CHAPTER 5

Discrete groups and the thick–thin decomposition

Suppose we have a complete hyperbolic structure on an orientable 3-manifold. Then the developing map $D\colon \widetilde{M} \to \mathbb{H}^3$ is a covering map, by theorem 3.19. Since $\widetilde{M}$ and $\mathbb{H}^3$ are both simply connected, it follows that the developing map is an isometry. Thus we may view $\mathbb{H}^3$ as the universal cover of M. The covering transformations are then the elements of the holonomy group $\rho(\pi_1(M)) = \Gamma \leq \mathrm{PSL}(2,\mathbb{C})$. Hence M is homeomorphic to the quotient $M \cong \mathbb{H}^3/\Gamma$.

Subgroups Γ of $\mathrm{PSL}(2,\mathbb{C})$ can have very nice properties, and have been investigated since the 1960s and 1970s. In this chapter, we discuss some classical results in the area and their consequences for hyperbolic 3-manifolds. Some of our discussion follows closely work of Jørgensen and Marden; we recommend the book [**Marden, 2007**] for more details, generalizations, and consequences.

5.1. Discrete subgroups of hyperbolic isometries

5.1.1. Isometries and subgroups. In theorem 2.16 we classified elements of $\mathrm{PSL}(2,\mathbb{C})$ as elliptic, parabolic, or loxodromic depending on their fixed points. One of the first things we need is an extension of that theorem.

Before we give the extension, recall that we can view an element of $\mathrm{PSL}(2,\mathbb{C})$ as a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ with } a,b,c,d \in \mathbb{C} \text{ and } ad - bc = 1,$$

and the matrix is well-defined up to multiplication by $\pm\mathrm{Id}$. In this chapter, we will frequently write an isometry of $\mathbb{H}^3$ as a 2 by 2 matrix with determinant 1, omitting and ignoring the $\pm$ sign. The sign very rarely affects our arguments, but the reader should be aware that we are suppressing it, for example in the following definition.

Definition 5.1. We say $A \in \mathrm{PSL}(2,\mathbb{C})$ is *conjugate* to $B \in \mathrm{PSL}(2,\mathbb{C})$ if there exists $U \in \mathrm{PSL}(2,\mathbb{C})$ such that $A = UBU^{-1}$. The *trace* of A is the trace of its normalized matrix:

$$\mathrm{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d.$$

Note there is a sign ambiguity in our definition of trace; again this will not affect our arguments. Note also that conjugate elements have the same trace.

LEMMA 5.2. *For* $A \in \mathrm{PSL}(2, \mathbb{C})$,

- A *is* parabolic *if and only if* $\mathrm{tr}(A) = \pm 2$*, and if and only if* A *is conjugate to*
$$z \mapsto z + 1.$$
- A *is* elliptic *if and only if* $\mathrm{tr}(A) \in (-2, 2) \subset \mathbb{R} \subset \mathbb{C}$*, and if and only if* A *is conjugate to*
$$z \mapsto e^{2i\theta} z, \qquad \text{with } 2\theta \neq 2\pi n \text{ for any } n \in \mathbb{Z}.$$
- A *is* loxodromic *if and only if* $\mathrm{tr}(A) \in \mathbb{C} - [-2, 2]$*, and if and only if* A *is conjugate to*
$$z \mapsto \rho^2 z, \quad \text{with } |\rho| > 1.$$

PROOF. Exercise 5.2 □

DEFINITION 5.3. A subgroup of $\mathrm{PSL}(2, \mathbb{C})$ is said to be *discrete* if it contains no sequence of distinct elements converging to the identity element. A discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$ is often called a *Kleinian group*.

An example of a discrete group is a subgroup generated by a single loxodromic element, or a single parabolic element. These are the simplest such groups. They are so simple that they are examples of what are called *elementary groups*; see definition 5.11. Examples of discrete groups in general can be quite complicated. In proposition 5.10, we will prove that the holonomy group of any complete hyperbolic 3-manifold is always a discrete group. Meanwhile, consider the example of the figure-8 knot complement.

EXAMPLE 5.4. Let K be the figure-8 knot, and give $S^3 - K$ its complete hyperbolic structure by gluing two regular ideal tetrahedra, with face pairings as in figure 4.10. We will find generators of the holonomy group, which is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$, as we will see in proposition 5.10. These are obtained by face pairing isometries, as follows.

Place the two ideal tetrahedra in $\mathbb{H}^3$, putting ideal vertices for one tetrahedron at 0, 1, ω, and ∞, where $\omega = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, and putting the ideal vertices of the other tetrahedron at 1, ω, $\omega + 1$, and ∞. This glues the faces labeled A along the ideal triangle with vertices 1, ω, and ∞, to obtain one connected fundamental region for the knot complement, shown in figure 5.1.

The manifold $S^3 - K$ is obtained by gluing the remaining faces labeled B, C, and D. These gluings, or face-pairings, correspond to holonomy isometries, which we will denote by T_B, T_C, and T_D, respectively. A calculation (exercise 5.3) shows that the gluing isometries are given by:

$$T_B = \begin{pmatrix} \omega & \omega \\ \omega & 1 \end{pmatrix}, \quad T_C = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}, \quad T_D = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}. \tag{5.1}$$

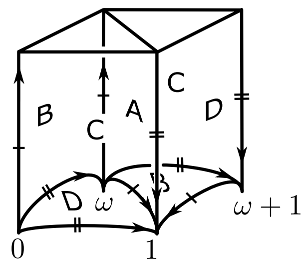

FIGURE 5.1. A connected fundamental region for the figure-8 knot complement

These three gluing isometries generate the holonomy group for $S^3 - K$. In fact, T_B can be written as a (somewhat complicated) product involving T_C and T_D and their inverses.

Riley was the first to prove that $S^3 - K$ has a hyperbolic structure [**Riley, 1975**]. He did so by taking a presentation of the fundamental group of $S^3 - K$ with two generators, and finding an explicit representation of the fundamental group into $\mathrm{PSL}(2,\mathbb{C})$. Exercises 5.4, 5.5, and 5.6 explore this work a little further.

We finish this section with one quick condition equivalent to a group being discrete.

LEMMA 5.5. *A subgroup* $G \leq \mathrm{PSL}(2,\mathbb{C})$ *is discrete if and only if it does not contain an infinite sequence of distinct elements that converges to some element* $A \in \mathrm{PSL}(2,\mathbb{C})$.

PROOF. One implication is trivial: If G is not discrete, by definition it contains an infinite sequence of distinct elements converging to the identity in $\mathrm{PSL}(2,\mathbb{C})$.

For the other direction, suppose $\{A_n\} \subset G$ is an infinite sequence of distinct elements of G converging to $A \in \mathrm{PSL}(2,\mathbb{C})$. Consider $\{A_{n+1}A_n^{-1}\} \subset G$. Note the sequence converges to the identity. To show G is not discrete, it remains to show that $\{A_{n+1}A_n^{-1}\}$ contains infinitely many distinct elements. Suppose not. Then $A_{n+1} = CA_n$ for some fixed $C \in G$ and some subsequence. Since $A_{n+1}A_n^{-1} \to \mathrm{Id}$, we must have $C = \mathrm{Id}$, and thus $A_{n+1} = A_n$. This contradicts the fact that $\{A_n\}$ is a sequence of distinct elements. Thus G is not discrete. □

5.1.2. Sequences of isometries. We can learn a lot about subgroups of $\mathrm{PSL}(2,\mathbb{C})$ by considering sequences of group elements. For example, note that the definition of a discrete group involves sequences. We also have the following result, which will be used later in the chapter.

LEMMA 5.6. *Let* $\{A_n\}$ *be a sequence of elements of* $\mathrm{PSL}(2,\mathbb{C})$. *Then either a subsequence of* $\{A_n\}$ *converges to some* $A \in \mathrm{PSL}(2,\mathbb{C})$, *or there exists a point* $q \in \partial\mathbb{H}^3$ *such that for all* $x \in \mathbb{H}^3$, *the sequence* $\{A_n(x)\}$ *has a subsequence converging to* q.

PROOF. Let p_n, q_n denote the fixed points of A_n; note we could have $p_n = q_n$. Then $\{p_n\}$ and $\{q_n\}$ are sequences in $\partial\mathbb{H}^3 \cong S^2$, which is compact, so they have convergent subsequences. Replace A_n, p_n, q_n by a subsequence such that $p_n \to p$ and $q_n \to q$. Again note that p could equal q.

Case 1. If $p \neq q$, then for large enough n, $p_n \neq q_n$. Consider an isometry R_n of $\mathbb{H}^3$ mapping p_n to 0 and q_n to ∞. For concreteness, we may take R_n to map some point $y \in \partial\mathbb{H}^3$ to 1, where y is disjoint from $\{p_n\}$, $\{q_n\}$, p, and q. If we view R_n as a sequence of matrices for example, we see that R_n converges to the hyperbolic isometry $R \in \mathrm{PSL}(2,\mathbb{C})$ taking p to 0, q to ∞, and y to 1. Consider $B_n = R_n A_n R_n^{-1}$. This is an isometry in $\mathrm{PSL}(2,\mathbb{C})$ fixing 0 and ∞. Hence it has the form $B_n(z) = a_n z$ for $a_n \in \mathbb{C}$. If $\{|a_n|\}$ has a bounded subsequence, then some subsequence a_n converges to $a \in \mathbb{C}$. Hence there is a subsequence B_n with $B_n \to B$, where B is the hyperbolic isometry $B(z) = az$. This is an element of $\mathrm{PSL}(2,\mathbb{C})$. It follows that $A_n = R_n^{-1} B_n R_n$ converges to $A = R^{-1}BR \in \mathrm{PSL}(2,\mathbb{C})$.

If $|a_n| \to \infty$, then for any $z \in \partial\mathbb{H}^3$, $B_n(z) \to \infty$. Thus for any point $x \in \mathbb{H}^3$, $B_n(x) \to \infty$. It follows that for all $x \in \mathbb{H}^3$, $A_n(x) = R_n^{-1} B_n R_n(x)$ converges to $q \in \partial\mathbb{H}^3$.

Case 2. Now suppose $p = q$. Then again we will conjugate A_n by an isometry R_n taking q_n to infinity. For concreteness, choose y_1 and y_2 disjoint from $\{p_n\}$, $\{q_n\}$, and q. Let R_n be the isometry taking y_1 to 1, y_2 to 0, and q_n to ∞. Then R_n converges to the isometry R taking y_1, y_2, and q to 1, 0, and ∞, respectively. Finally let $B_n = R_n A_n R_n^{-1}$. Note B_n fixes ∞, hence it is of the form $B_n = a_n z + b_n$ for a_n, $b_n \in \mathbb{C}$. If $a_n = 1$, B_n is parabolic and has unique fixed point ∞. Otherwise, the other fixed point of B_n is $b_n/(1-a_n)$.

If $\{|b_n|\}$ has a bounded subsequence, then some subsequence $b_n \to b$. In that case, either $a_n = 1$ for large n, or since p_n, q_n converge to $p = q$, the fixed point $b_n/(1-a_n)$ converges to ∞. Thus a_n converges to 1. In any case, $B_n(z)$ converges to $B(z) = z + b$. This is an element of $\mathrm{PSL}(2,\mathbb{C})$. It follows that $A_n = R_n^{-1} B_n R_n$ converges to $R^{-1}BR \in \mathrm{PSL}(2,\mathbb{C})$.

If $\{|b_n|\}$ has no bounded subsequence, then $b_n \to \infty$. We know $b_n/(1-a_n) \to \infty$ because it is a fixed point of B_n, so $(1-a_n)/b_n \to 0$. Rewrite B_n to have the form

$$B_n(z) = b_n\left(\frac{(a_n-1)z}{b_n} + 1\right) + z.$$

Then as $n \to \infty$, $B_n(z) \to \infty$ for all $z \in \partial\mathbb{H}^3$. Thus $B_n(x) \to \infty$ for all $x \in \mathbb{H}^3$. It follows that $A_n(x) = R_n^{-1} B_n R_n(x)$ converges to q for all $x \in \mathbb{H}^3$. □

5.1.3. Action of groups of isometries. We return to the problem of showing that holonomy groups of complete hyperbolic 3-manifolds are discrete. We will show this by considering the action of these groups on $\mathbb{H}^3$.

DEFINITION 5.7. The action of a group $G \leq \mathrm{PSL}(2,\mathbb{C})$ on $\mathbb{H}^3$ is *properly discontinuous* if for every closed ball $B \subset \mathbb{H}^3$, the set $\{\gamma \in G \mid \gamma(B) \cap B \neq \emptyset\}$ is a finite set.

DEFINITION 5.8. The action of a group $G \leq \mathrm{PSL}(2,\mathbb{C})$ is *free* if the identity element of G is the only element to have a fixed point in $\mathbb{H}^3$.

Note that parabolics and loxodromics have fixed points on $\partial\mathbb{H}^3$, but not in the interior of $\mathbb{H}^3$. However, elliptics have fixed points in the interior of $\mathbb{H}^3$. Thus the action of G is free if and only if G contains no elliptics.

LEMMA 5.9. *A subgroup of* $\mathrm{PSL}(2,\mathbb{C})$ *is discrete if and only if its action on* $\mathbb{H}^3$ *is properly discontinuous.*

PROOF. Suppose G is a subgroup of $\mathrm{PSL}(2,\mathbb{C})$ that is not discrete, so there exists a sequence $\{A_n\}$ in G with $A_n \to \mathrm{Id}$. Then for all $x \in \mathbb{H}^3$, the hyperbolic distance $d(x, A_n x) \to 0$. Let B be any closed ball about x with radius $R > 0$. For n such that $d(x, A_n x) < R$, the set

$$\{A \in G \mid A(B) \cap B \neq \emptyset\}$$

contains A_n. Since this is true for infinitely many A_n, the action is not properly discontinuous.

Now suppose that for $G \leq \mathrm{PSL}(2,\mathbb{C})$, there exists a closed ball B of radius R such that the set $\{A \in G \mid A(B) \cap B \neq \emptyset\}$ is infinite. Let $\{A_n\}$ be a sequence of distinct elements in this set. Note that for $x \in B$, the hyperbolic distance $d(x, A_n x)$ is bounded by $4R$, for all n. Thus $\{A_n x\}$ has no subsequence converging to a point on $\partial\mathbb{H}^3$. Lemma 5.6 implies that $\{A_n\}$ has a subsequence converging to $A \in \mathrm{PSL}(2,\mathbb{C})$. Then lemma 5.5 implies G is not discrete. ☐

We are now ready to prove the main result in this section, namely that a complete hyperbolic 3-manifold has a discrete holonomy group, and conversely a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$ that acts freely gives rise to a complete hyperbolic 3-manifold.

PROPOSITION 5.10. *The action of a group* $G \leq \mathrm{PSL}(2,\mathbb{C})$ *on* $\mathbb{H}^3$ *is free and properly discontinuous if and only if* $\mathbb{H}^3/G$ *is a hyperbolic 3-manifold with covering projection* $\mathbb{H}^3 \to \mathbb{H}^3/G$.

PROOF. Suppose the action of G on $\mathbb{H}^3$ is free and properly discontinuous. Let $x \in \mathbb{H}^3/G$, and let $\widetilde{x} \in \mathbb{H}^3$ be a point that projects to x under the map $\mathbb{H}^3 \to \mathbb{H}^3/G$. Because the action of G is properly discontinuous, there is a closed ball B_x that intersects only finitely many of its translates. Because the action is free, we may shrink B_x until all its translates are disjoint. Then the interior of B_x maps isometrically to a neighborhood of x in $\mathbb{H}^3/G$, so $\mathbb{H}^3/G$ is a hyperbolic manifold. Moreover, this neighborhood is evenly covered (by translates of the interior of B_x), and so the quotient map is a covering projection.

Conversely, suppose $\mathbb{H}^3/G$ is a hyperbolic manifold and $p\colon \mathbb{H}^3 \to \mathbb{H}^3/G$ is a covering projection. For any $x \in \mathbb{H}^3$, the action of G permutes the preimages $\{p^{-1}p(x)\}$. Only the identity of G fixes x, so the action is free.

Let $B \subset \mathbb{H}^3$ be a closed ball. Consider the compact set $B \times B$. For any $(x, y) \in B \times B$, we claim there exist neighborhoods U_{xy} of x and V_{xy} of y such that $g(U_{xy}) \cap V_{xy} \neq \emptyset$ for at most one $g \in G$. To see this, if y is not in the orbit of x, then $p(x)$ and $p(y)$ have disjoint neighborhoods in $\mathbb{H}^3/G$. Shrink these neighborhoods to be evenly covered, and let U_{xy} and V_{xy} be neighborhoods of x and y respectively homeomorphic to the disjoint neighborhoods of $p(x)$ and $p(y)$. For any $g \in G$, $g(U_{xy}) \cap V_{xy} = \emptyset$ in this case. On the other hand, if $y = g_1(x)$ for some $g_1 \in G$, then take U_{xy} to be homeomorphic to an evenly covered neighborhood of $p(x) = p(y)$ in $\mathbb{H}^3/G$, and let $V_{xy} = g_1(U_{xy})$. Then $g(U_{xy}) \cap V_{xy} \neq \emptyset$ only when $g = g_1$.

Now $B \times B$ is compact, and the set $\{U_{xy} \times V_{xy}\}_{(x,y)\in B\times B}$ forms an open cover. Thus there is a finite subcover $\{U_1 \times V_1, \dots, U_n \times V_n\}$, where $U_i \times V_i$ has the property that $g(U_i) \cap V_i \neq \emptyset$ only when $g = g_i \in G$.

If $\gamma \in G$ is such that there exists $x \in \gamma(B) \cap B$, then consider $(\gamma^{-1}(x), x) \in B \times B$. There must be some $U_i \times V_i$ containing $(\gamma^{-1}(x), x)$. Since $x \in \gamma(U_i) \cap V_i$, it follows that $\gamma = g_i$. Thus γ must be one of the elements $g_1, \dots, g_n$ associated to the finite covering. It follows that the action is properly discontinuous. ☐

Proposition 5.10 implies that if $\mathbb{H}^3/G$ is a hyperbolic 3-manifold, then G contains no elliptics. For this reason, we will exclude elliptic elements from discrete groups G whenever possible to simplify our proofs in the rest of the chapter. In fact, many results below also hold for discrete groups that contain elliptics. Details can be found, for example, in Marden [**Marden, 2007**].

5.2. Elementary groups

Definition 5.11. A subgroup $G \leq \mathrm{PSL}(2, \mathbb{C})$ is *elementary* if one of the following holds.

(1) The union of all fixed points of all nontrivial elements of G is a single point on $\partial\mathbb{H}^3$.
(2) The union of all fixed points of all nontrivial elements of G consists of exactly two points on $\partial\mathbb{H}^3$.
(3) There exists $x \in \mathbb{H}^3$ such that for all $g \in G$, $g(x) = x$.

The group is *nonelementary* if it is not elementary.

Elementary groups will be important subgroups of the discrete groups we study. Because of that, we will need to know more about their form.

Proposition 5.12. *Let G be a discrete nontrivial elementary subgroup of* $\mathrm{PSL}(2, \mathbb{C})$ *without elliptics. Then either*

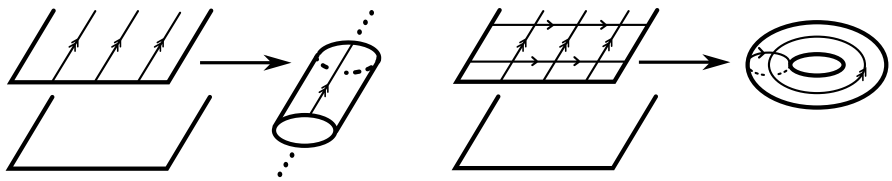

FIGURE 5.2. Left: The quotient of horosphere ∂C under the group $\mathbb{Z}$ generated by a single parabolic gives a cylinder, or annulus. Right: The quotient of ∂C under $\mathbb{Z} \times \mathbb{Z}$ is a torus

(1) the union of fixed points of nontrivial elements of G is a single point on $\partial\mathbb{H}^3$, G is isomorphic to $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$, and G is generated by parabolics (fixing the same point on $\partial\mathbb{H}^3$), or

(2) the union of fixed points of nontrivial elements of G consists of two points on $\partial\mathbb{H}^3$, G is isomorphic to $\mathbb{Z}$, and G is generated by a single loxodromic leaving invariant the line between the fixed points.

PROOF. If the union of all fixed points of nontrivial elements of G consists of a single point on $\partial\mathbb{H}^3$, then G must contain only parabolics fixing that point. Conjugate so that the fixed point is ∞ in $\mathbb{H}^3$. Then G fixes a horosphere about ∞, which is isometric to the Euclidean plane P. The group G acts on P by Euclidean translations. Since G is discrete, G must be generated by either one translation, in which case $G \cong \mathbb{Z}$, or two linearly independent translations, in which case $G \cong \mathbb{Z} \times \mathbb{Z}$.

If the union of all fixed points of nontrivial elements of G consists of two points, then G contains only loxodromics fixing the axis between them. The group G acts on the axis; the fact that the group is discrete means that there is some finite minimal translation distance τ under this group action. Let $A \in G$ realize the minimal translation distance, i.e. $d(x, Ax) = \tau$ for x on the axis. We claim $G = \langle A \rangle$. First, we show all $C \in G$ translate by distance $n\tau$ for some $n \in \mathbb{Z}$, for if some $C \in G$ has translation distance that is not a multiple of τ, then $C(x)$ lies between $A^n(x)$ and $A^{n+1}(x)$ for any x on the axis. But then $CA^{-n} \in G$ translates $A^n(x)$ a distance strictly less than τ, which is a contradiction. Thus all $C \in G$ translate along the axis a distance equal to a multiple of τ. Now suppose $C \in G$ translates by $n\tau$ for some integer n. Then CA^{-n} fixes the axis pointwise. Because G contains no elliptics, $C = A^n$. So G is cyclic generated by A. □

Consider the first case of proposition 5.12.

DEFINITION 5.13. If G is an infinite elementary discrete group in $\mathrm{PSL}(2, \mathbb{C})$ fixing a single point on $\partial\mathbb{H}^3$, we may conjugate G so that fixed point is the point at infinity. Let H be the closed horoball of height 1:

$$H = \{(x, y, z) \mid z \geq 1\}.$$

Proposition 5.12 tells us that G is isomorphic to $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$.

If $G \cong \mathbb{Z}$, the quotient of the horoball H/G is homeomorphic to the space $A \times [1, \infty)$, where A is an annulus, or cylinder; see figure 5.2. We say that H/G is a *rank-1 cusp*.

If $G \cong \mathbb{Z} \times \mathbb{Z}$, the quotient of the horoball H/G is homeomorphic to $T \times [1, \infty)$, where T is a Euclidean torus; see figure 5.2, right. We say that H/G is a *rank-2 cusp*.

Proposition 5.12 has an immediate corollary giving information about $\mathbb{Z} \times \mathbb{Z}$ subgroups of discrete groups, which will be used in later chapters.

Corollary 5.14 ($\mathbb{Z}\times\mathbb{Z}$ subgroups). *Suppose a discrete group G without elliptics has a subgroup isomorphic to $\mathbb{Z}\times\mathbb{Z}$. Then the subgroup is generated by two parabolic elements fixing the same point on the boundary of $\mathbb{H}^3$ at infinity.*

Proof. Let A and B denote the generators of the subgroup of G isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Since A and B commute, they must have the same fixed points on the boundary of $\mathbb{H}^3$ at infinity (exercise). Thus $H \cong \langle A, B \rangle$ is an elementary discrete group isomorphic to $\mathbb{Z} \times \mathbb{Z}$. By proposition 5.12, H must be generated by parabolics fixing the same point on $\partial\mathbb{H}^3$. □

Discrete elementary groups are often defined in terms of the set of accumulation points of the group on $\partial\mathbb{H}^3$; for example this is the definition in [**Thurston, 1979**]. We review that definition here as well.

Definition 5.15. Let $G \leq \mathrm{PSL}(2, \mathbb{C})$ be a discrete group, and let $x \in \mathbb{H}^3$ be any point. The *limit set* $\Lambda(G)$ is defined to be the set of accumulation points on $\partial\mathbb{H}^3$ of the orbit $G(x)$.

Lemma 5.16. *The limit set $\Lambda(G)$ is well-defined, independent of choice of x.*

Proof. Suppose $\{A_n\} \subset G$ is a sequence such that $A_n(x)$ converges to a point $p \in \Lambda(G) \subset \partial\mathbb{H}^3$. Let $y \in \mathbb{H}^3$. Then the distance between x and y is a constant, equal to the distance between $A_n(x)$ and $A_n(y)$ for all n. Thus as $n \to \infty$, $A_n(x)$ and $A_n(y)$ lie a bounded distance apart, but $A_n(x)$ approaches p. This is possible only if $A_n(y)$ approaches the same point p on $\partial\mathbb{H}^3$. □

Consider a few examples of groups G and limit sets $\Lambda(G)$. If G is generated by a single loxodromic element g, then its limit set $\Lambda(G)$ consists of the two fixed points of g on $\partial\mathbb{H}^3$: one is an accumulation point for $g^n(x)$, and the other for $g^{-n}(x)$. If G is generated by a single parabolic element, then $\Lambda(G)$ consists of a single point. If G contains both a loxodromic element g and a parabolic element h, then $\Lambda(G)$ contains the fixed points of g on $\partial\mathbb{H}^3$, as well as the fixed points of $h^n \circ g$ for all n; this is a countably infinite set. Finally, if G is the identity group, consisting only of the identity element, then $\Lambda(G)$ is empty.

The following is often given as the definition of an elementary discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$.

LEMMA 5.17. *A discrete subgroup $G \leq \mathrm{PSL}(2,\mathbb{C})$ with no elliptics is elementary if and only if $\Lambda(G)$ consists of* 0, 1, *or* 2 *points.* □

Proposition 5.12 classifies elementary discrete groups without elliptics. We also need the following result giving more information on *nonelementary* discrete groups without elliptics.

LEMMA 5.18. *If G is a nonelementary discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$ that contains no elliptics, then the following hold.*

1. *G is infinite.*
2. *For any nontrivial $A \in G$, there exists a loxodromic $B \in G$ that has no common fixed points with A.*
3. *If $B \in G$ is loxodromic, then there is no nontrivial $C \in G$ that has exactly one fixed point in common with B.*
4. *G contains two loxodromic elements with no fixed points in common.*

PROOF. The group G must be nontrivial; since it contains no elliptics it must contain a loxodromic or parabolic. Such an element has infinite order, so G is infinite, proving (1).

Next we show (3). Suppose B is loxodromic, and C has exactly one fixed point in common with B; we will show that the group generated by B and C is indiscrete, contradicting the fact that G is discrete. Conjugate the group. Lemma 5.2 implies we may assume $B = \begin{pmatrix} \rho & 0 \\ 0 & 1/\rho \end{pmatrix}$, and since C has exactly one fixed point in common with B, it has the form $C = \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}$. Then

$$B^n C B^{-n} C^{-1} = \begin{pmatrix} 1 & ab\,(\rho^{2n}-1) \\ 0 & 1 \end{pmatrix}.$$

If $|\rho| < 1$, let $n \to \infty$. If $|\rho| > 1$, let $n \to -\infty$. In either case, $B^n C B^{-n} C^{-1}$ approaches the parabolic $\begin{pmatrix} 1 & -ab \\ 0 & 1 \end{pmatrix}$. Lemma 5.5 now implies that the subgroup generated by B and C is not discrete, therefore G is not discrete.

Now we show (2). There are two cases depending on whether A is parabolic or loxodromic. Note that if we can show the result for a conjugate group UGU^{-1} for $U \in \mathrm{PSL}(2,\mathbb{C})$, then the result holds for G, so in both cases we will replace G by a conjugate group at the first step.

Case 1. Suppose A is parabolic. Then by lemma 5.2, A is conjugate to $z \mapsto z+1$, so we may assume $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and A fixes ∞. Because G is nonelementary, there exists $C \in G$ that does not fix ∞. If C is loxodromic, we are done. If not, C must be parabolic, and $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \neq 0$. Note that $A^n C$ cannot fix ∞ for any integer n, and $\mathrm{tr}(A^n C) = a + nc + d = nc \pm 2$. For $|n|$ sufficiently large, this cannot be in $[-2, 2]$, so $A^n C$ is the desired loxodromic by lemma 5.2.

Case 2. Suppose A is loxodromic. Then after conjugating, lemma 5.2 implies we may assume $A = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix}$ with $|\rho| > 1$, so A fixes 0 and ∞. Because G is nonelementary and discrete, (3) implies there is $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ that does not fix either 0 or ∞. If C happens to be loxodromic, we are done. If not, C is parabolic, so $a + d = \pm 2$. Then $A^n C$ also has distinct fixed points from those of A for any integer n, and $\mathrm{tr}(A^n C) = a\rho^n + d\rho^{-n}$. For $|n|$ large, this lies outside $[-2, 2]$, hence $A^n C$ is loxodromic by lemma 5.2.

Finally, to prove part (4), we use part (2). Suppose $A \in G$ is not the identity. Then (2) implies there is a loxodromic $B \in G$ with distinct fixed points from A. If A is also loxodromic, we are done. Otherwise, apply (4) to B, to obtain a loxodromic C with no fixed points in common with B. Then B and C are the desired loxodromics. □

Before we close this section, we give one more result we will need on convergence of nonelementary discrete groups. The following theorem is due to Jørgensen and Klein [**Jørgensen and Klein, 1982**], using previous work of Jørgensen [**Jørgensen, 1976**].

Theorem 5.19 (Jørgensen and Klein, 1982). *Let*

$$G_n = \langle A_{1,n}, A_{2,n}, \dots, A_{r,n} \rangle$$

be a sequence of r-generator, nonelementary, discrete subgroups of $\mathrm{PSL}(2,\mathbb{C})$ *such that $A_k = \lim_{n\to\infty} A_{k,n}$ exists and is an element of* $\mathrm{PSL}(2,\mathbb{C})$ *for each k. Then $G = \langle A_1, A_2, \dots, A_r \rangle$ is also nonelementary and discrete. Moreover, for sufficiently large n, the map $A_k \to A_{k,n}$ for each k extends to a homomorphism from G to G_n.*

The proof of theorem 5.19 follows from an analysis of various properties of elements of $\mathrm{PSL}(2,\mathbb{C})$ and discrete subgroups. Its proof is not unlike many of the other results proved in this chapter. However, its proof would lead us a little further afield than we wish to go, into technicalities of $\mathrm{PSL}(2,\mathbb{C})$. The full proof can be found in the original papers; Marden also gives an exposition closely following the original proof in [**Marden, 2007**]. We will refer the interested reader to those references.

Meanwhile, we don't actually need the full strength of theorem 5.19; we only need the following immediate consequence.

Corollary 5.20. *Suppose $\{\langle A_n, B_n \rangle\}$ is a sequence of nonelementary discrete subgroups of* $\mathrm{PSL}(2,\mathbb{C})$ *such that* $\lim A_n = A$ *and* $\lim B_n = B$ *in* $\mathrm{PSL}(2,\mathbb{C})$. *Then $\langle A, B \rangle$ is a nonelementary discrete subgroup of* $\mathrm{PSL}(2,\mathbb{C})$. □

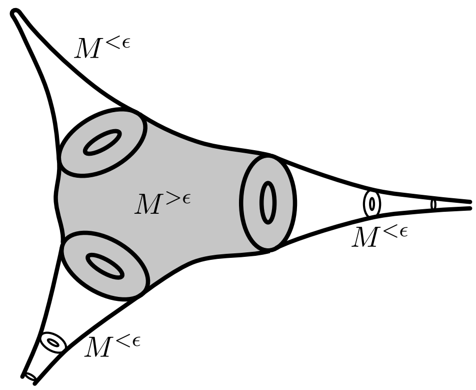

FIGURE 5.3. A schematic picture of a hyperbolic 3-manifold M, with $M^{<\epsilon}$ a collection of cusps and tubes

5.3. Universal elementary neighborhoods

We are now ready to put together facts about elementary and nonelementary discrete groups to prove a remarkable result on the geometry and topology of hyperbolic 3-manifolds, namely that any such manifold decomposes into a thick part and completely classified thin parts. To state the result precisely, we give a few definitions.

DEFINITION 5.21. Suppose M is a complete hyperbolic 3-manifold and $x \in M$. The *injectivity radius* of x, denoted $\mathrm{injrad}(x)$, is defined to be the supremal radius r such that a metric r-ball around x is embedded.

DEFINITION 5.22. Let M be a complete hyperbolic 3-manifold, and let $\epsilon > 0$. Define the *ϵ-thin part* of M, denoted $M^{<\epsilon}$ to be

$$M^{<\epsilon} = \{x \in M \mid \mathrm{injrad}(x) < \epsilon/2\}.$$

Similarly, the *ϵ-thick part*, denoted $M^{>\epsilon}$ is defined to be

$$M^{>\epsilon} = \{x \in M \mid \mathrm{injrad}(x) > \epsilon/2\}.$$

We also have closed versions $M^{\geq\epsilon}$ and $M^{\leq\epsilon}$ defined in the obvious way.

THEOREM 5.23 (Structure of thin part). *There exists a universal constant $\epsilon_3 > 0$ such that for $0 < \epsilon \leq \epsilon_3$, the ϵ-thin part of any complete, orientable, hyperbolic 3-manifold M consists of tubes around short geodesics, rank-1 cusps, and/or rank-2 cusps.*

A cartoon illustrating theorem 5.23 is given in figure 5.3.

DEFINITION 5.24. The supremum of all constants ϵ_3 satisfying theorem 5.23 is called the *Margulis constant*. More generally, given a complete hyperbolic 3-manifold M, a number $\epsilon > 0$ is said to be a *Margulis number for M* if $M^{<\epsilon}$ satisfies the conclusions of theorem 5.23, i.e. $M^{<\epsilon}$ consists of tubes around short geodesics, rank-1, and/or rank-2 cusps. The Margulis constant is therefore the infimum over all complete hyperbolic 3-manifolds M of the supremum of all Margulis numbers for M.

As of the writing of this book, the optimal Margulis constant is still unknown, although there are bounds on its value. The following result summarizes the current state of knowledge.

THEOREM 5.25. *The Margulis constant is at least* 0.104*, and is less than* 0.616*. Moreover,* 0.292 *is a Margulis number for every every non-closed hyperbolic 3-manifold. In addition,* 0.29 *is a Margulis number for all but finitely many hyperbolic 3-manifolds.*

PROOF. R. Meyerhoff gave what is currently the best lower bound on ϵ_3 in [**Meyerhoff, 1987**, Section 9]. As for the upper bound, M. Culler has discovered a closed hyperbolic 3-manifold with Margulis number less than 0.616 using SnapPea [**Weeks, 2005**]. Culler and P. Shalen proved 0.292 is a Margulis number for non-closed 3-manifolds [**Culler and Shalen, 2012**]. Using that result and a limiting argument, Shalen proved the final statement, that 0.29 is a Margulis number for all but finitely many hyperbolic 3-manifolds [**Shalen, 2011**]. □

We save the proof of theorem 5.23 until the end of this section. We will see that it is a consequence of a well-known theorem concerning the structure of discrete groups of isometries, commonly called the Margulis lemma, which appears in a paper of Každan and Margulis [**Každan and Margulis, 1968**]. The actual Margulis lemma is very general, concerning discrete groups acting on symmetric spaces. We restrict to the case of discrete subgroups of $\mathrm{PSL}(2,\mathbb{C})$ acting freely on hyperbolic space. The consequence we will need is the following.

THEOREM 5.26 (Universal elementary neighborhoods). *There is a universal constant $\epsilon_3 > 0$ such that for all $x \in \mathbb{H}^3$, and for any discrete group $G \leq \mathrm{PSL}(2,\mathbb{C})$ without elliptics, if H denotes the subgroup of G generated by all elements of G that translate x distance less than ϵ_3, then H is elementary.*

A few remarks are in order. First, the Margulis lemma holds when we allow elliptics; this appears in Wang [**Wang, 1969**] in the full generality of the theorem of Každan and Margulis. Second, the form of theorem 5.26 above, concerning discrete subgroups of $\mathrm{PSL}(2,\mathbb{C})$, is due to Jørgensen and Marden, only their result is more general in that it also includes elliptics. Their proof appears in [**Marden, 2007**], and is the basis for proof that we include here.

PROOF OF THEOREM 5.26. First we establish some notation. For fixed $x \in \mathbb{H}^3$ and $A \in \mathrm{PSL}(2,\mathbb{C})$, let $d(x, Ax)$ denote the distance in $\mathbb{H}^3$ between x and Ax. For fixed $r > 0$, let $G(r, x)$ denote the set

$$G(r, x) = \{A \in G \mid d(x, Ax) < r\}.$$

The group generated by $G(r, x)$ will be denoted by $\langle G(r, x)\rangle$.

Our goal is to show that there exists $r > 0$ such that for all discrete G and for all x, the group $\langle G(r, x)\rangle$ is elementary.

As a first step, we show that if we fix a discrete group G with no elliptics and fix x, then there exists $r > 0$ such that the group $\langle G(r, x)\rangle$ is elementary. For suppose this is not the case. Then for a sequence $r_n \to 0$, each $\langle G(r_n, x)\rangle$ is nonelementary. It follows that there exists a sequence of distinct $A_n \in G(r_n, x)$ with $d(x, A_n x) < r_n$. But then lemma 5.6 implies that A_n must converge to some $A \in \mathrm{PSL}(2, \mathbb{C})$. Using lemma 5.5, we see that this contradicts the fact that G is a discrete group. So for $r > 0$ sufficiently small, $\langle G(r, x)\rangle$ is elementary, and it follows that $G(r, x)$ contains finitely many elements. By choosing $r > 0$ smaller than the translation distance of each of these elements, we find that $G(r, x)$ contains only the identity element. Note that the identity group is elementary.

Now we will prove the more general result, that there is a universal $r > 0$, independent of G and x, such that $\langle G(r, x)\rangle$ is always elementary. Again suppose not. Then there is a sequence $r_n \to 0$, a sequence of discrete groups $G_n \le \mathrm{PSL}(2, \mathbb{C})$ without elliptics, and a sequence of points $x_n \in \mathbb{H}^3$ such that $\langle G_n(r_n, x_n)\rangle$ is not elementary.

We will simplify the argument by replacing x_n with a fixed x for all n: choose any $x \in \mathbb{H}^3$, and let $R_n \in \mathrm{PSL}(2, \mathbb{C})$ be an isometry mapping x_n to x. Consider the group $R_n G_n R_n^{-1}$. Note that $A \in G_n(r_n, x_n)$ if and only if $R_n A R_n^{-1}$ is in $R_n G_n R_n^{-1}(r_n, x)$, and so $\langle R_n G_n R_n^{-1}(r_n, x)\rangle$ is nonelementary. Thus if we replace G_n by $R_n G_n R_n^{-1}$, we may work with a single fixed value of x. So we assume there is a fixed x and sequences $r_n \to 0$ and G_n so that $\langle G_n(r_n, x)\rangle$ is nonelementary.

Now fix n. Our next goal is to find A_n and B_n in $G_n(r_n, x)$ such that $\langle A_n, B_n\rangle$ is nonelementary. Because $\langle G_n(r_n, x)\rangle$ is nonelementary, lemma 5.18 implies that there exist loxodromics S_n and T_n with no common fixed points in $\langle G_n(r_n, x)\rangle$, and certainly they generate a nonelementary group. However, we need to take some care to ensure that A_n and B_n are actually in $G_n(r_n, x)$. To do this, we use the first part of this proof: consider the groups $\langle G_n(\rho, x)\rangle$ as ρ ranges between 0 and r_n. We have observed that for some $\rho_n < r_n$, the group $\langle G_n(\rho_n, x)\rangle$ will consist only of the identity element. As ρ increases, the sets $G_n(\rho, x)$ will be nested. There will be some value $0 < \mu_n \le r_n$ such that $\langle G_n(\rho, x)\rangle$ is elementary for $\rho < \mu_n$ but $\langle G_n(\mu_n, x)\rangle$ is nonelementary. We may assume $\mu_n = r_n$.

Moreover, there is some $\tau_n < r_n$ such that for $\tau_n \le \rho < r_n$, the groups $\langle G_n(\rho, x)\rangle$ are all elementary and isomorphic, equal to the group $\langle G_n(\tau_n, x)\rangle$.

Suppose that the elementary group $\langle G_n(\tau_n, x)\rangle$ is infinite with two fixed points on $\partial\mathbb{H}^3$. Then proposition 5.12 implies that it contains a loxodromic $A_n \in G_n(\tau_n, x)$ fixing a line ℓ. Since $\langle G_n(r_n, x)\rangle$ is not elementary, $G_n(r_n, x)$ must contain a loxodromic B_n that does not fix ℓ. Then A_n and B_n are loxodromics in $G_n(r_n, x)$ with no common fixed points. So $\langle A_n, B_n\rangle$ is not elementary.

Now suppose that the elementary group $\langle G_n(\tau_n, x)\rangle$ fixes a single point $\zeta \in \partial\mathbb{H}^3$. Then $G_n(\tau_n, x)$ contains a parabolic A_n. Since $\langle G_n(r_n, x)\rangle$ is not elementary, $G_n(r_n, x)$ contains some B_n that does not fix ζ. So again A_n and B_n are elements of $G_n(r_n, x)$ with no common fixed points, and $\langle A_n, B_n\rangle$ is not elementary.

Finally suppose that the elementary group $\langle G_n(\tau_n, x)\rangle$ consists only of the identity element. Since $\langle G_n(r_n, x)\rangle$ is nonelementary with no elliptics, the generating set $G_n(r_n, x)$ must contain two elements A_n and B_n with no common fixed point. Thus $\langle A_n, B_n\rangle$ is not elementary.

In all cases, we have a nonelementary subgroup with two generators, $\langle A_n, B_n\rangle$, and A_n, $B_n \in G_n(r_n, x)$. Note that $A_n(x) \to x$ and $B_n(x) \to x$, so lemma 5.6 implies there are subsequences of $\{A_n\}$ and $\{B_n\}$ converging to $A \in \mathrm{PSL}(2,\mathbb{C})$ and $B \in \mathrm{PSL}(2,\mathbb{C})$, respectively. Then corollary 5.20 implies that $\langle A, B\rangle$ is nonelementary.

On the other hand, $A_n, B_n \in G_n(r_n, x)$, so as $n \to \infty$, A_n and B_n must converge to elements of $\mathrm{PSL}(2,\mathbb{C})$ fixing x. Thus $\langle A, B\rangle$ fixes x, hence it is elementary by definition. This contradiction finishes the proof. □

To relate translation distance as in theorem 5.26 (Universal elementary neighborhoods) to injectivity radius as in theorem 5.23 (Structure of thin part), we give the following lemma.

LEMMA 5.27. *Let M be a complete, orientable, hyperbolic 3-manifold with $M \cong \mathbb{H}^3/\Gamma$ for a discrete group $\Gamma \leq \mathrm{PSL}(2,\mathbb{C})$. For any $x \in M$ with lift $\tilde{x} \in \mathbb{H}^3$,*

$$\mathrm{injrad}(x) = \frac{1}{2} \inf_{A \neq \mathrm{Id} \in \Gamma} \{d(\tilde{x}, A\tilde{x})\}.$$

Moreover, this is realized. That is, there exists nontrivial $A \in \Gamma$ such that $2\mathrm{injrad}(x) = d(\tilde{x}, A\tilde{x})$.

PROOF. A metric r-ball is embedded at x if and only if for all $A \neq \mathrm{Id} \in \Gamma$, the metric r-ball $B(r, \tilde{x})$ is disjoint from the metric r-ball $A(B(r, \tilde{x})) = B(r, A\tilde{x})$. This holds if and only if the translation distance $d(\tilde{x}, A\tilde{x})$ is at least $2r$ for all A.

Now suppose $\mathrm{injrad}(x) = b$. Then a metric b-ball is embedded, but for any $\epsilon > 0$, a metric $b + \epsilon$-ball is not embedded. Thus for each $\epsilon > 0$, there is $A_\epsilon \in \Gamma$ such that $d(\tilde{x}, A_\epsilon(\tilde{x})) < 2(b + \epsilon)$. If the set $\{A_\epsilon\}$ contains infinitely many distinct elements, then we obtain a sequence $\{A_n\}$ such that $A_n(\tilde{x})$ is of bounded distance from $\tilde{x}$. By lemma 5.6, $A_n \to A \in \mathrm{PSL}(2,\mathbb{C})$, implying Γ is not discrete by lemma 5.5. This is a contradiction. Thus $\{A_\epsilon\}$ is a finite set. Let $A \in \Gamma$ be such that $d(\tilde{x}, A\tilde{x})$ is minimal. This A satisfies the conclusion of the lemma. □

We are now ready to complete the proof of theorem 5.23.

PROOF OF THEOREM 5.23 (STRUCTURE OF THIN PART). Take $\epsilon_3 > 0$ as in theorem 5.26. Let $M \cong \mathbb{H}^3/\Gamma$ be a compete, orientable, hyperbolic 3-manifold, so $\Gamma \leq \mathrm{PSL}(2,\mathbb{C})$ is a discrete subgroup with no elliptics.

For $\epsilon \leq \epsilon_3$, if $x \in M^{<\epsilon}$, then by definition injrad$(x) < \epsilon/2$. By lemma 5.27, it follows that there exists $A \neq \mathrm{Id} \in \Gamma$ such that $d(\tilde{x}, A\tilde{x}) < \epsilon$ for any lift $\tilde{x}$ of x. But theorem 5.26 implies that the subgroup Γ_ϵ of Γ generated by all $A \in \Gamma$ such that $d(\tilde{x}, A\tilde{x}) < \epsilon$ is elementary. Since Γ_ϵ contains $A \neq \mathrm{Id}$, proposition 5.12 implies that Γ_ϵ either fixes a single point $\zeta \in \partial\mathbb{H}^3$ and is generated by parabolics fixing ζ, or Γ_ϵ is generated by a single loxodromic preserving an axis $\ell \subset \mathbb{H}^3$.

Suppose first that Γ_ϵ fixes a single point $\zeta \in \partial\mathbb{H}^3$. Then Γ_ϵ is generated by one or two parabolics (proposition 5.12), and $\tilde{x}$ lies on a horosphere H about ζ that is fixed by Γ_ϵ. Suppose $\tilde{y}$ lies in the horoball bounded by H. Then the height of $\tilde{y}$ is at least C: $\tilde{y}$ has coordinates $(a + b\,i, t)$ with $t \geq C$. A generator A of Γ_ϵ takes $\tilde{y}$ to a point with the same height t. A calculation in this case (exercise 5.14) shows that

$$\epsilon > d(\tilde{x}, A\tilde{x}) \geq d(\tilde{y}, A\tilde{y}),$$

and it follows that in the quotient $\mathbb{H}^3/\Gamma$, the point $\tilde{y}$ maps to $M^{<\epsilon}$. Since this is true for every point in the horoball bounded by H, $M^{<\epsilon}$ contains the quotient of a horoball under the elementary group Γ_ϵ; this is a rank-1 or rank-2 cusp.

Now suppose that H is generated by a single loxodromic A preserving the axis ℓ. Let R denote the distance from $\tilde{x}$ to the axis ℓ, and let T_R denote the set of points in $\mathbb{H}^3$ of distance R from the axis ℓ. Then T_R bounds a tube consisting of all points in $\mathbb{H}^3$ of distance at most R from ℓ. If $\tilde{y}$ is any point within this tube, then one can calculate (exercise 5.15) that $d(\tilde{y}, A\tilde{y}) \leq d(\tilde{x}, A\tilde{x}) < \epsilon$, so $M^{<\epsilon}$ contains the quotient of a tube about ℓ under the elementary group $\langle A \rangle$. This is a tube around a short geodesic. □

5.4. Hyperbolic manifolds with finite volume

In chapter 4 we gave a method that will allow us to compute (complete) hyperbolic structures on many 3-manifolds, including many knot complements. Once we have a hyperbolic structure on a 3-manifold, we have equipped the manifold with a Riemannian metric with very nice properties, for example the metric can be described in local coordinates by equation (2.4).

One of the simplest invariants we can compute from a hyperbolic metric is the volume of the underlying manifold. This gives a good measure of the "size" of the manifold. In chapter 13, we will discuss volumes in some detail, including how to compute volumes of hyperbolic 3-manifolds including knot and link complements. Meanwhile, we close this chapter with an application of the thick–thin decomposition of hyperbolic 3-manifolds to classifying those with finite volume.

THEOREM 5.28. *A hyperbolic 3-manifold M has finite volume if and only if M is closed (compact without boundary), or M is homeomorphic to the interior of a compact manifold $\overline{M}$ with torus boundary components.*

PROOF. If M is closed then a fundamental domain for M in its universal cover $\mathbb{H}^3$ is a compact set, hence has finite volume. If M is the interior of a manifold with torus boundary, then each such boundary component will be realized as a cusp in the complete hyperbolic structure on M. The complement of the cusps of M in M is compact, hence has finite volume. We now show that each cusp has finite volume.

Consider the universal cover $\mathbb{H}^3$. For any cusp C, we may apply an isometry to $\mathbb{H}^3$ so that the point at infinity projects to that cusp, and a horoball of height 1 projects to an embedded horoball neighborhood of the cusp. On the horosphere of height 1, some parallelogram A will be a fundamental region for the torus of the cusp, since the structure is complete (theorem 4.10). Then the volume of the cusp is given by

$$\int_C d\,\mathrm{vol} = \int_{t=1}^{\infty}\int_A d\,\mathrm{vol} = \int_{t=1}^{\infty}\int_A \frac{dx\,dy\,dt}{t^3} = \frac{1}{2}\,\mathrm{area}(A).$$

(See exercise 2.13.) Thus every cusp has finite volume. Since the volume of M is the sum of the volumes of the compact region with cusps removed, as well as a finite number of finite-volume cusps, the manifold M has finite volume.

To prove the converse, we use theorem 5.23. Suppose M is a complete hyperbolic manifold with finite volume. Fix $\epsilon > 0$ less than the universal constant ϵ_3 of theorem 5.23, and consider $M^{<\epsilon}$ and $M^{\geq\epsilon}$. By theorem 5.23, $M^{<\epsilon}$ consists of cusps and tubes. Note that a rank-1 cusp has infinite volume, hence since M has finite volume, $M^{<\epsilon}$ consists of rank-2 cusps and tubes, each of which has finite volume. On the other hand, $M^{\geq\epsilon}$ has finite volume. Moreover, any point in $M^{\geq\epsilon}$ is contained in an embedded ball of radius at least $\frac{1}{2}\epsilon$. If two points in $M^{\geq\epsilon}$ have distance at least ϵ, then the balls of radius $\frac{1}{2}\epsilon$ about each are disjointly embedded in $M^{\geq\epsilon}$. Thus a collection of points with pairwise distance at least ϵ in $M^{\geq\epsilon}$ leads to a pairwise disjoint collection of $\epsilon/2$-balls. Because M has finite volume there can only be finitely many of these. Starting with any such collection of points, we may complete the collection to a maximal collection of points of $M^{\geq\epsilon}$ of distance at least ϵ; there are finitely many of these and the $\epsilon/2$-balls around each are embedded. Then the closed ϵ-balls about the collection must contain $M^{\geq\epsilon}$. The union of these balls is a compact set, and $M^{\geq\epsilon}$ is a closed subset. Hence $M^{\geq\epsilon}$ is compact.

Now the union of $M^{\geq\epsilon}$ and any tubes of $M^{<\epsilon}$ is the union of compact sets, hence compact. This is a manifold with boundary homeomorphic to a finite collection of tori corresponding to the finite number of cusps of $M^{<\epsilon}$. Attach a closed collar neighborhood of each torus boundary component, and call the result N; each collar neighborhood is homeomorphic to $T^2 \times [0,1]$, where T^2 is a torus. Then by construction, the manifold M is homeomorphic to the interior of N. □

By theorem 5.28, the complement of any knot or link in S^3 with a hyperbolic structure must have finite hyperbolic volume.

5.5. Exercises

EXERCISE 5.1. Is a subgroup of $\mathrm{PSL}(2,\mathbb{C})$ generated by a single elliptic element always discrete? Prove it is discrete, or give a counterexample.

EXERCISE 5.2. Prove lemma 5.2, giving more properties of parabolic, elliptic, and loxodromics in $\mathrm{PSL}(2,\mathbb{C})$.

EXERCISE 5.3. Prove that the gluing isometries for the figure-8 knot complement are the elements of $\mathrm{PSL}(2,\mathbb{C})$ given in equation equation (5.1).

EXERCISE 5.4. R. Riley gave a presentation of the fundamental group of the figure-8 knot complement in [**Riley, 1975**]:

$$\pi_1(S^3 - K) = \langle a, b \mid yay^{-1} = b \rangle,$$

where $y = a^{-1}bab^{-1}$. He let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ -\sigma & 1 \end{pmatrix},$$

where σ is a primitive cube root of unity, and let

$$\rho\colon\ \pi_1(S^3 - K) \to \langle A, B \rangle \leq \mathrm{PSL}(2,\mathbb{C})$$

be the representation $\rho(a) = A$, $\rho(b) = B$. Prove the representation ρ gives an isomorphism of groups.

EXERCISE 5.5. Let A and B in $\mathrm{PSL}(2,\mathbb{C})$ be as in exercise 5.4. Find an explicit element $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $\mathrm{PSL}(2,\mathbb{C})$ such that Riley's A and B are conjugate via U to our isometries T_C and T_D^{-1}, respectively. That is, find U such that

$$A = UT_CU^{-1}, \quad B = UT_D^{-1}U^{-1}.$$

Even better: U can be written as a composition of a parabolic fixing infinity T, followed by a rotation R: $U = RT$. Find T and R.

EXERCISE 5.6. Note that Riley's isometries A and B of exercise 5.4 do not give face pairings of the fundamental domain in figure 5.1. Find a fundamental domain for the figure-8 knot such that A and B are face-pairing isometries.

Hint: exercise 5.5 might be helpful.

EXERCISE 5.7. If a group G acts on Euclidean space $\mathbb{R}^n$ or hyperbolic space $\mathbb{H}^n$, extend the definitions of properly discontinuous and free actions in the obvious way.

Show directly by definitions that each of the following groups G acts freely and properly discontinuously on the given space X.

(1) $X = \mathbb{R}^2$, G is generated by two translations $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $\psi\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $\phi(x, y) = (x + t, y)$ and $\psi(x, y) = (x, y + s)$ for $s, t \in \mathbb{R}$.
(2) $X = \mathbb{H}^2$, G is the holonomy group of the (complete) 3-punctured sphere.
(3) $X = \mathbb{H}^3$, G is generated by face-pairing isometries of an ideal polyhedron such that the face identifications give a complete hyperbolic 3-manifold.

EXERCISE 5.8. Show that the following give finite elementary groups.

(1) Cyclic groups fixing an axis in $\mathbb{H}^3$.
(2) Orientation preserving symmetries of an ideal platonic solid (tetrahedron, octahedron/cube, icosohedron/dodecahedron).
(3) Dihedral groups preserving an ideal polygon with n sides inscribed in a plane in $\mathbb{H}^3$.

EXERCISE 5.9. Show that the finite groups in exercise 5.8 are the only finite elementary groups.

EXERCISE 5.10. Let G be a subgroup of $\mathrm{PSL}(2, \mathbb{C})$. Show that the following are equivalent.

(1) G is discrete.
(2) G has no limit points in the interior of $\mathbb{H}^3$. That is, for any $x \in \mathbb{H}^3$, there is no $y \in \mathbb{H}^3$ and no sequence of distinct elements $\{A_n\}$ in G such that $A_n(y) = x$.

EXERCISE 5.11. Let A and B in $\mathrm{PSL}(2, \mathbb{C})$ be distinct from the identity. Prove that the following are equivalent.

(a) A and B commute.
(b) Either A and B have the same fixed points, or A and B have order 2 and each interchanges the fixed points of the other.
(c) Either A and B are parabolic with the same fixed point at infinity, or the axes of A and B coincide, or A and B have order 2 and their axes intersect orthogonally in $\mathbb{H}^3$.

EXERCISE 5.12. Suppose that A and B in $\mathrm{PSL}(2, \mathbb{C})$ are loxodromics with exactly one fixed point in common. Show that $\langle A, B\rangle$ is not discrete.

EXERCISE 5.13. State and prove a version of theorem 5.23, the structure of the thin part, for hyperbolic 2-manifolds.

EXERCISE 5.14. Suppose A is a parabolic fixing the point ζ and p is a point in $\mathbb{H}^3$ such that $d(p, A(p)) < \epsilon$. After applying an isometry, we may assume that $\zeta = \infty$, that $A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ for some $\alpha \in \mathbb{C}$, and p lies on a horosphere H_C that is a Euclidean plane of constant height $t = C$ for some $C > 0$:

$$H_C = \{(x + y\, i, C) \mid C > 0\}.$$

(a) Prove that if a point q lies inside the horoball bounded by H_C on a horosphere H_t of height $t \geq C$, then the Euclidean distance from q to $A(q)$ measured along H_t is at most the Euclidean distance from p to $A(p)$ measured along H.

(b) Prove that the hyperbolic distances, measured in $\mathbb{H}^3$, satisfy

$$\epsilon > d(p, A(p)) \geq d(q, A(q)).$$

EXERCISE 5.15. Suppose A is a loxodromic fixing an axis ℓ, and p is a point in $\mathbb{H}^3$ such that $d(p, A(p)) < \epsilon$.

(1) Prove that the distance from any $q \in \mathbb{H}^3$ to ℓ is the same as the distance from $A(q)$ to ℓ.

(2) We can use cylindrical coordinates in $\mathbb{H}^3$ about the geodesic ℓ. Let r denote the distance from ℓ, θ the rotation about ℓ (measured modulo 2π), and ζ the translation distance along ℓ. Finally, let $\widehat{\mathbb{H}}^3$ denote the cover of $\mathbb{H}^3$ in which θ is no longer measured modulo 2π, but is a real number.

Using these coordinates, it can be shown that the distance d between points p_1 and p_2 in $\widehat{\mathbb{H}}^3$ with cylindrical coordinates (r_1, θ_1, ζ_1) and (r_2, θ_2, ζ_2) with $|\theta_1 - \theta_2| < \pi$ is given by

$$\cosh d = \cosh(\zeta_1 - \zeta_2) \cosh r_1 \cosh r_2 - \cos(\theta_1 - \theta_2) \sinh r_1 \sinh r_2.$$

(See [**Gabai et al., 2001**, Lemma 2.1])

Using this formula, prove that if $x, y \in \mathbb{H}^3$ are such that $d(y, \ell) \leq d(x, \ell)$, then

$$d(y, A(y)) \leq d(x, A(x)).$$