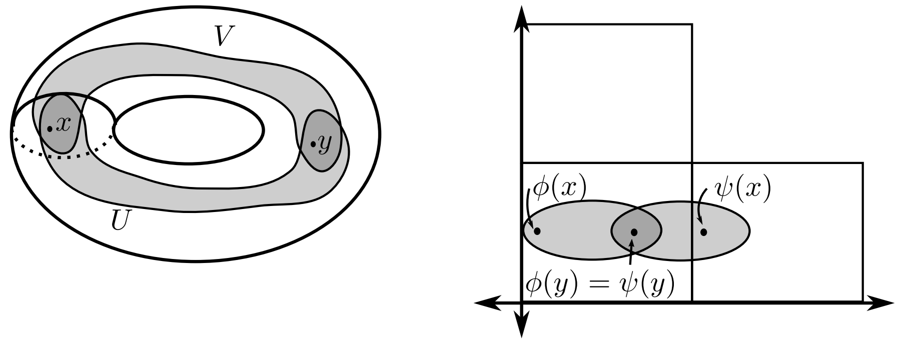

FIGURE 3.5. Neighborhoods U and V on the torus have two components of intersection, one containing x and one containing y. The map ϕ cannot be extended over V because it will not be well defined at x

So suppose (V, ψ) is another chart, and $y \in U \cap V$. Then

$$\gamma = \phi \circ \psi^{-1} \colon \ \psi(U \cap V) \to \phi(U \cap V)$$

is an element of G acting on $\psi(U \cap V)$. Because G is a group of real analytic diffeomorphisms, the element of G is uniquely determined in a neighborhood of $\psi(y)$, hence constant for all x in a neighborhood of y in $U \cap V$. We let $\gamma(y)$ denote this element of G. Then we may define a map $\Phi \colon U \cup V \to X$ by

$$\Phi(x) = \begin{cases} \phi(x) & \text{if } x \in U \\ \gamma(y)\psi(x) & \text{if } x \in V \end{cases}$$

Note that if $U \cap V$ is connected, then Φ is a well-defined homeomorphism, since for $x \in U \cap V$, we have $\phi(x) = \gamma(y)\psi(x)$. Thus in this case, Φ is an extension of ϕ. However, note that we may run into trouble when $U \cap V$ is not connected, for then $\phi(x)$ may not equal $\gamma(y)\psi(x)$ for x in a component disjoint from that containing y. Example 3.8 gives an example of this.

EXAMPLE 3.8. Consider the Euclidean torus obtained by gluing sides of a square with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$ in $\mathbb{R}^2$. Suppose the union of two simply connected neighborhoods U and V forms a neighborhood of a longitude for the torus, as in figure 3.5, such that $U \cap V$ has two components. Suppose y lies in one component of $U \cap V$. There exist charts (U, ϕ) and (V, ψ) sending y to the interior of the basic square in $\mathbb{R}^2$. Then the transition map $\gamma(y)$ is the identity element of G in this case, since $\phi(U)$ and $\psi(V)$ overlap in the component of $U \cap V$ containing y. However, the map Φ defined above is not well defined, for if x lies in the other component of $U \cap V$, $\phi(x)$ lies in the basic square, but $\gamma(y)\psi(x) = \psi(x)$ lies in the square with vertices $(1, 0)$, $(2, 0)$, $(1, 1)$, and $(1, 2)$.

Similarly, as we attempt to extend Φ by considering other coordinate neighborhoods overlapping U and V, the natural extensions using transition maps such as $\gamma(y)$ again may not be well-defined.

To overcome this problem, we use the universal cover of M. Recall that the universal cover $\widetilde{M}$ of M can be defined to be the space of homotopy classes of paths in M that start at a fixed basepoint x_0. See, for example [**Munkres, 2000**, Theorem 82.1] or [**Hatcher, 2002**, page 64]. Let $\alpha\colon [0,1] \to M$ be a path representing a point $[\alpha] \in \widetilde{M}$, and let the chart (U_0, ϕ_0) contain the basepoint x_0.

Now find $0 = t_0 < t_1 < \cdots < t_n = 1$ and charts (U_i, ϕ_i) such that $\alpha([t_i, t_{i+1}])$ is contained in U_i for $i = 0, 1, \dots, n-1$. Denote the points $\alpha(t_i)$ by $x_i \in M$. We extend ϕ_0 to all of α as follows. First, note that each x_i, for $i = 1, \dots, n-1$, is contained in the intersection of two charts, $x_i \in U_{i-1} \cap U_i$. Then the transition map $\gamma_{i-1,i} = \phi_{i-1} \circ \phi_i^{-1}$ gives an element $\gamma_{i-1,i}(x_i)$ in G. Thus at the first step, we may extend ϕ_0 to a function from $[0, t_2]$ to X by defining $\Phi_1\colon [0, t_2] \to X$ to be the function:

$$\Phi_1(t) = \begin{cases} \phi_0(\alpha(t)) & \text{if } t \in [0, t_1] \\ \gamma_{0,1}(x_1)\phi_1(\alpha(t)) & \text{if } t \in [t_1, t_2] \end{cases}$$

This will be well-defined on all of $[0, t_2]$, since $\phi_0(\alpha(t_1)) = \gamma_{0,1}(x_1)\phi_1(\alpha(t_1))$.

Extend inductively to $\Phi_i\colon [0, t_{i+1}] \to X$ by setting:

$$\Phi_i(t) = \begin{cases} \Phi_{i-1}(t) & \text{if } t \in [0, t_i] \\ \gamma_{0,1}(x_1)\gamma_{1,2}(x_2)\dots\gamma_{(i-1),i}(x_i)\phi_i(\alpha(t)) & \text{if } t \in [t_i, t_{i+1}] \end{cases}$$

Again this is well-defined, for we know $\phi_{i-1}(\alpha(t_i)) = \gamma_{(i-1),i}(x_i)\phi_i(\alpha(t_i))$. Thus by induction,

$$\begin{aligned} \Phi_{i-1}(t_i) &= \gamma_{0,1}(x_1)\gamma_{1,2}(x_2)\dots\gamma_{(i-2),(i-1)}(x_{i-1})\phi_{i-1}(\alpha(t_i)) \\ &= \gamma_{0,1}(x_1)\gamma_{1,2}(x_2)\dots\gamma_{(i-1),i}(x_i)\phi_i(\alpha(t_i)). \end{aligned}$$

After the $(n-1)$-st step, we have a map $\Phi_{n-1}\colon [0,1] \to X$. In fact, note that the definition of Φ_{n-1} actually provides a map $\Phi_{[\alpha]}\colon U \to X$, for some small neighborhood U of $\alpha(1)$, defined by

$$\Phi_{[\alpha]}(x) = \gamma_{0,1}(x_1)\gamma_{1,2}(x_2)\dots\gamma_{(n-2),(n-1)}(x_n)\phi_{n-1}(x).$$

The function $\Phi_{[\alpha]}$ defined in this manner, with fixed initial chart (U_0, ϕ_0) and fixed basepoint x_0, is an example of a function defined by *analytic continuation*. It is well known that analytic continuation gives a well-defined function, independent of choice of the charts $(U_1, \phi_1), \dots, (U_{n-1}, \phi_{n-1})$, independent of the choice of points $t_1, \dots, t_{n-1}$, and independent of the choice of path α in the homotopy class $[\alpha] \in \widetilde{M}$. For our particular application, we will leave this as an exercise.

DEFINITION 3.9. The *developing map* $D\colon \widetilde{M} \to X$ is the map

$$D([\alpha]) = \Phi_n(1) = \gamma_{0,1}(x_1)\gamma_{1,2}(x_2)\dots\gamma_{(n-2),(n-1)}(x_{n-1})\phi_{n-1}(\alpha(1)),$$

with notation given above.

PROPOSITION 3.10. *The developing map* $D\colon \widetilde{M} \to X$ *satisfies the following properties.*

(1) *For fixed basepoint* x_0 *and initial chart* (U_0, ϕ_0), *with* $x_0 \in U_0$, *the map* D *is well defined, independent of all other choices used to define it, including charts, points in the overlap of chart neighborhoods, and independent of choice of* α *in the homotopy class of* $[\alpha]$.
(2) D *is a local diffeomorphism.*
(3) *If we define a new map in the same way as* D, *except beginning with a new choice of basepoint and initial chart, the resulting map is equal to the composition of* D *with an element of* G.

PROOF. We leave parts (1) and (3) exercises. Part (2) follows immediately from part (1), the fact that each $\gamma_{i,(i+1)}(x_i)$ is a diffeomorphism and ϕ_n is a local diffeomorphism on M, and the topology on $\widetilde{M}$. □

Now consider the case that $[\alpha] \in \widetilde{M}$ is an element of the fundamental group of M. That is, $[\alpha]$ is a homotopy class of loops starting and ending at x_0. Analytic continuation along a loop gives a function $\Phi_{[\alpha]}$ whose domain is a neighborhood of the basepoint of the loop; this is a new chart defined in a neighborhood of the basepoint. Since ϕ_0 and $\Phi_{[\alpha]}$ are both charts defined in a neighborhood of the basepoint, these maps must differ by an element of G. Let $g_{[\alpha]} \in G$ be the element such that $\Phi_{[\alpha]} = g_{[\alpha]}\phi_0$.

Let $T_{[\alpha]}$ denote the covering transformation of $\widetilde{M}$ that corresponds to $[\alpha]$. It follows that

$$D \circ T_{[\alpha]} = g_{[\alpha]} \circ D.$$

Note also that for $[\alpha], [\beta] \in \widetilde{M}$,

$$D \circ T_{[\alpha]} \circ T_{[\beta]} = (g_{[\alpha]} \circ D) \circ T_{[\beta]} = g_{[\alpha]} \circ g_{[\beta]} \circ D.$$

It follows that the map $\rho\colon \pi_1(M) \to G$ defined by $\rho([\alpha]) = g_{[\alpha]}$ is a group homomorphism.

DEFINITION 3.11. The element $g_{[\alpha]}$ is the *holonomy* of $[\alpha]$. The group homomorphism ρ is called the *holonomy* of M. Its image is the *holonomy group* of M.

Note that ρ depends on the choices from the construction of D. When D changes, ρ changes by conjugation in G (exercise).

EXAMPLE 3.12. Pick a point x on the torus, say x lies at the intersection of a choice of meridian and longitude curves for the torus, and consider a nontrivial curve γ based at x. An example of a nontrivial curve γ on the torus is shown in figure 3.6.

Now consider a Euclidean structure on the torus. There exists a chart mapping x onto the Euclidean plane. We can take our chart to be an open

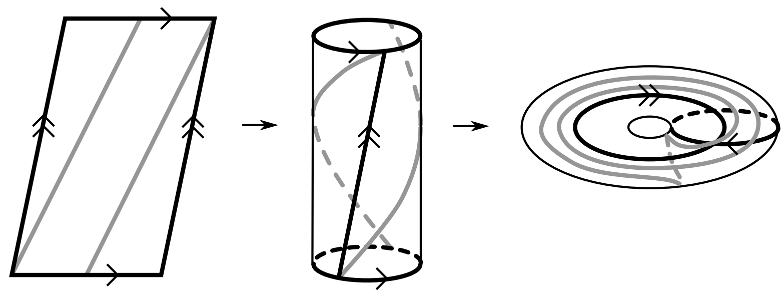

FIGURE 3.6. A nontrivial curve γ (lighter gray) on the torus. Meridian and longitude curves are shown in thicker black

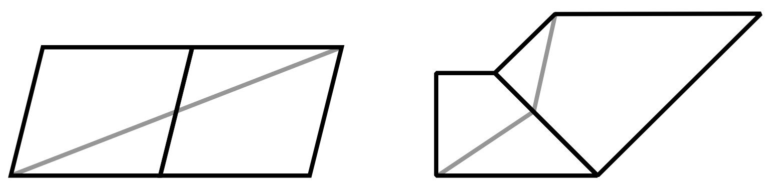

FIGURE 3.7. Left: developing a Euclidean torus. Right: developing an affine torus.

parallelogram about x, where boundaries of the parallelogram glue in the usual way to form the torus. As the curve γ passes over a meridian or longitude, in the image of the developing map we must glue a new parallelogram to the appropriate side of the parallelogram we just left. See figure 3.7, left, for an example. The tiling of the plane by parallelograms is the image of the developing map, or the developing image of the Euclidean torus.

As for the affine torus, example 3.5, each time a curve crosses a meridian or longitude we attach a rescaled, rotated, translated copy of our quadrilateral to the appropriate edge. Figure 3.7 right shows an example. Figure 3.3 shows (part of) the developing image of the affine torus.

3.2.2. Completeness of polygonal gluings. Now we return to the question of determining when a gluing of hyperbolic polygons gives a complete hyperbolic structure. We know there will be a hyperbolic structure provided the angle sum around finite vertices is 2π (lemma 3.7). The question of whether the structure is complete or not depends on what happens near ideal vertices.

Let M be an oriented hyperbolic surface obtained by gluing *ideal* hyperbolic polygons. An *ideal vertex* of M is an equivalence class of ideal vertices of the polygons, identified by the gluing.

Let v be an ideal vertex of M. Then v is identified to some ideal vertex v_0 of a polygon P_0. Let h_0 be a horocycle centered at v_0 on P_0, and extend h_0 counterclockwise around v_0. The horocycle h_0 will meet an edge e_0 of P_0, which is glued to an edge of some polygon P_1 meeting ideal vertex v_1 identified to v. Note h_0 meets e_0 at a right angle. It extends to a

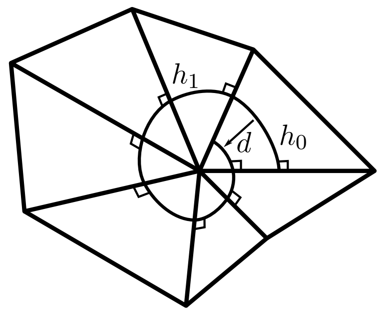

FIGURE 3.8. Extending a horocyle: view inside the manifold.

unique horocycle h_1 about v_1 in P_1. Continue extending the horocycle in this manner, obtaining horocycles $h_2, h_3, \ldots$. Since we only have a finite number of polygons with a finite number of vertices, eventually we return to the vertex v_0 of P_0, obtaining a horocycle h_n about that ideal vertex. Note h_n may not agree with the initial horocycle h_0. See figure 3.8.

DEFINITION 3.13. Let $d(v)$ denote the signed distance between h_0 and h_n on P_0. See figure 3.8. The sign is taken such that if h_n is closer to v_0 than h_0, then $d(v)$ is positive. This is the direction shown in the figure.

LEMMA 3.14. *The value $d(v)$ does not depend on the initial choice of horocycle h_0, nor on the initial choice of v_0 in the equivalence class of v.*

PROOF. Exercise. □

It may be easier to compute $d(v)$ if we look at polygons in $\mathbb{H}^2$, using terminology of developing map and holonomy.

Fix an ideal vertex on one of the polygons P. Put P in $\mathbb{H}^2$ with v at infinity. Now take h_0 to be a horocycle centered at infinity intersected with P. Follow h_0 to the right. When it meets the edge of P, a new polygon is glued. The developing map instructs us how to embed that new polygon as a polygon in $\mathbb{H}^2$, with one edge the vertical geodesic which is the edge of P. Continue along this horocycle, placing polygons in $\mathbb{H}^2$ according to their developing image. Eventually the horocycle will meet P again with v at infinity. When this happens, the developing map will instruct us to glue a copy of P to the given edge. This copy of P will be isometric to the original copy of P, where the isometry is the holonomy of the closed path which encircles the ideal vertex v once in the clockwise direction (why?). This holonomy isometry, call it T, takes the horocycle h_0 on our original copy of P to a horocycle $T(h_0)$, and $T(h_0)$ will be of distance $d(v)$ from the extended horocycle that began with h_0. See figure 3.9.

PROPOSITION 3.15. *Let S be a surface with hyperbolic structure obtained by gluing hyperbolic polygons. Then the metric on S is complete if and only if $d(v) = 0$ for each ideal vertex v.*

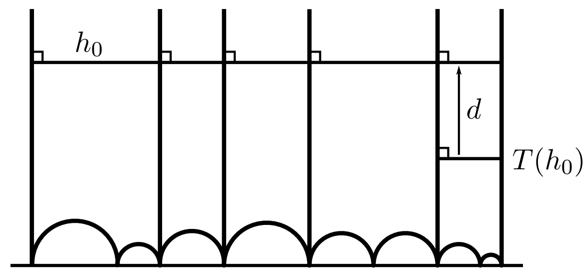

FIGURE 3.9. Extending a horocycle.

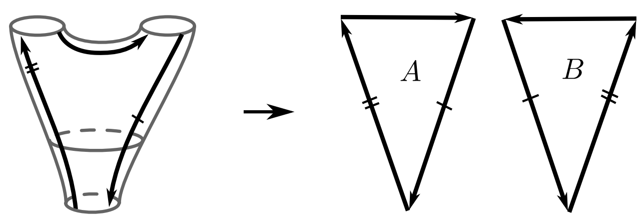

FIGURE 3.10. Topological polygonal decomposition for the 3-punctured sphere.

Before we prove this proposition, let's look at an example.

EXAMPLE 3.16 (Complete 3-punctured sphere). A topological polygonal decomposition for the 3-punctured sphere consists of two ideal triangles. See figure 3.10.

Let's try to construct a geometric polygonal decomposition by building the developing image. We can put one of the ideal triangles in $\mathbb{H}^2$ as the triangle with vertices at 0, 1, ∞. If we glue the other triangle immediately to the right, we have two vertices at 1 and at ∞, but the third can go to any point x, where $x > 1$. See figure 3.11. These two triangles on the left, labeled A and B, give a fundamental region for the 3-punctured sphere. The developing image will be created by gluing additional copies of these two triangles to edges in the figure by holonomy isometries.

We may choose the position of the next copy of the triangle A glued to the right, putting its vertex at the point y as in figure 3.11. After this choice, notice we cannot choose where the next vertex of B to the right will go. This is because the choice y determines an isometry of $\mathbb{H}^2$ taking the triangle A on the left to the triangle labeled A on the right. This isometry is exactly the holonomy element corresponding to the closed curve running once around the vertex at infinity. The same isometry, which has been determined with the choice of y, must take B in the middle to the next triangle glued to the right in our figure. In fact, now that we know this holonomy element, we may apply it and its inverse successively to the triangles of figure 3.11, and we obtain the entire developing image of all triangles adjacent to infinity.

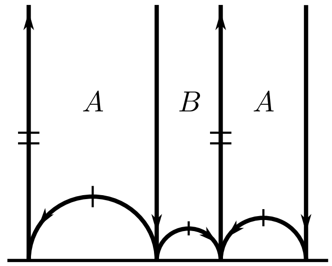

FIGURE 3.11. We may choose any $x > 1$, $y > x$ when finding a hyperbolic structure.

Recall that we want our hyperbolic structure to be complete. By proposition 3.15, we need to look at horocycles. Pick a collection of horocycles about the vertices 0, 1, and ∞. Each of these horocycles extends to give a new horocycle about another copy of A. Each copy of A is obtained by applying a holonomy isometry to the original triangle with vertices at 0, 1, and ∞. We want the horocycles obtained under these holonomy isometries to agree with the horocycles obtained by extending the original horocycles. This is the condition for completeness.

Here is one way to determine complete structures. Let ℓ_1 denote the distance in $\mathbb{H}^2$ between the horocycle at infinity and the horocycle at 0. See figure 3.12. The holonomy element ψ, corresponding to the group element fixing the ideal vertex at infinity, is an isometry of $\mathbb{H}^2$, hence it preserves distances. Thus, under this isometry, the distance between the image of the horocycle at infinity and the horocycle at $\psi(0) = x$ must also be ℓ_1. If the structure is complete, then the horocycle about infinity is preserved by ψ. Thus the horocycle at x must have the same (Euclidean) diameter as the horocycle at 0.

Now consider the length of the edge between horocycles at 0 and 1, labeled ℓ_3 in figure 3.12. There is another holonomy isometry ϕ mapping the geodesic edge between 0 and 1 to one between x and 1, corresponding to the group element encircling the ideal vertex at 1. Again completeness implies that the horocycle at 1 is fixed by ϕ. The horocycle at 0 maps to the horocycle centered at x, and again because ϕ is an isometry, the distance between horocycles centered at 1 and x must still be ℓ_3. We already determined the fact that the horocycle at x has the same (Euclidean) diameter as the one at 0. The only possible way that the distance ℓ_3 will also be preserved is if $x = 2$, and the picture is symmetric across the edge from 1 to infinity.

Note at this point that the holonomy ψ is completely determined: It fixes ∞, takes 0 to 2, and maps a point $i\,h$ on a horocycle about infinity to the point $2 + i\,h$. This is the translation $\psi(z) = z + 2$. Similarly, the holonomy ϕ is also completely determined, as it fixes 1, maps 0 to 2 and takes a point on a horocycle on the edge of length ℓ_3 to a determined point on the edge

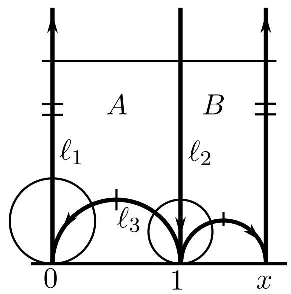

FIGURE 3.12. Lengths between horocycles

from x to 1. Because the fundamental group of the 3-punctured sphere is generated by the two loops corresponding to ψ and ϕ, this determines the complete structure. We have therefore shown:

PROPOSITION 3.17. *There is a unique complete hyperbolic structure on the 3-punctured sphere. A fundamental region for the structure is given by two ideal triangles with vertices* 0, 1, *and* ∞ *and* 1, 2, *and* ∞, *respectively.* □

EXAMPLE 3.18 (Incomplete structure on 3-punctured sphere). What if we choose a different value for x besides $x = 2$? Say we let $x = 3/2$. To simplify things, let's keep the length of the edge between horocycles at 0 and 1 constant as we extend horocycles. Choose horocycles at 0 and 1 of (Euclidean) radius $1/2$, so that these horocycles are tangent along the edge between 0 and 1, hence the distance between horocycles is 0. This distance will remain equal to 0 under each holonomy element, so there will be a horocycle at $x = 3/2$ tangent to the horocycle about 1, to preserve distance 0. This determines where the image of the triangle A must go under the holonomy fixing infinity: its third vertex (called y above) must have a horocycle about it of the same (Euclidean) size as the horocycle at $3/2$. This determines the holonomy isometry about the vertex at infinity. Apply this holonomy isometry successively, and we obtain a pattern of triangles as in figure 3.13.

Notice that the edges of the triangles approach a limit — the thick line shown on the far right of the figure. Notice also that this line is not part of the developing image of the 3-punctured sphere.

This hyperbolic structure is incomplete: for any horocycle about infinity in $\mathbb{H}^2$, the sequence of points at the intersection of the horocycle and the edges of the developing images of ideal triangles projects to a Cauchy sequence that does not converge. Alternately, the value $d(v)$ is nonzero for v the ideal vertex lifting to the point at infinity.

An incomplete metric space may be completed by adjoining points corresponding to limits of Cauchy sequences, and giving the resulting space the metric topology. In our case, the completion of this incomplete 3-punctured sphere is obtained by attaching a geodesic of length $d(v)$: each point of the

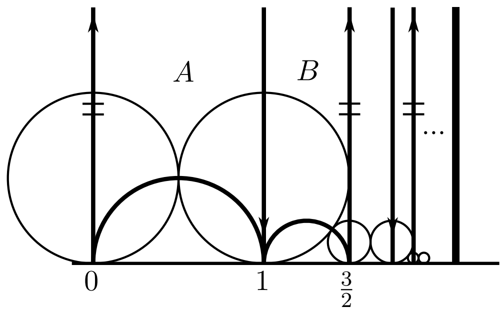

FIGURE 3.13. Part of developing image of an incomplete structure on a 3-punctured sphere.

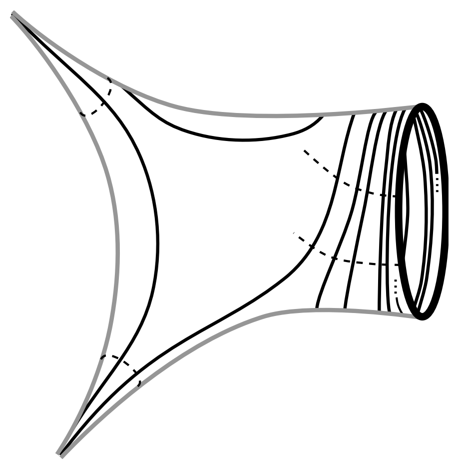

FIGURE 3.14. The completion of an incomplete structure on a 3-punctured sphere. Attach a geodesic of length $d(v)$. Ideal edges spin arbitrarily close to the attached geodesic without meeting it. Horocycles (dashed) run directly into the geodesic. (This example has two complete cusped ends and one incomplete end.)

thick geodesic on the right of figure 3.13 corresponds to the limiting point of the Cauchy sequence given by a horocycle about infinity at the appropriate height.

Note that horocycles about infinity run straight into this thick geodesic, meeting it at right angles. On the other hand, these horocycles meet infinitely many edges of ideal triangles on their way into the geodesic, and none of these ideal edges meets the geodesic. It follows that the ideal edges of the two triangles A and B become arbitrarily close to the geodesic attached in the completion, without ever meeting it. Geometrically, it appears that the edges of the ideal triangles spin around the geodesic infinitely many times, while horocycles run directly into it. See figure 3.14.

Proof of proposition 3.15. Suppose $d(v)$ is nonzero. Then take a sequence of points on a horocycle about v, one point for each intersection of the horocycle with an ideal edge. This gives a Cauchy sequence that does not converge. Therefore, the metric is not complete.

Now suppose $d(v) = 0$ for each ideal vertex v. Then some horocycle closes up around each ideal vertex, so we may remove the interior horoball from each polygon. After this removal, the closure of the remainder is a compact manifold with boundary. For any $t > 0$, let S_t be the compact manifold obtained by removing interiors of horocycles of distance t from our original choice of horocycle. Then the compact subsets S_t of S satisfy $\bigcup_{t\in\mathbb{R}^+} S_t = S$ and S_{t+a} contains a neighborhood of radius a about S_t. Any Cauchy sequence must be contained in some S_t for sufficiently large t. Hence by compactness of S_t, the Cauchy sequence must converge. □

3.3. Developing map and completeness

Here is a better condition for completeness that works in all dimensions and all geometries.

Theorem 3.19. *Let M be an n-manifold with a (G, X)-structure, where G acts transitively on X, and X admits a complete G-invariant metric. Then the metric on M inherited from X is complete if and only if the developing map $D\colon \widetilde{M} \to X$ is a covering map.*

Proof. Suppose first that the developing map $D\colon \widetilde{M} \to X$ is a covering map. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in M. For n large enough, x_n will be contained in an ϵ-ball in M that is evenly covered in $\widetilde{M}$. Thus the sequence lifts to a Cauchy sequence $\{\widetilde{x}_n\}$ in $\widetilde{M}$. Since D is a local isometry, $\{D(\widetilde{x}_n)\}$ is a Cauchy sequence in X. Finally since X is complete, $\{D(\widetilde{x}_n)\}$ converges to $y \in X$. Now, because D is a covering map, there is a neighborhood U of y that is evenly covered by D. Lift this to a neighborhood $\widetilde{U}$ of $\widetilde{M}$ containing infinitely many points of the sequence $\{\widetilde{x}_n\}$. The lift of y in this neighborhood, call it $\widetilde{y}$, must be a limit point of $\{\widetilde{x}_n\}$. Then the projection of $\widetilde{y}$ to M is a limit point of the sequence $\{x_n\}$, so M is complete.

For the converse, we appeal to a proof by Thurston [**Thurston, 1997**, Proposition 3.4.15]. Suppose M is complete. To show $D\colon \widetilde{M} \to X$ is a covering map, we show that any path α_t in X lifts to a path $\widetilde{\alpha}_t$ in $\widetilde{M}$. Since D is a local homeomorphism, this implies that D is a covering map.

First, if M is complete, then $\widetilde{M}$ must also be complete, where the metric $\widetilde{M}$ is the lift of the metric on M, as follows. The projection to M of any Cauchy sequence gives a Cauchy sequence in M, with limit point x. Then x has a compact neigbhorhood which is evenly covered in $\widetilde{M}$, hence there is a compact neighborhood in $\widetilde{M}$ containing all but finitely many points of the Cauchy sequence and also containing a lift of x. Thus the sequence converges in $\widetilde{M}$.

Let α_t be a path in X. Because D is a local homeomorphism, we may lift α_t to a path $\widetilde{\alpha_t}$ in $\widetilde{M}$ for $t \in [0, t_0)$, some $t_0 > 0$. By completeness of $\widetilde{M}$, the lifting extends to $[0, t_0]$. But because D is a local homeomorphism, a lifting to $[0, t_0]$ extends to $[0, t_0 + \epsilon)$. Hence the lifting extends to all of α_t and D is a covering map. □

Corollary 3.20. *If X is simply connected, and M is a manifold with a (G, X)-structure as in theorem 3.19, then M is complete if and only if the developing map is an isometry of X.*

Proof. The developing map is a local isometry by construction. Theorem 3.19 shows that M is complete if and only if the developing map is a covering map. Since X and $\widetilde{M}$ are simply connected, the developing map is a covering map if and only if it is a covering isomorphism. A covering isomorphism that is a local isometry must be an isometry. □

3.4. Exercises

Exercise 3.1. We have seen that Euclidean structures on a torus are determined by a parallelogram.

(a) Show that by applying translation and rotation isometries of $\mathbb{E}^2$, we may assume that the parallelogram has vertices $(0, 0)$, $(x_1, 0)$, (x_2, y), and $(x_1 + x_2, y)$ where $x_1 > 0$ and $y > 0$.
(b) Show that up to rescaling, a parallelogram has vertices $(0, 0)$, $(1, 0)$, (x, y), and $(x + 1, y)$ for some $(x, y) \in \mathbb{R}^2$ with $y > 0$.

Exercise 3.2. If X is a metric space, and G is a group of isometries acting transitively on X, and M is a manifold admitting a (G, X)-structure, show that M inherits a metric from X. That is, explain how to define a metric on M from that on X, and show that the metric is well-defined.

Exercise 3.3. (Induced structures [**Thurston, 1997**, Exercise 3.1.5]). Let $\pi\colon N \to M$ be a local homeomorphism from a topological space N into a manifold M with a (G, X)-structure. Prove N has a (G, X)-structure that is preserved by π. As a corollary, show that any covering space of M admits a (G, X)-structure.

Exercise 3.4. Prove item (1) of proposition 3.10. That is, show that for a fixed basepoint x_0 and fixed initial chart (U_0, ϕ_0) with $x_0 \in U_0$, the map $D\colon \widetilde{M} \to X$ is well defined, independent of choice of other charts, points $t_1, \dots, t_n$, and independent of the choice of α in the homotopy class of $[\alpha] \in \widetilde{M}$.

Exercise 3.5. Prove item (3) of proposition 3.10. Show that if we define a new map in the same way as D, except we change the basepoint x_0 or the initial chart (U_0, ϕ_0), then the resulting map is equal to the composition of D with an element of G.

EXERCISE 3.6. Let T be the affine torus obtained by identifying the sides of the trapezoid with vertices $(0,0)$, $(1,0)$, $(0,1)$, and $(1/2,1)$.

(a) Compute the holonomy elements of T corresponding to meridian and longitude (i.e. the loop running along the horizontal edge of the trapezoid and the loop running along the vertical edge of the trapezoid). What is the holonomy group of T?

(b) For basepoint $(0,0)$ and initial chart chosen so that the trapezoid is mapped by the identity into $\mathbb{R}^2$, compute explicitly the developing images of various curves, including the following:
- The curve running twice along the meridian (based at $(0,0)$).
- The curve running twice along the longitude.
- The curve running twice along the meridian and three times along the longitude.

EXERCISE 3.7. Let T be the affine torus of exercise 3.6, obtained by identifying sides of the trapezoid with vertices $(0,0)$, $(1,0)$, $(0,1)$, and $(1/2,1)$, and let $\widetilde{T}$ denote its universal cover. Prove that the developing image $D(\widetilde{T}) \subset \mathbb{R}^2$ misses exactly one point.

EXERCISE 3.8. Generalize exercise 3.7: Let T be any affine torus. Prove that either the developing map $D\colon \widetilde{T} \to \mathbb{R}^2$ is a covering map, and T is a Euclidean torus, or the image of the developing map misses a single point in $\mathbb{R}^2$.

EXERCISE 3.9. Fix an example of your favorite quadrilateral that is not a parallelogram, and let T be the torus obtained by identifying sides. Use a computer to create a picture such as figure 3.3 for your quadrilateral.

EXERCISE 3.10. Prove lemma 3.14, that $d(v)$ is independent of initial choice of horocycle, and independent of choice of v_0 in the equivalence class of v.

EXERCISE 3.11. Prove the holonomy group of the complete structure on a 3-punctured sphere is generated by

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

EXERCISE 3.12. How many incomplete hyperbolic structures are there on a 3-punctured sphere? How can they be parameterized? Give a geometric interpretation of this parameterization. That is, relate the parameterization to the developing image of the associated hyperbolic structure.

EXERCISE 3.13. A torus with 1 puncture has a topological polygonal decomposition consisting of two triangles.

(a) Find a complete hyperbolic structure on the 1-punctured torus and prove your structure is complete.

(b) Find all complete hyperbolic structures on the 1-punctured torus. How are they parameterized?

EXERCISE 3.14. A sphere with 4 punctures has a topological polygonal decomposition consisting of four triangles. Repeat exercise 3.13 for the 4-punctured sphere.