CHAPTER 9

Volume and angle structures

Those hyperbolic 3-manifolds that admit a triangulation by positively oriented geometric triangulations exhibit many additional nice properties. The existence of such a triangulation often gives a simpler way to prove many results in hyperbolic geometry. We present some of the techniques and consequences in this chapter.

In the theory of knots and links, these tools have been applied to great effect to an infinite class of knots and links called 2-bridge links, which we will describe (and triangulate) in the next chapter.

9.1. Hyperbolic volume of ideal tetrahedra

Ideal tetrahedra are building blocks of many complete hyperbolic manifolds. In this section, we will calculate volumes of ideal tetrahedra.

Recall that a hyperbolic ideal tetrahedron is completely determined by $z \in \mathbb{C}$ with positive imaginary part, as in definition 4.4. It is also determined by three dihedral angles, as the following lemma shows.

LEMMA 9.1. *Let α, β, γ be angles in $(0,\pi)$ such that $\alpha+\beta+\gamma=\pi$. Then α, β, and γ determine a unique hyperbolic ideal tetrahedron up to isometry of $\mathbb{H}^3$. Conversely, any hyperbolic tetrahedron determines unique $\{\alpha,\beta,\gamma\} \subset (0,\pi)$ with $\alpha+\beta+\gamma=\pi$.*

PROOF. First we prove the converse. Given an ideal tetrahedron with ideal vertices on $\partial\mathbb{H}^3$ at 0, 1, ∞, and z, note that a horosphere about ∞ intersects the tetrahedron in a Euclidean triangle. Let α, β, γ denote the interior angles of the triangle; these are dihedral angles of the tetrahedron. Each angle α, β, γ lies in $(0,\pi)$, and the sum $\alpha+\beta+\gamma=\pi$, as desired. Exercise 2.11 shows that taking a different collection of vertices to 0, 1, and ∞ will give the same dihedral angles α, β, γ, so these three angles are uniquely determined by the tetrahedron.

Now, suppose α, β, and γ in $(0,\pi)$ are given, with $\alpha+\beta+\gamma=\pi$. Then these three numbers determine a Euclidean triangle, uniquely up to scale, with interior angles α, β, γ. View the triangle as lying in $\mathbb{C}$; we may adjust such a triangle so that it has vertices at 0, 1, and some $z \in \mathbb{C}$ with positive imaginary part. This determines a tetrahedron with edge parameter z. If we rotate and scale the triangle so that different vertices map to 0 and 1, this corresponds to mapping different ideal vertices of the tetrahedron to 0 and 1. The parameter z will be adjusted as in lemma 4.6, but the tetrahedron will be the same up to isometry. □

Lemmas 9.1 and 4.6 give two different ways of uniquely describing an ideal tetrahedron, either by a single complex number z or by a triple of angles α, β, γ with $\alpha + \beta + \gamma = \pi$. We will compute volumes of an ideal tetrahedron, and we choose to compute volumes using a parameterization by angles rather than edge parameter, although computations can be done either way. (See exercises.)

DEFINITION 9.2. The *Lobachevsky function* $\Lambda(\theta)$ is the function defined by

$$\Lambda(\theta) = -\int_0^\theta \log|2\sin u|\,du.$$

THEOREM 9.3. *Suppose α, β, and γ are angle measures strictly between 0 and π, and suppose $\alpha + \beta + \gamma = \pi$, so they determine a hyperbolic ideal tetrahedron $\Delta(\alpha, \beta, \gamma)$. Then the volume* $\mathrm{vol}(\Delta(\alpha, \beta, \gamma))$ *is equal to*

$$\mathrm{vol}(\Delta(\alpha, \beta, \gamma)) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma),$$

where Λ is the Lobachevsky function of definition 9.2.

EXAMPLE 9.4. The figure-8 knot complement has complete hyperbolic structure built of two regular ideal tetrahedra. Therefore the volume of the figure-8 knot complement is $6\Lambda(\pi/3)$, which can be numerically calculated to be approximately 2.0299.

Our proof of theorem 9.3 follows that given by Milnor in [**Milnor, 1982**] and also in [**Thurston, 1979**, Chapter 7]. Milnor, in turn, credits Lobachevsky for several of his calculations.

First, we need a lemma concerning the Lobachevsky function.

LEMMA 9.5. *The Lobachevsky function $\Lambda(u)$ satisfies:*

(1) It is well-defined and continuous on $\mathbb{R}$ (even though the defining integral is improper).
(2) $\Lambda(-\theta) = -\Lambda(\theta)$, i.e. $\Lambda(\theta)$ is odd.
(3) $\Lambda(\theta)$ is periodic of period π.
(4) It satisfies the expression $\Lambda(2\theta) = 2\Lambda(\theta) + 2\Lambda(\theta + \pi/2)$.

PROOF. To prove the lemma, we will relate the Lobachevsky function to the well-known *dilogarithm function*

$$\psi(z) = \sum_{n=1}^{\infty} z^n/n^2 \quad \text{for } |z| \leq 1. \tag{9.1}$$

For more information on the dilogarithm, see for example [**Zagier, 2007**]. Note that for $|z| < 1$, the derivative of $\psi(z)$ satisfies

$$\psi'(z) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n} = \frac{1}{z}\left(\sum_{n=1}^{\infty} \frac{z^n}{n}\right).$$

The sum on the right hand side is a well-known Taylor series:

$$-\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \quad \text{for } |z| < 1.$$

Thus the analytic continuation of $\psi(z)$ is given by

$$\psi(z) = -\int_0^z \frac{\log(1-u)}{u}\, du \quad \text{for } z \in \mathbb{C} - [1,\infty). \tag{9.2}$$

For $0 < u < \pi$, consider $\psi(e^{2iu}) - \psi(1)$. Although the integral formula equation (9.2) above is not defined at $z = 1$, the summation of equation (9.1) is defined and continuous at $z = 1$ (in fact, it can be shown that $\psi(1) = \pi^2/6$), so we may write

$$\psi(e^{2iu}) - \psi(1) = -\int_1^{e^{2iu}} \frac{\log(1-w)}{w}\, dw.$$

Substitute $w = e^{2i\theta}$ into this expression to obtain

$$\begin{aligned}
\psi(e^{2iu}) - \psi(1) &= -\int_{\theta=0}^{u} \log(1-e^{2i\theta})\,(2i)\,d\theta \\
&= -\int_0^u \log\left(-2ie^{i\theta}\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)\right)(2i)\,d\theta \\
&= -\int_0^u 2i(\log(-i) + \log(e^{i\theta}) + \log(2\sin\theta))\,d\theta \\
&= -\int_0^u (\pi - 2\theta + 2i\log(2\sin\theta))\,d\theta.
\end{aligned}$$

Take the imaginary parts of both sides of the above equation. Note $\psi(1)$ is real, hence

$$\Im(\psi(e^{2iu}) - \psi(1)) = \Im(\psi(e^{2iu})) = \Im\left(\sum_{n=1}^{\infty} \frac{e^{2inu}}{n^2}\right) = \sum_{n=1}^{\infty} \frac{\sin(2nu)}{n^2}.$$

On the other side, this equals

$$\Im(\psi(e^{2iu}) - \psi(1)) = 2\int_0^u -\log(2\sin\theta)\,d\theta = 2\Lambda(u).$$

Thus for $0 \leq u \leq \pi$, we have the uniformly convergent Fourier series for $\Lambda(u)$ given by

$$\Lambda(u) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{\sin(2nu)}{n^2} \quad \text{for } 0 \leq u \leq \pi. \tag{9.3}$$

This shows $\Lambda(u)$ is well-defined and continuous for $0 \leq u \leq \pi$. It also shows that $\Lambda(u)$ can be defined on $-\pi \leq u \leq 0$, and it is an odd function on this range. Finally, it shows that $\Lambda(0) = \Lambda(\pi) = 0$.

Notice now that the derivative $d\Lambda(\theta)/d\theta = -2\log|2\sin\theta|$ is periodic of period π. Then for $\theta > \pi$,

$$\begin{aligned}\Lambda(\theta) = \int_0^{\theta} \Lambda'(u)\,du &= \int_0^{\pi} \Lambda'(u)\,du + \int_{\pi}^{\theta} \Lambda'(u)\,du \\ &= \Lambda(\pi) + \int_0^{\theta-\pi} \Lambda'(u)\,du = \Lambda(\theta - \pi),\end{aligned}$$

by the periodicity of Λ', and the fact that $\Lambda(\pi) = 0$. This shows that Λ is well-defined and continuous for $\theta \geq 0$; a similar result implies it is well-defined and continuous for $\theta \leq 0$, and it will be odd everywhere.

It only remains to show the last item of the lemma. To do so, begin with the identity

$$2\sin(2\theta) = 4\sin\theta\cos\theta = (2\sin\theta)(2\sin(\theta + \pi/2)).$$

Then note that

$$\begin{aligned}\Lambda(2\theta) &= \int_0^{2\theta} -\log|2\sin u|\,du \\ &= 2\int_0^{\theta} -\log|2\sin(2w)|\,dw \quad \text{(letting } w = u/2\text{)} \\ &= 2\int_0^{\theta} -\log|2\sin w|\,dw + 2\int_0^{\theta} -\log|2\sin(w+\pi/2)|\,dw \\ &= 2\Lambda(\theta) + 2\int_{\pi/2}^{\theta+\pi/2} -\log|2\sin v|\,dv \\ &= 2\Lambda(\theta) + 2\Lambda(\theta + \pi/2) - 2\Lambda(\pi/2).\end{aligned}$$

Finally, note that if we substitute $u = \pi/2$ into equation (9.3), we obtain $\Lambda(\pi/2) = 0$. This finishes the proof of the lemma. □

REMARK 9.6. Item (4) of lemma 9.5 is a special case of more general identities known as the *Kubert identities*, which have the following form. For any nonzero integer n,

$$\Lambda(n\theta) = \sum_{k=0}^{n-1} n\Lambda(\theta + k\pi/n).$$

You are asked to prove these identities in the exercises.

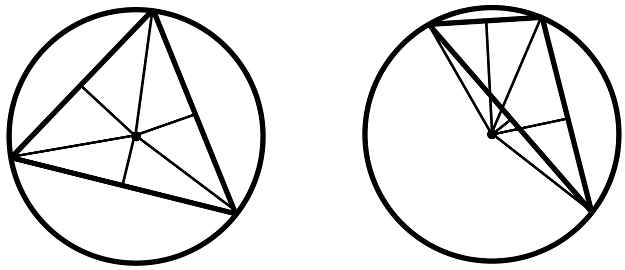

FIGURE 9.1. Left is a tetrahedron for which the point $(0,0)$ lies in the interior of the triangle on $\mathbb{C}$, right is one for which it is exterior. Both show subdivisions into six triangles.

To prove theorem 9.3, we will subdivide our ideal tetrahedron into six 3-dimensional simplices, each simplex with some finite and some infinite vertices. Such a simplex will be described by a region in $\mathbb{H}^3$. To obtain the volume, we integrate the hyperbolic volume form $d\,\mathrm{vol} = dx\,dy\,dz/z^3$ over the region describing the simplex, and then sum the six results.

More carefully, given an ideal tetrahedron in $\mathbb{H}^3$, we have been viewing the tetrahedron as having vertices 0, 1, ∞, and z. The three points 0, 1, and z determine a Euclidean circle on $\mathbb{C}$, which is the boundary of a Euclidean hemisphere, giving a hyperbolic plane in $\mathbb{H}^3$. To this picture, apply a hyperbolic isometry that takes the circle on $\mathbb{C}$ through 0, 1, z to the unit circle in $\mathbb{C}$, taking 0, 1, z to some points p, q, r on $S^1 \subset \mathbb{C}$.

Now, drop a perpendicular from ∞ to the hemisphere; this will be a vertical ray from $(0,0,1) \in \mathbb{H}^3$ to ∞. There will be two cases to consider: the case that the point $(0,0) \in \mathbb{C}$ is interior to the triangle determined by p, q, r, and the case that the point $(0,0)$ is exterior to that triangle. The cases are shown in figure 9.1.

Consider first the case that the point $(0,0)$ is interior to the triangle determined by p, q, and r. Then the ray from $(0,0,1)$ to ∞ lies interior to the tetrahedron. Now, on the hemisphere whose boundary is the unit sphere, draw perpendicular arcs from $(0,0,1)$ to each edge of the tetrahedron lying on that hemisphere. Also draw arcs from $(0,0,1)$ to the vertices of the tetrahedron, as shown in figure 9.1. Now cone to ∞. This divides the original tetrahedron up into six simplices.

LEMMA 9.7. *Each of the six simplices obtained as above has the following properties:*

(1) It has two finite vertices and two ideal vertices.
(2) Three of its dihedral angles are $\pi/2$, the other dihedral angles are ζ, ζ, and $\pi/2 - \zeta$ for some $\zeta \in (0, \pi/2)$.

PROOF. Note that by construction, the two ideal vertices are at ∞ and one of p, q, r, i.e. one of the vertices of the original ideal tetrahedron. The other vertices are at $(0,0,1)$, and some point on the unit hemisphere where

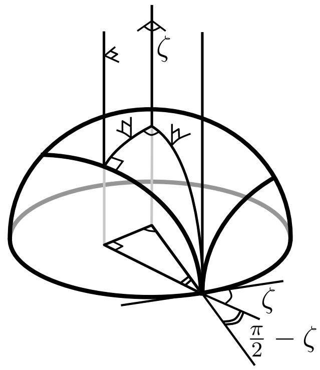

FIGURE 9.2. One of the six simplices obtained from subdividing an ideal tetrahedron

the line from $(0, 0, 1)$ meets an edge of the original tetrahedron in a right angle.

Consider the dihedral angles of the faces meeting infinity. Each of these is a cone (to ∞) over an edge on the unit hemisphere. The dihedral angles agree with the dihedral angles of the vertical projection of the simplex to $\mathbb{C}$, which is the triangle T shown in figure 9.1; these angles are $\pi/2$, ζ, and $\pi/2 - \zeta$ for some $\zeta \in (0, \pi/2)$. The fourth face of the tetrahedron lies on the hemisphere. It meets both vertical faces through $(0, 0, 1)$ in right angles. The final face is a subset of a vertical plane whose boundary on $\mathbb{C}$ is a line containing a side of the projection triangle T. The angle this vertical plane meets with the unit hemisphere is obtained by measuring the angles between the line and a tangent to the unit circle at the points where these intersect. Notice this angle is complementary to $\pi/2 - \zeta$, hence is ζ. □

Lemma 9.7 is illustrated in figure 9.2.

A simplex with the form of lemma 9.7 is called an *orthoscheme*, named by Scläfli in the 1950s [**Schläfli, 1950, Schläfli, 1953**]. Around that time, he computed volumes of orthoschemes.

LEMMA 9.8. *Let $S(\zeta)$ denote a simplex obtained as above, with properties of lemma 9.7. Then the volume of $S(\zeta)$ is*

$$\mathrm{vol}(S(\zeta)) = \frac{1}{2}\Lambda(\zeta).$$

PROOF. The proof is a computation.

Apply an isometry to $\mathbb{H}^3$ so that one ideal vertex of $S(\zeta)$ lies at ∞, the other on the unit circle, with one of the finite vertices at $(0, 0, 1)$; this is the same position of the simplex in the proof of lemma 9.7 above. When we project vertically to $\mathbb{C}$, we obtain a triangle T with one vertex at 0, one on the unit circle, and the last some $v \in \mathbb{C}$. The angle at v is $\pi/2$, and the other two angles are ζ and $\pi/2 - \zeta$. By applying a Möbius transformation that rotates and reflects (but does not affect volume), we may assume v is the point $\cos(\zeta) \in \mathbb{R} \subset \mathbb{C}$, and the third point, on the unit circle, is the point $\cos(\zeta) + i\,\sin(\zeta)$.

Now the triangle T is described by the region

$$0 \leq x \leq \cos(\zeta) \quad \text{and} \quad 0 \leq y \leq x\tan(\zeta).$$

Then $\mathrm{vol}(S(\zeta))$ is given by

$$\mathrm{vol}(S(\zeta)) = \int_T \int_{z \geq \sqrt{1-x^2-y^2}} d\,\mathrm{vol} = \int_0^{\cos(\zeta)} \int_0^{x\tan(\zeta)} \int_{\sqrt{1-x^2-y^2}}^{\infty} \frac{dz\,dy\,dx}{z^3}$$

Integrating with respect to z, we obtain

$$\mathrm{vol}(S(\zeta)) = \int_0^{\cos(\zeta)} \int_0^{x\tan(\zeta)} \frac{dx\,dy}{2(1-x^2-y^2)},$$

which we rewrite

$$\mathrm{vol}(S(\zeta)) = \int_0^{\cos(\zeta)} \int_0^{x\tan(\zeta)} \frac{dx\,dy}{2((\sqrt{1-x^2})^2-y^2)},$$

and integrate with respect to y:

$$\begin{aligned}
\mathrm{vol}(S(\zeta)) &= \int_0^{\cos(\zeta)} \frac{1}{4\sqrt{1-x^2}} \log\left(\frac{\sqrt{1-x^2}+x\tan\zeta}{\sqrt{1-x^2}-x\tan\zeta}\right) dx \\
&= \int_0^{\cos(\zeta)} \frac{1}{4\sqrt{1-x^2}} \log\left(\frac{\sqrt{1-x^2}\cos(\zeta)+x\sin(\zeta)}{\sqrt{1-x^2}\cos(\zeta)-x\sin(\zeta)}\right) dx.
\end{aligned}$$

Using the substitution $x = \cos(\theta)$, the integral becomes

$$\begin{aligned}
\mathrm{vol}(S(\zeta)) &= \int_{\pi/2}^{\zeta} \frac{1}{4} \log\left(\frac{\sin\theta\cos\zeta+\cos\theta\sin\zeta}{\sin\theta\cos\zeta-\cos\theta\sin\zeta}\right) (-d\theta) \\
&= -\frac{1}{4}\left(\int_{\pi/2}^{\zeta} \log\left(\frac{2\sin(\theta+\zeta)}{2\sin(\theta-\zeta)}\right) d\theta\right) \\
&= \frac{1}{4}\left(\int_{\pi/2}^{\zeta} -\log(2\sin(\theta+\zeta))\,d\theta - \int_{\pi/2}^{\zeta} -\log(2\sin(\theta-\zeta))\,d\theta\right) \\
&= \frac{1}{4}\left(\int_{\pi/2+\zeta}^{2\zeta} -\log(2\sin(u))\,du - \int_{\pi/2-\zeta}^{0} -\log(2\sin(u))\,du\right) \\
&= \frac{1}{4}(\Lambda(2\zeta) - \Lambda(\pi/2+\zeta) + \Lambda(\pi/2-\zeta)).
\end{aligned}$$

To finish, we use lemma 9.5. Since $\Lambda(\theta)$ is periodic of period π, note that $\Lambda(\pi/2-\zeta) = \Lambda(-\pi/2-\zeta)$. Since Λ is an odd function, $\Lambda(-\pi/2-\zeta) = -\Lambda(\pi/2+\zeta)$. Finally, since $\Lambda(2\zeta) = 2\Lambda(\zeta) + 2\Lambda(\zeta+\pi/2)$, the above becomes

$$\mathrm{vol}(S(\zeta)) = \frac{1}{4}(2\Lambda(\zeta) + 2\Lambda(\zeta+\pi/2) - 2\Lambda(\pi/2+\zeta)) = \frac{1}{2}\Lambda(\zeta). \qquad \square$$

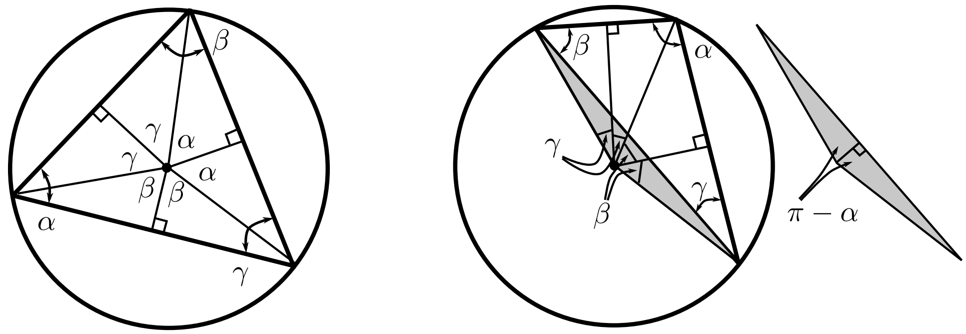

FIGURE 9.3. Left: angles of subsimplies when perpendicular ray lies interior to the tetrahedron. Right: angles when it is exterior

PROOF OF THEOREM 9.3. For an ideal tetrahedron with dihedral angles α, β, and γ, place the tetrahedron in $\mathbb{H}^3$ with vertices at ∞, and at p, q, r all on the unit circle in $\mathbb{C}$. As above, drop a perpendicular ray to the unit hemisphere.

Case 1. Suppose first that the ray lies in the interior of the ideal tetrahedron. Then subdivide the tetrahedron into six simplices as before. Each of the simplices has the properties of lemma 9.7, and is determined by some $\zeta \in (0, \pi/2)$. By lemma 9.8, its volume is determined by ζ as well, so it remains to calculate ζ for each of the six simplices making up the ideal tetrahedron. Project vertically to the complex plane $\mathbb{C}$; the angles determining the simplex can then be easily computed using Euclidean geometry. In particular, there are two with angle α, two with angle β, and two with angle γ. See the left of figure 9.3.

Then the volume of the tetrahedron $\Delta(\alpha, \beta, \gamma)$ is

$$\begin{aligned}\mathrm{vol}(\Delta(\alpha,\beta,\gamma)) &= 2\,\mathrm{vol}(S(\alpha)) + 2\,\mathrm{vol}(S(\beta)) + 2\,\mathrm{vol}(S(\gamma)) \\ &= \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma)\end{aligned}$$

Case 2. Now suppose that the ray from ∞ to the point $(0,0,1)$ lies outside of the ideal tetrahedron. We may still draw perpendicular lines from $(0,0,1)$ to the edges of the ideal tetrahedron on the unit hemisphere, and lines from $(0,0,1)$ to vertices of the ideal tetrahedron; the right of figure 9.3 shows the projection to $\mathbb{C}$ and the corresponding angles. Note that we may still cone to ∞, obtaining six simplices with the properties of lemma 9.7, only now they overlap. However, by adding and subtracting volumes of overlapping simplices, we still will obtain the volume of the ideal tetrahedron. In particular, we have the following.

$$\begin{aligned}\mathrm{vol}(\Delta(\alpha,\beta,\gamma)) &= 2\,\mathrm{vol}(S(\gamma)) + 2\,\mathrm{vol}(S(\beta)) - 2\,\mathrm{vol}(S(\pi-\alpha)) \\ &= \Lambda(\gamma) + \Lambda(\beta) - \Lambda(\pi-\alpha)\end{aligned}$$

Since Λ is an odd function and has period π, $-\Lambda(\pi-\alpha)=\Lambda(\alpha)$. Hence $\mathrm{vol}(\Delta(\alpha,\beta,\gamma))=\Lambda(\alpha)+\Lambda(\beta)+\Lambda(\gamma)$ in this case as well. □

The formula for volume of a tetrahedron has the following useful consequences.

THEOREM 9.9. *Let $\mathcal{A}$ be the set of possible angles on a tetrahedron:*

$$\mathcal{A}=\{(\alpha,\beta,\gamma)\in(0,\pi)^3 \mid \alpha+\beta+\gamma=\pi\}.$$

Then the function vol: $\mathcal{A}\to\mathbb{R}$ *given by*

$$\mathrm{vol}(\alpha,\beta,\gamma)=\Lambda(\alpha)+\Lambda(\beta)+\Lambda(\gamma)$$

is strictly concave down on $\mathcal{A}$. Moreover, we can compute its first two derivatives. For $a=(a_1,a_2,a_3)\in\mathcal{A}$ a point and $w=(w_1,w_2,w_3)\in T_a\mathcal{A}$ a nonzero tangent vector, the first two derivatives of vol *in the direction of w satisfy*

$$\frac{\partial\,\mathrm{vol}}{\partial w}=\sum_{i=1}^{3}-w_i\log\sin a_i, \qquad \frac{\partial^2\,\mathrm{vol}}{\partial w^2}<0.$$

PROOF. First, note that since w is a tangent vector to $\mathcal{A}$, and the sum of the three coordinates of each point in $\mathcal{A}$ is π, it follows that $w_1+w_2+w_3=0$.

Next, by theorem 9.3, the directional derivative of vol at a in the direction of w is given by

$$\begin{aligned}\frac{\partial\,\mathrm{vol}}{\partial w}&=\sum_{i=1}^{3}-w_i\log|2\sin a_i|\\&=\sum_{i=1}^{3}w_i(-\log 2)+\sum_{i=1}^{3}-w_i\log|\sin a_i|\\&=0+\sum_{i=1}^{3}-w_i\log\sin a_i.\end{aligned}$$

The last line holds since $w_1+w_2+w_3=0$ and since $a_i\in(0,\pi)$, hence $\sin a_i>0$.

For the second derivative, we know $a_1+a_2+a_3=\pi$, so at least two of a_1,a_2,a_3 are strictly less than $\pi/2$. Without loss of generality, say a_1 and a_2 are less than $\pi/2$.

Then the second derivative is

$$\frac{\partial^2\,\mathrm{vol}}{\partial w^2}=\sum_{i=1}^{3}-w_i^2\cot a_i.$$

Since $a_3=\pi-a_1-a_2$ and $w_3=-w_1-w_2$, we may write

$$w_3^2\cot a_3=(w_1+w_2)^2\cot(\pi-a_1-a_2)=-(w_1+w_2)^2\frac{\cot a_1\cot a_2-1}{\cot a_1+\cot a_2},$$

where the last equality is an exercise in trig identities.

Then we obtain

$$-\frac{\partial^2 \operatorname{vol}}{\partial w^2} = w_1^2 \cot a_1 + w_2^2 \cot a_2 - (w_1+w_2)^2 \frac{\cot a_1 \cot a_2 - 1}{\cot a_1 + \cot a_2}$$
$$= \frac{(w_1+w_2)^2 + (w_1 \cot a_1 - w_2 \cot a_2)^2}{\cot a_1 + \cot a_2}.$$

The denominator of the last fraction is positive, because $a_1, a_2 \in (0, \pi/2)$. The numerator is the sum of squares, hence at least zero. In fact, if it equals zero, then we have $w_1 = -w_2$ and $\cot a_1 = -\cot a_2$. But $a_1, a_2 \in (0, \pi/2)$, so this is impossible. Thus numerator and denominator are strictly positive, and so $\partial^2 \operatorname{vol} / \partial w^2$ is strictly negative, hence strictly concave down. □

Theorem 9.10. *The regular ideal tetrahedron, with dihedral angles $\alpha = \beta = \gamma = \pi/3$, maximizes volume over all ideal tetrahedra.*

Proof. Because vol is continuous, we know it obtains a maximum on the cube $[0, \pi]^3$. First we consider the boundary of that cube, and we show the maximum cannot occur there. If any angle is π, then $\alpha + \beta + \gamma = \pi$ implies the other two angles are 0. Thus to show the maximum does not occur on the boundary of the cube, it suffices to show the maximum does not occur when one of the angles is zero. So suppose $\alpha = 0$. Since $\Lambda(0) = \Lambda(\pi) = 0$ by equation (9.3), and since $\Lambda(\beta) + \Lambda(\pi - \beta) = \Lambda(\beta) + \Lambda(-\beta) = 0$ by lemma 9.5, the volume in this case will be 0. So the maximum does not occur on the boundary.

Thus we seek a maximum in the interior. We maximize $\operatorname{vol}(\alpha, \beta, \gamma) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma)$ subject to the constraint $\pi = \alpha + \beta + \gamma =: f(\alpha, \beta, \gamma)$. The theory of Lagrange multipliers tells us that at the maximum, there is a scalar λ such that

$$\nabla \operatorname{vol} = \lambda \nabla f, \quad \text{or}$$
$$\log \sin \alpha = \log \sin \beta = \log \sin \gamma = \lambda.$$

This will be satisfied when $\sin \alpha = \sin \beta = \sin \gamma$. Since $\alpha, \beta, \gamma \in (0, \pi)$, and $\alpha + \beta + \gamma = \pi$, it follows that $\alpha = \beta = \gamma = \pi/3$, and the tetrahedron is regular. □

9.2. Angle structures and the volume functional

Note that in theorem 9.3, we showed that the volume of an ideal tetrahedron can be computed given only its dihedral angles. A dihedral angle can be obtained by taking the imaginary part of the log of a tetrahedron's edge invariant. Thus the imaginary parts alone of the edge invariants allow us to assign a volume to the structure. These are exactly the angles of an angle structure.

Recall from definition 8.21 that we defined an angle structure on an ideal triangulation $\mathcal{T}$ of a manifold M to be a collection of (interior) dihedral angles satisfying:

(0) Opposite edges of a tetrahedron have the same angle.

(1) Dihedral angles lie in $(0, \pi)$.
(2) The sum of angles around any ideal vertex of any tetrahedron is π.
(3) The sum of angles around any edge class of M is 2π.

The set of all angle structures for a triangulation $\mathcal{T}$ is denoted by $\mathcal{A}(\mathcal{T})$. For M is an orientable 3-manifold with boundary consisting of tori, and $\mathcal{T}$ a triangulation of M, we will study the set of angle structures $\mathcal{A}(\mathcal{T})$.

PROPOSITION 9.11. *Let $\mathcal{T}$ be an ideal triangulation of a 3-manifold M consisting of n tetrahedra, and as usual denote the set of angle structures by $\mathcal{A}(\mathcal{T})$. If $\mathcal{A}(\mathcal{T})$ is nonempty, then it is a convex, finite-sided, bounded polytope in $(0, \pi)^{3n} \subset \mathbb{R}^{3n}$.*

PROOF. For each tetraheron of $\mathcal{T}$, an angle structure selects three dihedral angles lying in $(0, \pi)$. Thus $\mathcal{A}(\mathcal{T})$ is a subset of $(0, \pi)^{3n}$. The equations coming from conditions (2) and (3) are linear equations whose solution set is an affine subspace of $\mathbb{R}^{3n}$. When we intersect the solution space with the cube $(0, \pi)^{3n}$, we obtain a bounded, convex, finite-sided polytope. $\square$

There is no guarantee that $\mathcal{A}(\mathcal{T})$ is nonempty. However, proposition 9.11 implies that if it is nonempty, then we may view a point of $\mathcal{A}(\mathcal{T})$ as a point in $(0, \pi)^{3n}$. We write $a \in \mathcal{A}(\mathcal{T})$ as $a = (a_1, \ldots, a_{3n})$.

DEFINITION 9.12. The *volume functional* $\mathcal{V}\colon \mathcal{A}(\mathcal{T}) \to \mathbb{R}$ is defined by

$$\mathcal{V}(a_1, \ldots, a_{3n}) = \sum_{i=1}^{3n} \Lambda(a_i).$$

Thus $\mathcal{V}(a)$ is the sum of volumes of hyperbolic tetrahedra associated with the angle structure a.

A reason angle structures are so useful comes from the following two theorems.

THEOREM 9.13 (Volume and angle structures). *Let M be an orientable 3-manifold with boundary consisting of tori, with ideal triangulation $\mathcal{T}$. If a point $A \in \mathcal{A}(\mathcal{T})$ is a critical point for the volume functional $\mathcal{V}$ then the ideal hyperbolic tetrahedra obtained from the angle structure A give M a complete hyperbolic structure.*

The converse is also true:

THEOREM 9.14. *If M is finite volume hyperbolic 3-manifold with boundary consisting of tori such that M admits a positively oriented hyperbolic ideal triangulation $\mathcal{T}$, then the angle structure $A \in \mathcal{A}(\mathcal{T})$ giving the angles of $\mathcal{T}$ for the complete hyperbolic structure is the unique global maximum of the volume functional $\mathcal{V}$ on $\mathcal{A}(\mathcal{T})$.*

The two theorems are attributed to Casson and Rivin, and follow from proofs in [**Rivin, 1994**]. The first direct proof of the results are written in Chan's honours thesis [**Chan, 2002**], using work of Neumann and Zagier

[**Neumann and Zagier, 1985**]. A very nice self-contained exposition and proof of both theorems is given in [**Futer and Guéritaud, 2011**]. We will follow the ideas of Futer and Guéritaud to show theorem 9.13 in section 9.3.

To prove the converse, we will follow a simple proof of Chan using the Schläfli formula for the variation of volumes of ideal tetrahedra. Chan credits his proof to unpublished ideas of Schlenker.

9.3. Leading–trailing deformations

LEMMA 9.15. *Let M be an orientable 3-manifold with boundary consisting of tori, with ideal triangulation $\mathcal{T}$ consisting of n tetrahedra. Then the volume functional $\mathcal{V}\colon \mathcal{A}(\mathcal{T}) \to \mathbb{R}$ is strictly concave down on $\mathcal{A}(\mathcal{T})$. For $a = (a_1, \dots, a_{3n}) \in \mathcal{A}(\mathcal{T})$ and $w = (w_1, \dots, w_{3n}) \in T_a\mathcal{A}(\mathcal{T})$ a non-zero tangent vector, the first two directional derivatives of $\mathcal{V}$ satisfy*

$$\frac{\partial \mathcal{V}}{\partial w} = \sum_{i=1}^{3n} -w_i \log \sin a_i \quad \text{and} \quad \frac{\partial^2 \mathcal{V}}{\partial w^2} < 0.$$

PROOF. Because the volume functional $\mathcal{V}$ is the sum of volumes of ideal tetrahedra, the formulas for derivatives follow by linearity from theorem 9.9. Because the second derivative is strictly negative, the volume functional is strictly concave down. □

We will need to take derivatives in carefully specified directions. To that end, we now define a vector $w = (w_1, \dots, w_{3n}) \in \mathbb{R}^{3n}$ and show that w lies in $T_a\mathcal{A}(\mathcal{T})$. Again the ideas follow from [**Futer and Guéritaud, 2011**].

DEFINITION 9.16. Let C be a cusp of M with a cusp triangulation corresponding to the ideal tetrahedra of $\mathcal{T}$. Let ζ be an oriented closed curve on C, isotoped to run monotonically through the cusp triangulation, as in definition 4.12. Let $\zeta_1, \dots, \zeta_k$ be the oriented segments of ζ in distinct triangles. For the segment ζ_i in triangle t_i, define the *leading corner* of t_i to be the corner of the triangle that is opposite the edge where ζ_i enters t_i, and define the *trailing corner* to be the corner opposite the edge where ζ_i exits.

Each corner of the triangle t_i is given a dihedral angle a_j in an angle structure, thus corresponds to a coordinate of $\mathcal{A}(\mathcal{T}) \subset \mathbb{R}^{3n}$. Similarly for any $a \in \mathcal{A}(\mathcal{T})$, each corner of t_i corresponds to a coordinate of the tangent space $T_a\mathcal{A}(\mathcal{T}) \subset \mathbb{R}^{3n}$.

We define a vector $w(\zeta_i) \in \mathbb{R}^{3n}$ by setting the coordinate corresponding to the leading corner of t_i equal to $+1$, and the coordinate corresponding to the trailing corner of t_i equal to -1. Set all other coordinates equal to zero. The *leading–trailing deformation* corresponding to ζ is defined to be the vector $w(\zeta) = \sum_i w(\zeta_i)$.

An example is shown in figure 9.4.

LEMMA 9.17. *Let σ be a curve encircling a vertex of the cusp triangulation on cusp C. Let μ be an embedded curve isotopic to a generator of the*

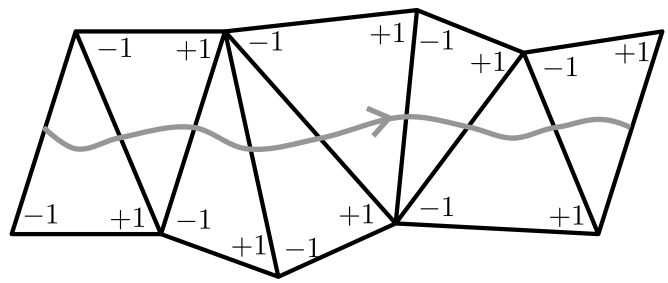

FIGURE 9.4. For the oriented curve ζ shown, leading corners are marked with $+1$ and trailing corners with -1

holonomy group of the cusp torus. Then the corresponding leading–trailing deformation vectors $w(\sigma)$ and $w(\mu)$ both lie in the tangent space $T_a\mathcal{A}(\mathcal{T})$, for any $a \in \mathcal{A}(\mathcal{T})$.

PROOF. The space $\mathcal{A}(\mathcal{T})$ is a submanifold of $\mathbb{R}^3$ cut out by linear equations corresponding to (2) and (3) of definition 8.21, namely that angles at each ideal vertex of a tetrahedron sum to π, and angles about an edge of M sum to 2π. Let $f_i(a) = a_i + a_{i+1} + a_{i+2}$ be the sum of angles of the i-th tetrahedron, and let $g_e(a) = \sum a_{e_i}$ be the sum of angles about the edge e. So $\mathcal{A}(\mathcal{T}) \subset (0, \pi)^{3n}$ is the space cut out by all equations $f_i = \pi$ and $g_e = 2\pi$. Thus to see that $w(\zeta)$ is a tangent vector to $\mathcal{A}(\mathcal{T})$ at a point a, we need to show that the vector is orthogonal to the gradient vectors ∇f_i and ∇g_e at a, for all i and all e.

Note ∇f_i is the vector $(0, \dots, 0, 1, 1, 1, 0, \dots, 0)$, with 0s away from the i-th tetrahedron t_i and 1s in the three positions corresponding to the angles of t_i. There are four cusp triangles coming from this tetrahedron, corresponding to its four ideal vertices. Suppose ζ is a curve in the cusp triangulation of C. If no segment of ζ runs through a triangle of t_i, then $w(\zeta)$ has only 0s in the position corresponding to the 1s of ∇f_i, hence $\nabla f_i \cdot w(\zeta) = 0$ in this case. So suppose that some segment ζ_j of ζ runs through a triangle of t_i. Then one corner of the triangle is a leading corner for ζ_j, and one is a trailing corner, so $w(\zeta_j)$ has one 0, one $+1$, and one -1 in the three positions corresponding to angles of t_i. Hence $\nabla f_i \cdot w(\zeta_j) = 0$. By linearity, $\nabla f_i \cdot w(\zeta) = 0$.

So it remains to show that for each edge e, $\nabla g_e \cdot w(\zeta) = 0$, where ζ is one of the curves σ or μ in the hypothesis of the lemma. Note that ∇g_e is a vector $(\epsilon_1, \dots, \epsilon_{3n})$, where ϵ_j is one of the integers 0, 1, or 2, counting the number of times a dihedral angle of a tetrahedron occurs in the gluing equation g_e. We will consider the segments of ζ one at a time. Note that any segment ζ_j of ζ contributes 0, $+1$, and -1 to opposite edges of exactly one tetrahedron t_j, as illustrated in figure 9.5, left.

If the edge e is not identified to any of the edges of the tetrahedron t_j, then $\nabla g_e \cdot w(\zeta_j) = 0$. Similarly, if e is identified only to one or both of the edges for which $w(\zeta_j)$ contributes a 0, then although the corresponding coordinate of ∇g_e will be 1 or 2, the dot product $\nabla g_e \cdot w(\zeta_j)$ will still be 0.

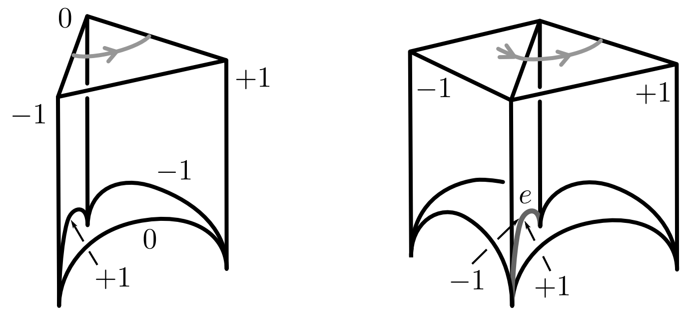

FIGURE 9.5. Left: Effect of $w(\sigma_j)$ on the edges of a tetrahedron. Right: If the lower edge is identified to e, then the contribution of $+1$ from $w(\zeta_j)$ to e cancels with the -1 contribution from $w(\zeta_{j-1})$.

If e is identified to one or both of the edges labeled with a $+1$ by $w(\zeta_j)$, then there will be a contribution of $+1$ or $+2$ (respectively) to $\nabla g_e \cdot w(\zeta_j)$ coming from these labels. We will show that in this case, there exists one or two (respectively) segments of ζ each contributing -1, so that the positive contributions cancel.

Suppose first that e is identified to the non-vertical edge of t_j labeled $+1$. Then consider the segment ζ_{j-1}. This lies in a tetrahedron t_{j-1} glued to t_j along a face containing the edge identified to e. In the cusp triangulation, ζ_{j-1} exits its cusp triangle at this face. Thus the opposite corner of the cusp triangle is a trailing corner, and is assigned a -1. This trailing corner corresponds to an edge opposite e. So e picks up a -1 from $w(\zeta_{j-1})$. See figure 9.5, right. Hence the $+1$ contribution of $w(\zeta_j)$ is canceled in this case with this -1 from $w(\zeta_{j-1})$.

Now suppose e is identified to the vertical edge of t_j labeled $+1$. Let $\zeta_j, \zeta_{j+1}, \dots, \zeta_{j+r}$ be a maximal collection of segments in cusp triangles adjacent to e. Note if $\zeta = \sigma$ encircles a vertex, then $r = 1$, i.e. there are just two segments. But if $\zeta = \mu$ is a generator of cusp homology, then $r \geq 1$. Because we are assuming ζ is embedded and meets each edge of the cusp triangulation at most once, we know $\zeta_j, \dots, \zeta_{j+r}$ do not encircle e completely. See figure 9.6.

For segments ζ_{j+k} with $0 < k < r$, note $w(\zeta_{j+k})$ contributes only 0s to the edge e. Since the segment after ζ_{j+r} is no longer adjacent to the vertical edge e, it follows that $w(\zeta_{j+r})$ contributes -1 to e. Then the $+1$ contribution from $w(\zeta_j)$ cancels with the -1 contribution from $w(\zeta_{j+r})$.

Finally, it could be the case that e is identified to both edges labeled $+1$ by $w(\zeta_j)$, so that ∇g_e has a 2 in that coordinate and $\nabla g_e \cdot w(\zeta_j)$ picks up a $+2$ from these two edges. But in this case, combining both arguments above implies that one of the $+1$ contributions is canceled by a -1 coming from $w(\zeta_{j-1})$ and one by a -1 coming from $w(\zeta_{j+r})$ for appropriate r. Thus both are canceled.

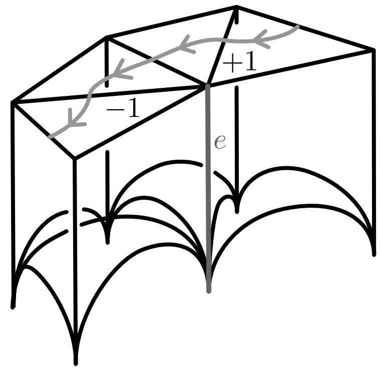

FIGURE 9.6. If $w(\zeta_j)$ contributes $+1$ to a vertical edge meeting e, there is a maximal collection of segments running through cusp triangles adjacent to e.

We have shown that for each j, each $+1$ contribution of $w(\zeta_j)$ to $\nabla g_e \cdot w(\zeta_j)$ is canceled by a -1 contribution from some $w(\zeta_k)$. Provided none of the -1 contributions from $w(\zeta_k)$ are repeated for distinct j, this shows that $\nabla g_e \cdot w(\zeta) \leq 0$. The fact that these contributions are not repeated follows from the uniqueness of the choice of ζ_{j-1} and ζ_{j+r}.

A similar argument implies $\nabla g_e \cdot w(\zeta) \geq 0$. Thus $\nabla g_e \cdot w(\zeta) = 0$, as desired. □

LEMMA 9.18. *Let ζ be one of the curves σ or μ of lemma 9.17, and let $w(\zeta) \in T_a\mathcal{A}(\mathcal{T})$ be the corresponding leading–trailing deformation vector. Let $H(\zeta)$ be the complex number associated to the curve ζ given in definition 4.12 (completeness equations). Then*

$$\frac{\partial \mathcal{V}}{\partial w(\zeta)} = \Re(\log H(\zeta)).$$

PROOF. Let $\zeta_1, \dots, \zeta_k$ denote segments of ζ in cusp triangles $t_1, \dots, t_k$, respectively. Label the dihedral angles of triangle t_i by α_i, β_i, γ_i, in clockwise order, so that α_i is the angle cut off by t_i. By definition 4.12,

$$\Re(\log H(\zeta)) = \sum_i \epsilon_i \Re(\log |z(\alpha_i)|),$$

where $z(\alpha_i)$ is the edge invariant associated with the edge labeled α_i, and $\epsilon_i = +1$ if α_i is to the left of ζ_i and $\epsilon_i = -1$ if α_i is to the right of ζ_i.

On the other hand, comparing figure 4.13 and figure 9.4, we see that when α_i is to the left of ζ_i, the vector $w(\zeta_i)$ has a $+1$ in the position corresponding to β_i and a -1 in the position corresponding to γ_i, and when α_i is to the right of ζ_i, the vector $w(\zeta_i)$ has a -1 in the position corresponding to β_i and a +1 in the position corresponding to γ_i. Then lemma 9.15 implies that

$$\begin{aligned}\frac{\partial \mathcal{V}}{\partial w} &= \sum_{j=1}^{3n} -w_j \log\sin a_j \\ &= \sum_i -\epsilon_i \log\sin\beta_i + \epsilon_i \log\sin\gamma_i \\ &= \sum_i \epsilon_i \log\left(\frac{\sin\gamma_i}{\sin\beta_i}\right) \\ &= \sum_i \epsilon_i \Re(\log|z(\alpha_i)|), \quad \text{by equation (8.1)}\end{aligned}$$

This is what we needed to show. □

We now have the tools we need to prove theorem 9.13, to show that a critical point $a \in \mathcal{A}(\mathcal{T})$ of the volume functional corresponds to a complete hyperbolic structure on the manifold M.

Proof of theorem 9.13. Suppose $a \in \mathcal{A}(\mathcal{T})$ is a critical point of the volume functional $\mathcal{V}$. Then a assigns a dihedral angle to each tetrahedron of $\mathcal{T}$, giving each ideal tetrahedron a unique hyperbolic structure. By theorem 4.7, gluing these tetrahedra will give a hyperbolic structure on M if and only if the edge gluing equations are satisfied for each edge. By theorem 4.10, the hyperbolic structure will be complete if and only if the induced geometric structure on each cusp torus is a Euclidean structure, and we obtain a Eulidean structure when the completeness equations are satisfied by proposition 4.15.

Consider first the edge gluing equations. Notice that any angle structure gives hyperbolic ideal tetrahedra satisfying the imaginary part of the gluing equations, so we need to show that our tetrahedra satisfy the real part. Fix an edge of the triangulation, and let σ be a curve on a cusp torus encircling an endpoint of that edge. The real part of the gluing equation corresponding to this edge will be satisfied if and only if $\Re(\log H(\sigma)) = 0$. But lemma 9.18 implies that $\Re(\log H(\sigma)) = \frac{\partial \mathcal{V}}{\partial w(\sigma)}$, and this is zero because our angle structure is a critical point of the volume functional. So the gluing equations hold.

As for the completess equations, for any cusp torus C, and μ_1 and μ_2 generators of the first homology group of C, the completeness equations require that $H(\mu_1) = H(\mu_2) = 1$. By lemma 9.18, we know

$$\Re\log(H(\mu_i)) = \frac{\partial \mathcal{V}}{\partial w(\mu_i)} = 0,$$

since a is a critical point. Thus the real part of each of these completeness equations is satisfied.

Consider the developing image of a fundamental domain for the cusp torus C. Because we know the angles given by a satisfy the gluing equations, the structure on C is at least an affine structure on the torus. Therefore the developing image of the fundamental domain is a quadrilateral in $\mathbb{C}$. Since the real parts of the completeness equations for μ_1 and μ_2 are both satisfied, it follows that the holonomy elements corresponding to μ_1 and μ_2 do not scale either side of the fundamental domain. But then the holonomy elements cannot effect a non-trivial rotation either; a quadrilateral in $\mathbb{C}$ whose opposite sides are the same length is a parallelogram. Thus the developing image of a fundamental domain is a parallelogram, the holonomy elements corresponding to μ_1 and μ_2 must be pure translations, and the cusp torus admits a Euclidean structure. So the completeness equations hold. □

Theorem 9.13 gives us a way of proving not only that a 3-manifold M is hyperbolic, but also that it admits a positively oriented geometric triangulation. To use the theorem, first, fix a triangulation $\mathcal{T}$. Then show the space of angle structures $\mathcal{A}(\mathcal{T})$ is nonempty. Finally, show that the volume functional achieves its maximum in the interior of $\overline{\mathcal{A}(\mathcal{T})}$. The last step can often be accomplished by considering angle structures on the boundary $\overline{\mathcal{A}(\mathcal{T})} - \mathcal{A}(\mathcal{T})$ and proving such structures cannot maximize volume. We will follow exactly this procedure for 2-bridge knots in chapter 10.

The following proposition is a useful tool for examining the maximum of the volume functional on the boundary $\overline{\mathcal{A}(\mathcal{T})} - \mathcal{A}(\mathcal{T})$.

Proposition 9.19. *Suppose an angle structure $a \in \overline{\mathcal{A}(\mathcal{T})}$ maximizes the volume functional $\mathcal{V}$. Suppose that for some tetrahedron Δ_i, one of the three angles of Δ_i in the angle structure a is 0. Then two of the angles are 0 and the third is π.*

Proof. Suppose instead that one angle, say a_i is 0, but the other two angles of Δ_i are nonzero: $a_{i+1} \neq 0$ and $a_{i+2} \neq 0$. We will find a path through $\mathcal{A}(\mathcal{T})$ with endpoint the angle structure a, and we will show that the derivative of this path is positive, and in fact unbounded, as it approaches the endpoint corresponding to a. It will follow that a cannot be a maximum, which contradicts our assumption on a.

The space of angle structures $\mathcal{A}(\mathcal{T})$ is a bounded open convex subset of $\mathbb{R}^{3n}$. Its tangent space can be extended to its boundary. We may choose a tangent vector w in $T_a\overline{\mathcal{A}(\mathcal{T})}$ pointing into the interior of $\mathcal{A}(\mathcal{T})$, and take the path corresponding to geodesic flow in the direction of this tangent vector. Theorem 9.9 implies that the derivative of the volume functional along this path is the sum of terms of the form $\sum_{i=1}^{3n} -w_i \log\sin(a_i)$.

Consider the contribution from Δ_i. The terms

$$w_{i+1} \log\sin(a_{i+1}) \text{ and } w_{i+2} \log\sin(a_{i+2})$$

are bounded, since a_{i+1} and a_{i+2} are bounded away from zero. However, as the path approaches the angle structure a, the term coming from $-w_i \log\sin(a_i)$ approaches positive infinity. Thus such a point cannot be a maximum. □

9.4. The Schläfli formula

In this short section, we prove theorem 9.14, the converse to theorem 9.13. Our proof uses the Schläfli formula for ideal tetrahedra, which can be stated as follows.

THEOREM 9.20 (Schläfli's formula for ideal tetrahedra). *Let P be an ideal tetrahedron. Let $H_1, \dots, H_n$ be a collection of horospheres centered on the ideal vertices of P. For each edge e_{ij}, running between the i-th to the j-th ideal vertices of P, let $\ell(e_{ij})$ denote the signed distance between H_i and H_j (that is, $\ell(e_{i,j})$ is defined to be negative if $H_i \cap H_j \neq \emptyset$). Finally, let θ_{ij} denote the dihedral angle along edge $e_{i,j}$. Then the variation in the volume of P satisfies*

$$d\mathcal{V}(P) = -\frac{1}{2}\sum_{i,j} \ell(e_{ij}) d\theta_{i,j}. \tag{9.4}$$

Schläfli's formula was originally proved for finite spherical simplices by Schläfli in the 1850s. It has been extended in many directions, including to finite and ideal polyhedra in spaces of constant curvature. A proof of a formula that contains the result in theorem 9.20 can be found in [**Milnor, 1994**]; see also [**Rivin, 1994**]. These sources note that the right hand side of equation (9.4) is independent of the choice of horospheres.

Using this, we can finish the proof.

PROOF OF THEOREM 9.14. In the complete hyperbolic structure on M, choose a horosphere about each cusp. Because the hyperbolic structure is complete, this choice gives a well-defined horosphere about each ideal vertex of each ideal tetrahedron in the positively oriented hyperbolic ideal triangulation $\mathcal{T}$. Thus for each tetrahedron, we may use this choice to define the edge lengths $\ell(e_{ij})$ of theorem 9.20.

Now note that because the total angle around each edge is a constant 2π, the contributions to the variation of the volume coming from each simplex add to zero for each edge. Thus the right hand side of equation (9.4) is zero. It follows that the complete structure is a critical point for the volume functional.

On the other hand, since the volume functional is strictly concave down on $\mathcal{A}$, theorem 9.9, it must follow that the complete structure is the unique global maximum. □

9.5. Consequences

Theorem 9.13 and its converse have a number of important immediate consequences. We leave many proofs as exercises.

COROLLARY 9.21 (Lower volume bounds, angle structures). *Suppose M has ideal triangulation $\mathcal{T}$ such that the volume functional $\mathcal{V}\colon \mathcal{A}(\mathcal{T}) \to \mathbb{R}$ has* *a critical point $p \in \mathcal{A}(\mathcal{T})$. Then for any other point $q \in \overline{\mathcal{A}(\mathcal{T})}$, the volume functional satisfies*

$$\mathcal{V}(q) \leq \mathrm{vol}(M),$$

with equality if and only if $q = p$, i.e. q also gives the complete hyperbolic metric on M.

Proof. The volume functional $\mathcal{V}$ is strictly concave down on $\overline{\mathcal{A}(\mathcal{T})}$ by lemma 9.15, and so for any point $q \in \overline{\mathcal{A}(\mathcal{T})}$, $\mathcal{V}(q)$ is at most the maximum value of the volume functional, which is the value $\mathcal{V}(p)$ by hypothesis, and with equality if and only if $q = p$. By theorem 9.13, $\mathrm{vol}(M) = \mathcal{V}(p)$. □

More is conjectured to be true. Corollary 9.21 only gives a bound when there is a known critical point of the volume functional in the interior of the space of angle structures. If the maximum of the volume functional occurs on the boundary, it still seems to be the case in practice that the maximum is bounded by the volume of the complete hyperbolic structure. However, the following conjecture is currently still open.

Conjecture 9.22 (Casson's conjecture). *Let M be a cusped hyperbolic 3-manifold, and let $\mathcal{T}$ be any ideal triangulation of M. If the space of angle structures $\mathcal{A}(\mathcal{T})$ is nonempty, then the maximum value for the volume functional on $\overline{\mathcal{A}(\mathcal{T})}$ is at most the volume of the complete hyperbolic structure on M.*

We may use angle structures to find hyperbolic Dehn fillings of triangulated 3-manifolds as well. Recall from chapter 6 that the (p, q) Dehn filling on a triangulated manifold satisfies equation equation (6.1):

$$p \log H(\mu) + q \log H(\lambda) = 2\pi i.$$

Theorem 9.23 (Angle structures and Dehn fillings). *Let M be a manifold with torus boundary components $T_1, \ldots, T_n$, with generators μ_j, λ_j of $\pi_1(T_j)$ for each j. For each j, let (p_j, q_j) denote a pair of relatively prime integers. Let $\mathcal{A}_{(p_1,q_1),\ldots,(p_n,q_n)} \subset \mathcal{A}$ be the set of all angle structures that satisfy the imaginary part of the Dehn filling equations:*

$$\Im(p_j \log H(\mu_j) + q_j \log H(\lambda_j)) = 2\pi.$$

Then a critical point of the volume functional $\mathcal{V}$ on $\mathcal{A}_{(p_1,q_1),\ldots,(p_n,q_n)}$ gives the complete hyperbolic structure on the Dehn filling $M((p_1, q_1), \ldots, (p_n, q_n))$ of M.

Proof. As in the proof of theorem 9.13, we will have a complete hyperbolic structure on the Dehn filling if and only if each edge gluing equation is satisfied and additionally each Dehn filling equation

$$p_j \log H(\mu_j) + q_j \log H(\lambda_j) = 2\pi i$$

is satisfied.

The proof that edge gluing equations are satisfied follows exactly as in the proof of theorem 9.13. As for the Dehn filling equations, the imaginary part of each equation is satisfied by the given constraint on the space of angle structures. By lemma 9.18, the real part satisfies

$$\Re(p_j \log(H(\mu_j)) + q_j \log(H(\lambda_j))) = p_j \frac{\partial \mathcal{V}}{\partial w(\mu_j)} + q_j \frac{\partial \mathcal{V}}{\partial w(\lambda_j)} = 0,$$

because this is a critical point for the volume functional. Thus each Dehn filling equation is satisfied. □

Corollary 9.21 and theorem 6.13 imply the following (weaker) version of Jørgensen's theorem, theorem 6.25.

Corollary 9.24. *Let M be as in theorem 9.23. If s is a slope in the neighborhood of ∞ provided by Thurston's hyperbolic Dehn filling theorem, theorem 6.13, then the volume of $M(s)$ is strictly smaller than the volume of M.*

Proof. Exercise 9.9. □

We also have the tools to prove a rigidity theorem originally du to Weil. The following follows from Mostow–Prasad rigidity, theorem 6.1, but was first proved over a decade before that theorem, and can now be proved easily using angle structures.

Corollary 9.25 (Weil rigidity theorem). *Suppose M is a 3-manifold with boundary consisting of tori, and suppose the interior of M admits a complete hyperbolic metric. Then the metric is locally rigid, i.e. there is no local deformation of the metric through complete hyperbolic structures.*

Proof. Exercise 9.10. □

9.6. Exercises

Exercise 9.1. Find a formula for volume of a tetrahedron with ideal vertices 0, 1, ∞ and z in terms of z alone.

Exercise 9.2. Give a proof that the figures of figure 9.3 are correct. That is, given a Euclidean triangle with vertices on the unit circle, and angles α, β, and γ, prove that the angles around the origin are given as shown in the figure.

Exercise 9.3. Prove the Kubert identities for the Lobachevsky function:

$$\Lambda(n\theta) = \sum_{k=0}^{n-1} n\Lambda(\theta + k\pi/n).$$

Hint: cyclotomic identities:

$$2\sin(n\theta) = \prod_{k=0}^{n-1} 2\sin\left(\theta + \frac{k\pi}{n}\right).$$

Exercise 9.4. Find an explicit convex polytope describing the set of angle structures on the complement of the figure-8 knot.

Exercise 9.5. Find an explicit convex polytope describing the set of angle structures on the complement of the 5_2 knot.

Exercise 9.6. Suppose an ideal tetrahedron has dihedral angles α, β, γ in clockwise order. Prove equation (8.1): that the edge invariant of the tetrahedron assigned to the edge with angle α is

$$z(\alpha) = \frac{\sin(\gamma)}{\sin(\beta)}\, e^{i\alpha}.$$

Exercise 9.7. An ideal octahedron can be obtained by gluing two identical ideal pyramids over an ideal quadrilateral base along the ideal quadrilateral. We triangulate this by running an edge from the ideal point opposite the quadrilateral on one pyramid, through the quadrilateral, to the ideal point opposite the quadrilateral on the other pyramid, and then we stellar subdivide. Using angle structures on this collection of ideal tetrahedra, prove that the maximal volume ideal hyperbolic octahedron is the regular one: the one for which the quadrilateral base is a square.

Exercise 9.8. Generalize exercise 9.7 to the ideal object obtained by gluing two ideal pyramids over an ideal n-gon base. Using stellar subdivision and angle structures, prove that the volume of the ideal double pyramid with base an n-gon is maximized when the n-gon is regular.

Exercise 9.9. Prove that volume decreases locally under Dehn filling, corollary 9.24.

Exercise 9.10. Prove the Weil rigidity theorem, corollary 9.25. First prove it for manifolds admitting an angle structure. Extend to all manifolds using the following theorem of Luo, Schleimer, and Tillmann, which can be found in [**Luo et al., 2008**]. (You may assume the theorem for the exercise.)

Theorem 9.26 (Geometric triangulations exist virtually). *Let M be a 3-manifold with boundary consisting of tori, such that the interior of M admits a complete hyperbolic structure. Then M has a finite cover N such that N decomposes into positively oriented ideal tetrahedra.*