Part 3

Hyperbolic knot invariants

CHAPTER 13

Estimating volume

We have seen that hyperbolic 3-manifolds have finite volume if and only if they are compact or the interior of a compact manifold with finitely many torus boundary components (theorem 5.27). However, it is not completely straightforward to estimate volumes of large classes of manifolds, including knot complements. There are many open questions concerning the relationship of volume of a hyperbolic manifold to other invariants, such as knot invariants. In this chapter, we discuss different ways to estimate volumes of hyperbolic 3-manifolds that are defined topologically or combinatorially, such as knot complements.

13.1. Summary of bounds encountered so far

13.1.1. Upper bounds. It is usually an easier problem to give upper bounds on the volume of a hyperbolic 3-manifold than lower bounds, although there are exceptions, especially when sharp upper bounds are needed. Here we review two methods we have already encountered that can give upper bounds on volume.

13.1.1.1. *Volume bounds from polyhedra.* Recall from theorem 9.10 that the maximal volume tetrahedron is the regular ideal tetrahedron. Its volume is the value $3\Lambda(\pi/3) := v_{\text{tet}} = 1.0149\ldots$. In various chapters, we have found decompositions of several different knot and link complements into ideal tetrahedra. The volume of such a knot or link is therefore bounded by v_{tet} times the number of tetrahedra in its decomposition.

For example, this can be used to show the following theorem, originally proved by Agol and D. Thurston in the appendix to [**Lackenby, 2004**].

THEOREM 13.1. *A fully augmented link L with $t(L)$ crossing circles has volume at most $10v_{\text{tet}}(t(L)-1)$.*

PROOF. In chapter 7, we saw that a fully augmented link has a decomposition into two right angled ideal polyhedra P_1 and P_2, with white and shaded faces, where shaded faces are ideal triangles coming from 2-punctured disks bounded by crossing circles, and white faces come from the plane of projection.

For the polyhedron P_1, add a finite vertex v_1 in the interior and cone to the faces of the polyhedra. Do the same for P_2, adding vertex v_2 and coning. Each shaded triangle in ∂P_1 gives rise to a tetrahedron. There are two shaded triangles per crossing circle in each of the two polyhedra, so $4t(L)$ tetrahedra arise in this way.

The white faces are coned to pyramids. Glue a pair of pyramids in P_1 and P_2 together across a matching white face, and perform stellar subdivision. That is, add an edge running from the finite vertex in one polyhedron through the center of the face to the finite vertex in the other polyhedron, then add triangles around the edge to divide the pyramids into tetrahedra. If the face has d edges, it is subdivided into d tetrahedra. Note each crossing circle contributes six edges to each polyhedron. Thus the total number of edges of the white faces will be $6t(L)$, and thus the white faces contribute $6t(L)$ tetrahedra to the decomposition.

This gives us $10t(L)$ tetrahedra, but these have finite vertices. We can improve the bound by choosing an ideal vertex w_1 in P_1, and collapsing the edge from w_1 to v_1. Similarly, choose the corresponding vertex w_2 in P_2, and collapse the edge from w_2 to v_2. Now simplify the triangulation by collapsing monogons to vertices, bigons to a single edge, and parallel triangles to a single triangle. Note that all tetrahedra adjacent to w_1 and w_2 are collapsed to triangles under this procedure. We count the number of these.

The ideal vertex w_1 is adjacent to two shaded triangles and two white faces. The white faces each have at least three edges, and so give rise to at least three tetrahedra to be collapsed, running between the two polyhedra. Thus there are at least six such tetrahedra arising from white faces. Each shaded face gives rise to one tetrahedron to be collapsed in each P_i, or four total. Thus there is an ideal triangulation with at most $10t(L) - 10$ tetrahedra.

Now apply theorem 9.10. The volume of each of the $10t(L)-10$ tetrahedra is at most v_{tet}. Thus the volume of the fully augmented link is at most $10v_{\text{tet}}(t(L) - 1)$. □

The bound of theorem 13.1 is *asymptotically sharp*, in the sense that there is a sequence of fully augmented links whose volumes approach the upper bound; this is proved in the first part of exercise 13.1.

13.1.1.2. *Jørgensen's theorem.* Recall Jørgensen's theorem, theorem 6.25: If M is hyperbolic with cusps $C_1, \dots, C_n$, and $s_1, \dots, s_n$ are slopes, one on each ∂C_j, such that $M(s_1, \dots, s_n)$ is hyperbolic, then

$$\mathrm{vol}(M) > \mathrm{vol}(M(s_1, \dots, s_n)).$$

This result can be combined with the previous to give an upper bound on the volume of knots in terms of the twist number, first observed in [**Lackenby, 2004**].

THEOREM 13.2. *Suppose K is a knot or link with a prime, twist-reduced diagram with twist number* $\mathrm{tw}(K) \geq 2$. *Then the volume of $S^3 - K$ satisfies*

$$\mathrm{vol}(S^3 - K) \leq 10v_{\text{tet}}(\mathrm{tw}(K) - 1).$$

Again the bound of theorem 13.2 is asymptotically sharp; this is proved in the second part of exercise 13.1.

Proof of theorem 13.2. Because the diagram of K is prime and twist-reduced, when we fully augment K by adding a crossing circle to each twist region, and then remove pairs of crossings to form a fully augmented link, the resulting link L is hyperbolic; see lemma 7.16 and lemma 7.17. It will have $t(L) = \mathrm{tw}(K)$ crossing circles. By theorem 13.1, the volume of the fully augmented link is at most $10v_{\mathrm{tet}}(\mathrm{tw}(K) - 1)$.

Now, we obtain $S^3 - K$ from $S^3 - L$ by Dehn filling the crossing circles, filling the i-th one along a slope $1/n_i$ where n_i is an integer such that $2n_i$ crossings were removed at that twist region to go from the diagram of K to that of L. By Jørgensen's theorem, theorem 6.25,

$$\mathrm{vol}(S^3 - K) < \mathrm{vol}(S^3 - L) \leq 10v_{\mathrm{tet}}(\mathrm{tw}(K) - 1). \qquad \square$$

13.1.2. Lower bounds via angle structures. In addition to previously obtaining results that lead to upper bounds on volume, we have also built the tools to give lower bounds on hyperbolic volume in special cases, in particular when the manifold admits an angle structure. Recall theorem 9.13: if the maximum of the volume functional over the set of all angle structures on a manifold M occurs in the interior of the set of angle structures, then that angle structure gives the unique complete hyperbolic metric on M. We have the following corollary.

Corollary 13.3. *Suppose M is an orientable 3-manifold with boundary consisting of tori, with an ideal triangulation $\mathcal{T}$. Suppose that the volume functional takes its maximum on the interior of the set of all angle structures $\mathcal{A}(\mathcal{T})$. Let A be any structure in the closure of $\mathcal{A}(\mathcal{T})$. Then* $\mathrm{vol}(M) \geq \mathcal{V}(A)$.

Proof. The volume functional is strictly concave down, and is uniquely maximized at the complete hyperbolic structure in the interior. Thus any structure in the closure of $\mathcal{A}(\mathcal{T})$ gives a volume at most that of M. $\square$

Corollary 13.3 was used by Futer and Guéritaud to obtain bounds on the volumes of 2-bridge knots in terms of their continued fraction decompositions [**Guéritaud, 2006**]. They proved the following.

Theorem 13.4. *Let K be a reduced alternating diagram of a hyperbolic 2-bridge link K with* $\mathrm{tw}(K)$ *twist regions. Then*

$$\mathrm{vol}(S^3 - K) > 2v_{\mathrm{tet}}\mathrm{tw}(K) - 2.7066.$$

Moreover, the lower bound is asymptotically sharp.

Proof. We may suppose that the diagram of K is determined by a continued fraction $[0, a_{n-1}, \ldots, a_1]$, where $\mathrm{tw}(K) = n-1$, as in definition 10.7. The link complement has a geometric triangulation discussed in chapter 10, determined by real numbers $(z_1, z_2, \ldots, z_{C-2}, z_{C-1})$, where C is the crossing number of the diagram of K; see the proof of proposition 10.18. We will choose explicit values of the z_i that give a structure in the boundary of the space of angle structures. By corollary 13.3 the volume of the result gives a lower bound on the actual hyperbolic volume.

First assume that $\mathrm{tw}(K) \geq 3$, so $n \geq 2$. We let $z_1 = z_{C-1} = 0$, and we will choose $z_i = \pi/3$ for indices i such that $a_1 \leq i \leq C - a_{n-1}$. These choices for appropriate i satisfy the hinge equation of equation (10.1): $|z_{i+1} - z_{i-1}| = 0 < \pi - z_i = 2\pi/3$. They do not satisfy the strict inequality of the convexity equation of equation (10.1), but only satisfy the weak inequality: $2z_i \leq z_{i-1} + z_{i+1}$. When $a_1 < i < C - a_{n-1}$, these will assign values to x_i and y_i using table 10.1. Note the angles will take values of $\pi/3$ in the hinge case, but $2\pi/3$ and 0 in the non-hinge case, and thus there will be flat tetrahedra. However, our choices so far give rise to a structure on the boundary of $\mathcal{A}(\mathcal{T})$, which will be sufficient for our purposes. Each hinge index contributes volume v_{tet} to the structure, while non-hinges contribute nothing to volume. Note hinges occur between twist regions; there are $\mathrm{tw}(K) - 3$ hinge indices between a_1 and $C - a_{n-1}$.

We cannot choose $z_i = \pi/3$ for all the indices in the first and last fans, else even the weak inequality of the convexity equations equation (10.1) will not be satisfied near $i = 0$ or $i = C - 1$. Instead, in the first and last fans, interpolate between 0 and $\pi/3$ in a way that satisfies the weak versions of equation (10.1). Then again angles x_i and y_i will be determined by table 10.1. At the hinge indices $i = a_1$ or $i = C - a_{n-1}$, the angles will be:

$$\frac{\pi}{2} - z_{i-1}, \quad \frac{\pi}{6} + z_{i-1}, \quad \frac{\pi}{3}.$$

The volume defined by these angles is smallest when $z_{i-1} = 0$, which occurs when the three angles are $\pi/2$, $\pi/6$, $\pi/3$, and the volume is $0.84578\ldots$. Thus the four tetrahedra $T^1_{a_1}$, $T^2_{a_1}$, $T^1_{C-a_{n-1}}$, and $T^2_{C-a_{n-1}}$ each have volume at least 0.84578, and the volume of this structure satisfies

$$\mathcal{V} > 2v_{\mathrm{tet}}(\mathrm{tw}(K) - 3) + 4(0.84578) > 2v_{\mathrm{tet}}\mathrm{tw}(K) - 2.7066.$$

Finally, check that when $\mathrm{tw}(K) = 2$, $\mathcal{V} > 2(0.84578)$ still satisfies the theorem. □

13.2. Negatively curved metrics and Dehn filling

We now turn our attention to a new technique for bounding volume from below that has not arisen in previous chapters. This bound comes from differential geometric methods, in particular from finding volumes of hyperbolic manifolds under families of metrics, and showing that the hyperbolic metric maximizes volume among such a family.

A hyperbolic manifold has constant sectional curvature equal to -1. The metrics we will consider will have negative sectional curvature, not necessarily constant. If a 3-manifold admits such a metric, it actually follows from the Geometrization theorem that the manifold also admits a hyperbolic metric; see theorem 13.5. However, the proof of that fact gives no information on how the hyperbolic metric relates to the negatively curved one. Often we can build an explicit negatively curved metric, and we will use this metric to make conclusions about the hyperbolic geometry of the manifold. This section presents a number of results along these lines, particularly relating to volume.

THEOREM 13.5. *Suppose M is a compact orientable 3-manifold whose interior admits a Riemannian metric with negative sectional curvature. Then M admits a hyperbolic metric.*

PROOF. Because sectional curvature is negative, the Cartan–Hadamard theorem implies that the universal cover of M is homeomorphic to $\mathbb{R}^3$ (see for example [**Gallot et al., 2004**, Theorem 3.87]). It follows that the fundamental group of M is infinite. It also follows that M is irreducible, for any sphere in M lifts to a sphere in $\mathbb{R}^3$, which bounds a ball. Then the image of the sphere in M bounds the image of that ball in M, which is a ball.

In the case that M is closed, because M has strictly negative curvature, it is known that every abelian subgroup of its fundamental group is cyclic [**Preissmann, 1943**]. Thus there is no $\mathbb{Z}\times\mathbb{Z}$ subgroup of its fundamental group. So in this case, M is irreducible, with $\pi_1(M)$ infinite, containing no $\mathbb{Z}\times\mathbb{Z}$ subgroup. By the Geometrization Theorem, theorem 8.25, M admits a hyperbolic structure.

In the case that M has boundary, there may be a $\mathbb{Z}\times\mathbb{Z}$ subgroup of $\pi_1(M)$, but the fact that the curvature is strictly negative in the interior implies that the subgroup is peripheral, hence M is atoroidal [**Ballmann et al., 1985**]. If M is a Seifert fibered space with infinite fundamental group, then $\pi_1(M)$ contains a cyclic normal subgroup (theorem 8.22). But again, a complete negatively curved finite volume Riemannian manifold cannot have a cyclic normal subgroup [**Ballmann et al., 1985**]. It follows that M is hyperbolic. □

Notice that the proof of theorem 13.5 gives no information on the relationship between the negatively curved metric and the hyperbolic metric. For example, we can make no conclusions about the difference in volumes of the manifolds. In the closed case, work of Besson, Courtois, and Gallot [**Besson et al., 1995**] can be used to bound the volume of a manifold under one negatively curved metric in terms of the volume of other. This was extended to the finite volume case by Boland, Connell, and Souto [**Boland et al., 2005**]. In 3-dimensions, a special case of their work is the following theorem.

THEOREM 13.6. *Let σ and σ' be two complete, finite volume Riemannian metrics on the same 3-manifold N. Suppose the Ricci curvature of σ satisfies $\mathrm{Ric}_\sigma \geq -2\sigma$, and suppose the sectional curvatures of σ' lie in the interval $[-a,-1]$ for some constant $a\geq 1$. Then*

$$\mathrm{vol}(N,\sigma) \geq \mathrm{vol}(N,\sigma'),$$

with equality if and only if both metrics are hyperbolic. □

Our goal in this section is to bound the change in volume of a hyperbolic manifold under Dehn filling by constructing a negatively curved metric on the Dehn filling of a hyperbolic manifold, and then applying theorem 13.6. This volume estimate was first obtained in [**Futer et al., 2008**]. Along the way we will obtain additional important cansequences, for example the 2π-theorem [**Bleiler and Hodgson, 1996**].

13.2.1. Negatively curved metrics on a solid torus. We will construct metrics in this subsection, and use them to bound volume. The arguments here require a little more familiarity with Riemannian geometry than the rest of the book so far. However, these arguments are only needed in this section and will not be required elsewhere in the book. Thus a reader disinclined to work carefully through the calculations in Riemannian geometry at this time may accept the statements of the main results here and skip ahead to their applications in subsection 13.2.2.

The metrics we construct will have constant sectional curvatures away from a collection of solid tori, namely those we glue to perform Dehn filling. Within a solid torus, we will use cylindrical coordinates.

Definition 13.7. Let V be a solid torus, and let $\widetilde{V}$ be its universal cover. The *cylindrical coordinates* on $\widetilde{V}$ are given by (r, μ, λ), where $r \leq 0$ is the radial distance measured outward from ∂V, $0 \leq \mu \leq 1$ is measured around each meridional circle, and $-\infty < \lambda < \infty$ is measured in the longitudinal direction, orthogonal to μ. Normalize the coordinates so that the generator of the deck transformation group on $\widetilde{V}$ changes the λ coordinate by 1. These coordinates descend to *cylindrical coordinates* on V, where $r \leq 0$ is radial distance measured outward from ∂V, $0 \leq \mu \leq 1$ is in the meridional direction, and $0 \leq \lambda \leq 1$ is measured orthogonal to μ.

Lemma 13.8. *Let (r, μ, λ) by cylindrical coordinates on a solid torus or its universal cover. Then a metric of the form*

$$ds^2 = dr^2 + f(r)^2 d\mu^2 + g(r)^2 d\lambda^2, \tag{13.1}$$

where $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are smooth functions of r, satisfies the property that all sectional curvatures are convex combinations of

$$\frac{f''}{f}, \quad \frac{g''}{g}, \quad \frac{f'g'}{fg}.$$

Moreover, the metric is nonsingular if $f'(r_0) = 2\pi$, where $r_0 < 0$ is the root of f nearest 0 (if it exists).

Proof. The proof will be a standard calculation from Riemannian geometry, following [**Bleiler and Hodgson, 1996**].

For notational convenience, set $r = x_1$, $\mu = x_2$, $\lambda = x_3$. Our Riemannian metric can be written in coordinates as

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & f(r)^2 & 0 \\ 0 & 0 & g(r)^2 \end{pmatrix} \quad \text{and} \quad (g^{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & f(r)^{-2} & 0 \\ 0 & 0 & g(r)^{-2} \end{pmatrix}$$

The Christoffel symbols $\Gamma_{ij}^k = \sum_\ell \Gamma_{ij\ell} g^{\ell k}$ can be computed using

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ik}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k}\right).$$

Most of the 27 Γ_{ijk} are zero; the non-zero ones are $\Gamma_{122} = f \cdot f'$, $\Gamma_{133} = g \cdot g'$, $\Gamma_{212} = f \cdot f'$, $\Gamma_{221} = -f \cdot f'$, $\Gamma_{313} = g \cdot g'$, and $\Gamma_{331} = -g \cdot g'$. We then obtain the connection $\nabla_{\partial/\partial x_i}(\partial/\partial x_j) = \sum_k \Gamma_{ij}^k \cdot \partial/\partial x_k$ as follows.

	$\nabla_{\partial/\partial x_i}(\partial/\partial x_j)$	j: 1	2	3
	1	0	$f'/f \cdot \partial/\partial x_2$	$g'/g \cdot \partial/\partial x_3$
i	2	$f'/f \cdot \partial/\partial x_2$	$-f \cdot f' \cdot \partial/\partial x_1$	0
	3	$g'/g \cdot \partial/\partial x_3$	0	$-f \cdot f' \cdot \partial/\partial x_1$

The Riemannian curvature tensor is given by

$$R(X, Y, Z) = (\nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z,$$

and the sectional curvatures

$$K(X, Y) = -\frac{\langle R(X, Y, X), Y\rangle}{|X|^2 |Y|^2 - \langle X, Y\rangle^2}$$

are all convex combinations of the three sectional curvatures

$$K_{ij} = K(\partial/\partial x_i, \partial/\partial x_j)$$

for $\{i, j\} \subset \{1, 2, 3\}$. We compute

$$\begin{aligned}
K_{12} &= -\frac{\langle R(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_1), \partial/\partial x_2\rangle}{\langle \partial/\partial x_1, \partial/\partial x_1\rangle\langle \partial/\partial x_2, \partial/\partial x_2\rangle - \langle \partial/\partial x_1, \partial/\partial x_2\rangle^2} \\
&= -\frac{\langle \nabla_{\partial/\partial x_1} \cdot \nabla_{\partial/\partial x_2}(\partial/\partial x_1) - \nabla_{\partial/\partial x_2} \cdot \nabla_{\partial/\partial x_1}(\partial/\partial x_1), \partial/\partial x_2\rangle}{1 \cdot f^2 - 0^2} \\
&= -\frac{\langle \nabla_{\partial/\partial x_1}(f'/f \cdot \partial/\partial x_2), \partial/\partial x_2\rangle}{f^2} \\
&= -\frac{f''/f \cdot \langle \partial/\partial x_2, \partial/\partial x_2\rangle}{f^2} \\
&= -f''/f.
\end{aligned}$$

A symmetric calculation shows $K_{13} = -g''/g$. Finally

$$\begin{aligned} K_{23} &= -\frac{\langle \nabla_{\partial/\partial x_2} \cdot \nabla_{\partial/\partial x_3}(\partial/\partial x_2) - \nabla_{\partial/\partial x_3} \cdot \nabla_{\partial/\partial x_2}(\partial/\partial x_2), \partial/\partial x_3\rangle}{f^2 \cdot g^2} \\ &= -\frac{\langle -\nabla_{\partial/\partial x_3}(-f' \cdot f \cdot \partial/\partial x_1), \partial/\partial x_3\rangle}{f^2 \cdot g^2} \\ &= -\frac{f' \cdot f \cdot g'/g\langle \partial/\partial x_3, \partial/\partial x_3\rangle}{f^2} \\ &= -\frac{f' \cdot g'}{f \cdot g}. \end{aligned}$$

Finally, to ensure the metric is nonsingular, it must have a cone angle of 2π along the core, i.e. at the point $r = r_0 < 0$ nearest 0 such that $f(r_0) = 0$. If $f(r_0) = 0$, then

$$f'(r_0) = \lim_{r \to r_0} \frac{1}{r - r_0} \int_0^1 f(r) d\mu$$

gives the cone angle along the core circle of the solid torus. Thus we must ensure that $f'(r_0) = 2\pi$. □

LEMMA 13.9. *Suppose V is a solid torus with a prescribed Euclidean metric on ∂V such that a Euclidean geodesic representing a meridian has length $\ell_1 > 2\pi$. Then there exists a smooth Riemannian metric on V that is hyperbolic on a collar neighborhood of ∂V, has negative sectional curvature elsewhere, and the restriction of the metric to ∂V gives the prescribed Euclidean metric.*

PROOF. Let $\widetilde{V}$ denote the universal cover of V. We will assign a metric to $\widetilde{V}$ that has the form of equation (13.1). The functions f and g must satisfy a number of properties.

In order to obtain the prescribed Euclidean metric, we must have $f(0) = \ell_1$ and $g(0) = \ell_2$ where $\ell_2 = \text{area}(V)/\ell_1$. Then the deck transformation group on $\widetilde{V}$ is generated by

$$(r, \mu, \lambda) \mapsto (r, \mu + \theta, \lambda + 1),$$

where $\theta \in [0, 1)$ is appropriately chosen so that the fundamental domain of ∂V has the correct shape. The metric on $\widetilde{V}$ descends to give a smooth metric on V.

In order for the metric to be hyperbolic near ∂V, f and g must give sectional curvatures equal to -1 near $r = 0$. This will hold if $f(r) = \ell_1 e^r$ and $g(r) = \ell_2 e^r$ near $r = 0$. To ensure it is nonsingular, we need to ensure $f'(r_0) = 2\pi$ where r_0 is the negative root of f nearest 0.

To finish the proof of the lemma, we need to show that there exist functions f and g satisfying the above properties. For purposes of this lemma, it suffices to choose r_0 such that $-\ell_1/2\pi < r_0 < -1$ and define f and g near $r = r_0$ by $f(r) = 2\pi \sinh(r - r_0)$ and $g(r) = b \cosh(r - r_0)$,

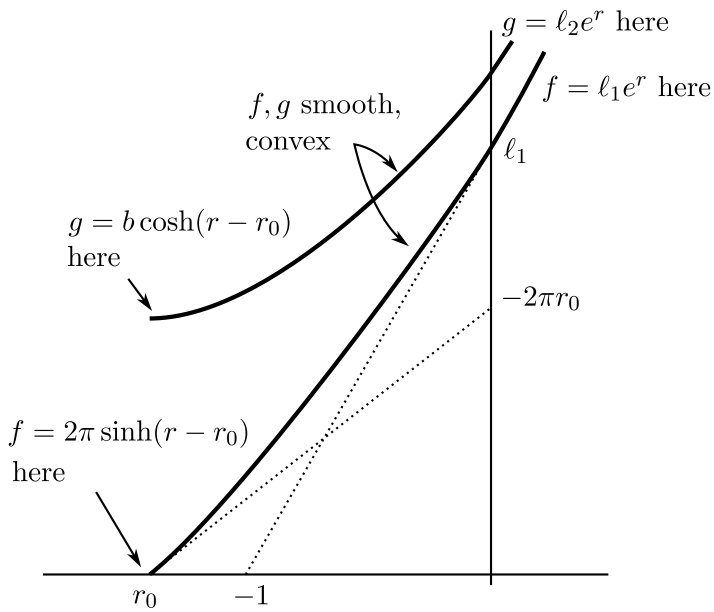

FIGURE 13.1. Extending f and g to be strictly convex, increasing, positve, smooth functions on $r_0 < r < 0$

for $0 < b < \ell_2$. Note that f and g give a metric of constant curvature -1 near r_0, and that $f'(r_0) = 2\pi$, so the metric will be nonsingular. To see that definitions of f and g can be extended, note that the tangent line to f at $r = r_0$ runs through the points $(r_0, 0)$ and $(0, -2\pi r_0)$, and the tangent line to f at $r = 0$ runs through points $(0, \ell_1)$ and $(-1, 0)$. Because $-2\pi r_0 < \ell_1$ and $r_0 < -1$, the function f can be extended to be strictly convex, increasing, positive and smooth on $r_0 < r < 0$; see figure 13.1. Similarly, $g'(r_0) = 0 < \ell_2 = g'(0)$, and $0 < b = g(r_0) < \ell_2 = g(0)$, so g can be extended to be strictly convex, increasing, positive and smooth on $r_0 < r < 0$. This gives the desired negatively curved metric. □

THEOREM 13.10 (2π-Theorem). *Suppose M is a hyperbolic 3-manifold with disjoint embedded cusps $C_1, \dots, C_n$ and slopes s_j on C_j such that a geodesic representative of each s_j on ∂C_j has length strictly greater than 2π in the induced Euclidean metric. Then the Dehn filled manifold $M(s_1, \dots, s_n)$ admits a metric of negative curvature. Thus it is hyperbolic.*

PROOF. Remove the cusps $C_1, \dots, C_n$. By lemma 13.9, there exists a negatively curved metric on a solid torus V_j such that the Euclidean metric on ∂V_j agrees with that of ∂C_j, and such that the metric is hyperbolic on a collar neighborhood of ∂V_j. Then put a metric on $M(S_1, \dots, s_n)$ by taking the hyperbolic metric on $M - (\bigcup_{i=1}^n)$, and gluing in solid tori with the metric from lemma 13.9. □

Note that there is flexibility in choosing the metric of lemma 13.9. For example, the value of b in the proof can be anything in a range of values. If we take a little more care to determine the metric, we can obtain bounds on additional geometric information. For example, we can determine the curvature more explicitly, using the following lemma, and bound the volume of the negatively curved solid torus, as in lemma 13.12.

LEMMA 13.11. *Let $k(r)$ be a smooth, increasing function that lies in $[0,1]$ for all r. Define f and g to be solutions to the differential equations $f''/f = k$ and $(f'g')/(fg) = k$, subject to initial conditions $f(0) = f'(0) = \ell_1$ and $g(0) = \ell_2$. Then the function g satisfies $g''/g = k + (f/f')k'$.*

PROOF. To check the formula for g''/g, note $g'/g = kf/f'$, and differentiate both sides of this equation.

$$\frac{g''}{g} - \left(\frac{g'}{g}\right)^2 = k\left(1 + \frac{f''}{f}\left(\frac{f}{f'}\right)^2\right) + \frac{f}{f'}k'.$$

Using the fact that $g'/g = kf/f'$ and $f''/f = k$, this simplifies to the desired equation:

$$\frac{g''}{g} = k + \frac{f}{f'}k'. \qquad \square$$

LEMMA 13.12. *Let $\ell_1 > 2\pi$, let k be a constant function $k(r) = t \in (0,1)$, and let f and g be defined by the differential equations in lemma 13.11. Let V be a solid torus with metric of equation* (13.1)*. Then:*

(1) Letting $r_0 = -\operatorname{arctanh}(\sqrt{t})/\sqrt{t}$, f and g have the form:

$$f(r) = \frac{\ell_1\sqrt{1-t}}{\sqrt{t}}\sinh(\sqrt{t}(r - r_0))$$
$$g(r) = \ell_2\sqrt{1-t}\cosh(\sqrt{t}(r - r_0))$$

(2) At r_0, $f(r_0) = 0$ and $f'(r_0) = \ell_1\sqrt{1-t}$. Thus the solid torus V has a nonsingular metric of negative curvature $-t$ when $t = 1-(2\pi/\ell_1)^2$.

(3) For any $t \in (0,1)$, the volume of the (possibly singular) solid torus V with metric $ds^2 = dr^2 + f(r)^2 d\mu^2 + g(r)^2 d\lambda^2$ is given by

$$\operatorname{vol}(V) = \int_{r_0}^{0} f(r)g(r)dr = \frac{\ell_1\ell_2}{2}.$$

PROOF. By lemma 13.11 and lemma 13.8, the metric will have negative sectional curvature. We need to show the additional properties. The proof is a series of calculations. First, solving the differential equation $f''/f = t$, the function f has the form

$$f(r) = c_1 e^{\sqrt{t}r} + c_2 e^{-\sqrt{t}r}.$$

Given initial conditions $f(0) = f'(0) = \ell_1$, we find

$$\begin{aligned} f(r) &= \frac{\ell_1}{2}\left(1 + \frac{1}{\sqrt{t}}\right) e^{\sqrt{t}r} + \frac{\ell_1}{2}\left(1 - \frac{1}{\sqrt{t}}\right) e^{-\sqrt{t}r} \\ &= \ell_1 \left(\cosh(r\sqrt{t}) + \frac{1}{\sqrt{t}} \sinh(r\sqrt{t})\right) \\ &= \frac{\ell_1\sqrt{1-t}}{\sqrt{t}} \sinh(\sqrt{t}(r - r_0)), \end{aligned}$$

where $r_0 = -\operatorname{arctanh}(\sqrt{t})/\sqrt{t}$, as claimed. Note that

$$f'(r) = \ell_1\sqrt{1-t}\cosh(\sqrt{t}(r - r_0)),$$

so when $r = r_0$, we have $f(r_0) = 0$ and $f'(r_0) = \ell_1\sqrt{1-t}$. Thus $f'(r_0) = 2\pi$ if $t = 1 - (2\pi/\ell_1)^2$, and the metric will be nonsingular in this case.

As for g, we may solve $g'/g = tf'/f = \sqrt{t}\tanh(\sqrt{t}(r - r_0))$ by integration, to obtain

$$g(r) = c_2 \cosh(\sqrt{t}(r - r_0)) = \ell_2\sqrt{1-t}\cosh(\sqrt{t}(r - r_0)),$$

using the initial condition $g(0) = \ell_2$ to determine the constant c_2.

Finally we compute the volume of a solid torus V with metric as in equation (13.1).

$$\begin{aligned} \operatorname{vol}(V) &= \int_{r_0}^{0} f(r)g(r)\,dr \\ &= \int_{r_0}^{0} \frac{\ell_1\ell_2(1-t)}{\sqrt{t}} \sinh(\sqrt{t}(r - r_0))\cosh(\sqrt{t}(r - r_0)) \\ &= \left[\frac{\ell_1\ell_2(1-t)}{2t}\sinh^2(\sqrt{t}(r - r_0))\right]_{r=r_0}^{0} \\ &= \frac{\ell_1\ell_2(1-t)}{2t}\sinh^2(\operatorname{arctanh}(\sqrt{t})) \\ &= \frac{\ell_1\ell_2(1-t)}{2t}\cdot\frac{t}{1-t} \\ &= \frac{\ell_1\ell_2}{2}. \end{aligned}$$

□

Now we would like to use the metric on the solid torus V obtained from lemma 13.12, along with theorem 13.6, to bound the volume of Dehn filled manifolds. However, at this point we have a problem. Although we have constructed a nonsingular Riemannian metric on the solid torus with nice curvature and volume, note that the metric does not give a hyperbolic metric, with sectional curvatures -1, on a collar neighborhood of the boundary of V. Thus we cannot glue the metric of lemma 13.12 to the metric of the cusped manifold with horoball neighborhoods removed to obtain a negatively curved metric on the Dehn filled manifold, as we did in theorem 13.10. The way

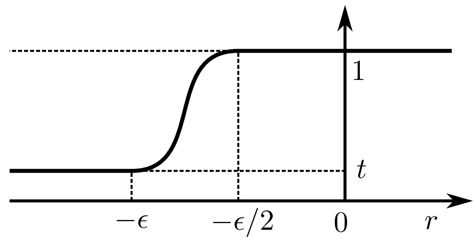

FIGURE 13.2. Graph of $k_{t,\epsilon}(r)$.

to fix this problem is to do a little deeper analysis, which is done in the following lemma.

LEMMA 13.13. *Suppose V is a solid torus with a prescribed Euclidean metric on ∂V, such that a Euclidean geodesic representing a meridian has length $\ell_1 > 2\pi$. Let $\zeta \in (0,1)$ be a constant. Then there exists a smooth, negatively curved Riemannian metric on V that satisfies the following properties.*

(1) The metric is hyperbolic on a collar neighborhood of ∂V, and its restriction to ∂V gives the prescribed Euclidean metric.
(2) The sectional curvatures are bounded above by $-\zeta(1-(2\pi/\ell_1)^2)$.
(3) The volume of V in this metric is at least $\frac{1}{2}\zeta\,\mathrm{area}(\partial V)$.

PROOF. We use the ideas of lemma 13.11 and lemma 13.12 to define f and g by differential equations. However, we do not choose $k(r)$ to be constant. We need $k(r) = 1$ near $r = 0$ to obtain the appropriate curvature estimates on the boundary of V. We have seen in lemma 13.12 that we may obtain nice volume and curvature results when $k(r) = t$ for some $t \in (0,1)$ for $r < 0$. So we define k to be a smooth bump function, depending on r, t, and $\epsilon > 0$, as follows. If $r \leq -\epsilon$, set $k_{t,\epsilon}(r) = t$. If $r \geq -\epsilon/2$, set $k_{t,\epsilon}(r) = 1$. For r between $-\epsilon$ and $-\epsilon/2$, the function $k_{t,\epsilon}(r)$ is smooth and strictly increasing. See figure 13.2 for a typical graph.

Then k is continuous in the three variables t, ϵ, r. We also define $k_{t,0}(r)$ to be the step function

$$k_{t,0}(r) = \lim_{\epsilon \to 0^+} k_{t,\epsilon}(r) = \begin{cases} t & \text{if } r < 0, \\ 1 & \text{if } r \geq 0. \end{cases}$$

Now for $\epsilon \geq 0$ and $t \in (0,1)$, define $f_{t,\epsilon}$ and $g_{t,\epsilon}$ by the differential equations

$$\frac{f''_{t,\epsilon}(r)}{f_{t,\epsilon}(r)} = k_{t,\epsilon}(r), \quad \frac{g'_{t,\epsilon}(r)}{g_{t,\epsilon}(r)} = k_{t,\epsilon}(r)\frac{f_{t,\epsilon}(r)}{f'_{t,\epsilon}(r)}.$$

The family of functions $f_{t,\epsilon}(r)$ and $g_{t,\epsilon}(r)$ can be shown to have a number of nice properties. Away from $\epsilon = 0$, these mostly follow by standard facts in differential equation. As $\epsilon \to 0^+$, a little more analysis is required, which we will omit here. For full details see [**Futer et al., 2008**]. In particular, the following hold.

Nonsingularity: For all $t \in (0,1)$ and $\epsilon \geq 0$, $f_{r,\epsilon}(r)$ has a unique root $r_0(t,\epsilon)$. The function $f'_{t,\epsilon}(r_0(t,\epsilon))$ is continuous in t and ϵ, and strictly decreasing in both variables. For every t between 0 and $1-(2\pi/\ell_1)^2 < 1$, there is a unique value $\epsilon(t) > 0$ such that $f'_{t,\epsilon(t)}(r_0(t,\epsilon(t)) = 2\pi$. This gives a nonsingular metric for every t. Moreover, as $t \to 1-(2\pi/\ell_1)^2$, $\epsilon(t) \to 0$.

Now let $t \in (0,1)$ and define $\tau(t)$ to be the nonsingular Riemannian metric given by the functions $f_t(r) = f_{t,\epsilon(t)}(r)$ and $g_t(r) = g_{t,\epsilon(t)}(r)$.

Sectional curvatures: The metric $\tau(t)$ has all sectional curvatures bounded above by $-t$. This follows from lemma 13.11, along with the fact that the function $f_t(r)$ is positive and increasing. Thus $f_t(r)/f'_t(r)$ is positive. Moreover, $k'_{t,\epsilon(t)}(r)$ is positive, since $k_{t,\epsilon(t)}(r)$ is increasing with r. Thus lemma 13.11 implies that $g''_t(r)/g_t(r)$ is least $k_{t,\epsilon(t)}(r) \geq t$. By definition, $f''_t(r)/f_t(r)$ and $(f'_t(r)g'_t(r))/(f_t(r)g_t(r))$ are equal to $k_{t,\epsilon(t)}(r) \geq t$. So all sectional curvatures are bounded above by $-t$.

Volumes: Recall we have fixed $\zeta > 0$. For notational purposes, define t_0 to be $t_0 = 1-(2\pi/\ell_1)^2$. Let t lie in the interval $(\zeta t_0, t_0)$. Then for the metric $\tau(t)$, we have

$$\lim_{t\to t_0} \mathrm{vol}(V,\tau(t)) = \frac{\ell_1\ell_2}{2} = \frac{1}{2}\,\mathrm{area}\,\partial V.$$

This follows from the fact that f_t and g_t converge uniformly to $f_{t_0,0}$ and $g_{t_0,0}$ as $t \to t_0$. Moreover, $r_0(t,\epsilon(t))$ converges to $r_0(t_0,0)$. Then the limit must be the limit of the differential equation in the case k is constant, which we computed in lemma 13.12. In particular, we have

$$\lim_{t\to t_0} \mathrm{vol}(V,\tau(t)) = \mathrm{vol}(V,t_0) = \frac{\ell_1\ell_2}{2}.$$

To finish the proof of the lemma, select $t \in (\zeta t_0, t_0)$ near enough to t_0 so that $\mathrm{vol}(V,\tau(t)) \geq \frac{1}{2}\zeta\,\mathrm{area}(\partial V)$. For this metric, sectional curvatures are bounded above by $-t \leq -\zeta t_0 = -\zeta(1-(2\pi/\ell_1)^2)$. Finally, the metric is nonsingular, and by choice of bump function and initial conditions, on a collar neighborhood of ∂V it is hyperbolic, with metric agreeing with the prescribed metric on ∂V. □

We are now ready to prove the main result of this section.

THEOREM 13.14 (Volume change under Dehn filling). *Let M be a complete, finite volume hyperbolic manifold with cusps. Suppose $C_1,\dots,C_n$ are disjoint embedded cusps with slopes s_j on C_j such that a geodesic representative of s_j on ∂C_j has length strictly greater than 2π. Denote the minimal slope length by $\ell_{\min}$. Then the Dehn filled manifold $M(s_1,\dots,s_n)$ is a hyperbolic manifold with*

$$\mathrm{vol}(M(s_1,\dots,s_n)) \geq \left(1-\left(\frac{2\pi}{\ell_{\min}}\right)^2\right)^{3/2}\mathrm{vol}(M).$$

PROOF. Fix an arbitrary constant $\zeta > 0$. Replace each cusp C_j by a solid torus V_j whose meridian is s_j. By lemma 13.13, the smooth Riemannian metric τ_j on V_j agrees with the hyperbolic metric on ∂C_j, so this gives a smooth Riemannian metric τ on $M(s_1, \ldots, s_n)$. Additionally, for each j, sectional curvatures on V_j are at most $-\zeta(1-(2\pi/\ell_{\min})^2)$, and the volume of V_j is at least $\zeta\,\mathrm{area}(\partial V_j)/2 = \zeta\,\mathrm{vol}(C_j)$ where $\mathrm{vol}(C_j)$ is the cusp volume in the hyperbolic metric. Note that sectional curvatures in V_j are also bounded below by some constant, since V_j is compact.

Thus the metric τ on $M(s_1, \ldots, s_n)$ has sectional curvatures bounded above by $-\zeta(1-(2\pi/\ell_{\min})^2)$ and bounded below by some constant. Moreover

$$\begin{aligned} \mathrm{vol}(M(s_1, \ldots, s_n)) &\geq \mathrm{vol}(M - \cup_{j=1}^n C_j) + \zeta \sum \mathrm{vol}(C_j) \\ &\geq \zeta\,\mathrm{vol}(M). \end{aligned}$$

We rescale the metric to obtain a metric with sectional curvatures bounded above by -1. To do this, replace τ by $\sigma = \sqrt{\zeta(1-(2\pi/\ell_{\min})^2)}\tau$. Note this multiplies all sectional curvatures by $(\zeta(1-(2\pi/\ell_{\min})^2))^{-1}$. The volume is rescaled by a factor of $(\zeta(1-(2\pi/\ell_{\min})^2))^{3/2}$. Thus under the metric σ, sectional curvatures of $M(s_1, \ldots, s_n)$ lie in $[-a, 1]$ for some $a \geq 1$, and $\mathrm{vol}(M(s_1, \ldots, s_n), \sigma) \geq \zeta^{5/2}(1-(2\pi/\ell_{\min})^2)^{3/2}\,\mathrm{vol}(M)$.

Now let S denote the set of all metrics on $M(s_1, \ldots, s_n)$ whose sectional curvatures lie in the interval $[-a, -1]$. Since ζ is arbitrary, by the above work the supremum of volumes of $M(s_1, \ldots, s_n)$ over all metrics in S satisfies:

$$\sup_{\sigma \in S} \mathrm{vol}(M(s_1, \ldots, s_n), \sigma) \geq \left(1 - \left(\frac{2\pi}{\ell_{\min}}\right)^2\right)^{3/2} \mathrm{vol}(M);$$

here $\mathrm{vol}(M(s_1, \ldots, s_n), \sigma)$ denotes the volume under the metric σ, and $\mathrm{vol}(M)$ denotes the volume of M under its given hyperbolic metric.

Now, theorem 13.6 implies that the hyperbolic metric σ_{hyp} on the Dehn filled manifold $M(s_1, \ldots, s_n)$ uniquely maximizes volume over the set S of all metrics whose sectional curvatures lie in the interval $[-a, 1]$. Thus we have

$$\begin{aligned} \mathrm{vol}(M(s_1, \ldots, s_n), \sigma_{\mathrm{hyp}}) &\geq \sup_{\sigma \in S} \mathrm{vol}(M(s_1, \ldots, s_n), \sigma) \\ &\geq \left(1 - \left(\frac{2\pi}{\ell_{\min}}\right)^2\right)^{3/2} \mathrm{vol}(M). \end{aligned}$$

□

13.2.2. Applications to knots. Recall the definitions of *twist-reduced* from definition 7.15 or definition 11.10, and *twist-number* from definition 11.13. We will denote the twist-number of a twist-reduced diagram K by $\mathrm{tw}(K)$. An application of theorem 13.14 is the following result, which first appeared in [**Futer et al., 2008**].

THEOREM 13.15 (Volume bounds for highly twisted links). *Let $K \subset S^3$ be a link with a prime, twist-reduced diagram. Assume the diagram has*

$\mathrm{tw}(K) \geq 2$ *twist regions, and that each twist region contains at least seven crossings. Then* K *is a hyperbolic link satisfying*

$$0.70734(\mathrm{tw}(K) - 1) \leq \mathrm{vol}(S^3 - K) < 10v_{\mathrm{tet}}(\mathrm{tw}(k) - 1),$$

where $v_{\mathrm{tet}} = 1.0149\ldots$ *is the volume of a regular ideal tetrahedron.*

The proof of theorem 13.15 uses a theorem due to Miyamoto, which in full generality gives a lower bound on the volume of an n-dimensional hyperbolic manifold with geodesic boundary [**Miyamoto, 1994**]. We state here only the 3-dimensional case, which is the case we will use.

THEOREM 13.16 (Miyamoto). *If* N *is a hyperbolic 3-manifold with totally geodesic boundary, then* $\mathrm{vol}(N) \geq -v_{\mathrm{oct}}\chi(N)$, *where* $v_{\mathrm{oct}} = 3.66\ldots$ *is the volume of a regular ideal octahedron.* □

Given Miyamoto's theorem, we prove theorem 13.15.

PROOF OF THEOREM 13.15. The fact that the link K is hyperbolic follows from theorem 8.47. The upper bound on volume comes from theorem 13.2.

To obtain the lower bound, we will consider fully augmented links. Let L be the fully augmented link obtained by adding a crossing circle encircling each twist region of K. By theorem 7.26, $S^3 - K$ is obtained from $S^3 - L$ by performing Dehn fillings on crossing circles, along a slopes of length at least $\sqrt{7^2 + 1} = \sqrt{50} > 2\pi$.

We will find a lower bound on the volume of $S^3 - L$. To do so, first remove all half-twists from the diagram of L. That is, recall L may have single crossings at twist regions. Replace L with a new fully augmented link L' that has no crossing at twist regions. Note the complement of L' is obtained from that of L by cutting along 2-punctured disks bounded by crossing circles and regluing, thus it follows from theorem 12.1 that the volume of $S^3 - L'$ is identical to the volume of $S^3 - L$.

Now cut $S^3 - L'$ along the plane of projection, separating it into two identical pieces, each with totally geodesic boundary coming from the white surface. Call one of these M. By Miyamoto's theorem, $\mathrm{vol}(M) \geq -v_{\mathrm{oct}}\chi(M)$. Note that M is homeomorphic to a ball in S^3 with a tube drilled out for each crossing circle, and there are $\mathrm{tw}(K)$ crossing circles. Thus the Euler characteristic of M is $\chi(M) = (1 - \mathrm{tw}(K))$. Because we form $S^3 - L'$ by taking two copies of M, the volume satisfies

$$\mathrm{vol}(S^3 - L) = \mathrm{vol}(S^3 - L') \geq -2v_{\mathrm{oct}}\chi(M) = 2v_{\mathrm{oct}}(\mathrm{tw}(K) - 1).$$

Now by theorem 13.14 (volume change under Dehn filling), the volume of $S^3 - K$ satisfies:

$$\begin{aligned}\mathrm{vol}(S^3 - K) &\geq \left(1 - \left(\frac{2\pi}{\sqrt{50}}\right)^2\right)^{3/2} \mathrm{vol}(S^3 - L) \\ &\geq \left(1 - \frac{2\pi^2}{25}\right)^{3/2} 2v_{\mathrm{oct}}(\mathrm{tw}(K) - 1) \\ &\geq 0.70735(\mathrm{tw}(K) - 1). \qquad \square\end{aligned}$$

13.3. Volume, guts, and essential surfaces

Theorem 13.15 gives volume bounds highly twisted knots and links, but only with at least seven crossings per twist region. A similar result holds for alternating knots and links, without a restriction on the number of crossings for twist regions, originally due to Lackenby [**Lackenby, 2004**]. The method of proof is different, but illustrates another tool for bounding hyperbolic volume from below, developed by Agol, Storm, and Thurston [**Agol et al., 2007**]. In this section, we explain the tool, and use it to bound volumes of alternating links.

The main theorem of the section is the following.

Theorem 13.17 (Volume bounds for alternating links). *Let K be a hyperbolic knot or link with a twist-reduced alternating diagram with twist number* $\mathrm{tw}(K)$. *Then*

$$\mathrm{vol}(S^3 - K) \geq \frac{v_{\mathrm{oct}}}{2}(\mathrm{tw}(K) - 2).$$

We will prove theorem 13.17 by considering again the checkerboard surfaces of the alternating link, and the bounded polyhedral decomposition of the link complement cut along those surfaces, theorem 11.6 and lemma 11.25. To describe our main tool, we need additional terminology.

First, recall the JSJ-decomposition of a 3-manifold, theorem 8.23 and definition 8.24. We will apply a special form of this decomposition to a 3-manifold M cut along an essential surface S. Recall that $M\backslash\backslash S$ is the closure of the manifold obtained by removing a regular neighborhood of S (definition 11.23). Its boundary consists of components of the parabolic locus, which are remnants of the torus boundary components of M, and $\widetilde{S}$.

Definition 13.18. Let M be a compact 3-manifold with torus boundary components, and let S be an essential surface properly embedded in M. The *double of* $M\backslash\backslash S$, denoted $D(M\backslash\backslash S)$ is the manifold obtained by taking two copies of $M\backslash\backslash S$ and gluing them by the identity map on $\widetilde{S}$.

Note the double of $M\backslash\backslash S$ will have torus boundary components coming from the parabolic locus of the boundary of $M\backslash\backslash S$. We will consider the JSJ-decomposition of the double, as in definition 8.24.

LEMMA 13.19. *Let M be a hyperbolic 3-manifold, homeomorphic to the interior of a compact manifold with torus boundary. Let S be a properly embedded essential surface in M. Consider the double $D(M\backslash\backslash S)$, and let $\mathcal{T}$ denote the JSJ-decomposition of $D(M\backslash\backslash S)$. Finally, slice $\mathcal{T}$ and $D(M\backslash\backslash S)$ along $\widetilde{S}$, obtaining two copies of $M\backslash\backslash S$. The following hold.*

1. *The tori in the collection $\mathcal{T}$ can be isotoped to be preserved by the reflection of $D(M\backslash\backslash S)$ in the surface $\widetilde{S}$; thus cutting $D(M\backslash\backslash S)$ along $\widetilde{S}$ cuts $\mathcal{T}$ and its complement into two identical pieces.*
2. *Each essential torus $T \in \mathcal{T}$ is sliced into essential annuli in $M\backslash\backslash S$ with boundary on $\widetilde{S}$.*
3. *The characteristic submanifold of $D(M\backslash\backslash S)$ intersects $M\backslash\backslash S$ in components that are either I-bundles over a subsurface of $\widetilde{S}$, or Seifert fibered solid tori.*

PROOF. The first item follows from the equivariant torus theorem, due to Holzmann [**Holzmann, 1991**]. It is an exercise to prove the remaining two items; exercise 13.5. □

DEFINITION 13.20. Let M, S, $\mathcal{T}$ be as in lemma 13.19. We say that the intersection of the characteristic submanifold of $D(M\backslash\backslash S)$ with $M\backslash\backslash S$ is the *characteristic submanifold of* $M\backslash\backslash S$. Its complement in $M\backslash\backslash S$ is the *guts* of S, denoted $\text{guts}(M\backslash\backslash S)$ or sometimes simply $\text{guts}(S)$.

EXAMPLE 13.21. Recall from example 12.26 that the figure-8 knot complement M contains a sufrace S that is a fiber, shown in figure 12.4. This surface S is essential and properly embedded. The manifold $M\backslash\backslash S$ is homeomorphic to $S \times I$. Thus in this case, all of $M\backslash\backslash S$ is an I-bundle. Thus $\text{guts}(M\backslash\backslash S)$ is empty.

By contrast, later in this section we will find examples of alternating knot complements M and surfaces S such that $\text{guts}(M\backslash\backslash S) = M\backslash\backslash S$, that is the window is empty.

LEMMA 13.22. *For M, S, and $\mathcal{T}$ as in lemma 13.19, the manifold* $\text{guts}(M\backslash\backslash S)$ *admits a hyperbolic metric with totally geodesic boundary.*

PROOF. If we double $\text{guts}(M\backslash\backslash S)$ along the portion of the boundary on $\widetilde{S}$, i.e. take two copies of $\text{guts}(M\backslash\backslash S)$ and glue by the identity along their common boundary on $\widetilde{S}$, we obtain the complement of the characteristic submanifold of the manifold $D(M\backslash\backslash S)$. This admits a finite volume hyperbolic metric. It also admits an involution fixing the surface $\text{guts}(M\backslash\backslash S) \cap \widetilde{S}$ pointwise. It follows from the proof of the Mostow–Prasad rigidity theorem that an embedded surface fixed pointwise by an involution of a finite volume hyperbolic 3-manifold must be totally geodesic. Hence cutting along it yields a hyperbolic structure on $\text{guts}(M\backslash\backslash S)$ with totally geodesic boundary. □

The following theorem, from [**Agol et al., 2007**], gives us a tool to bound volumes from below using the guts of surfaces.

THEOREM 13.23 (Agol, Storm, and Thurston). *Let S be a π_1-essential surface properly embedded in an orientable hyperbolic 3-manifold M. Then*

$$\mathrm{vol}(M) \geq -v_{\mathrm{oct}}\chi(\mathrm{guts}(M\backslash\backslash S)),$$

where $v_{\mathrm{oct}} = 3.66\ldots$ is the volume of a regular ideal octahedron.

PROOF SKETCH. The essential surface S can be isotoped to be minimal in M. Cut along the minimal surface isotopic to S, and denote $M\backslash\backslash S$ by N. Note that N inherits from M a Riemannian metric for which the mean curvature on its boundary $\widetilde{S}$ is 0. Denote this metric by g. Then $\mathrm{vol}(M) = \mathrm{vol}(N, g)$.

Let $D(N)$ denote the double of N, i.e. the manifold obtained by taking two copies of N and identifying them along their common boundary. By lemma 13.22, $D(\mathrm{guts}(N)) \subset D(N)$ inherits a complete hyperbolic metric with $S \cap \mathrm{guts}(N)$ a totally geodesic surface embedded in $D(\mathrm{guts}(N))$; denote this metric by h. On the other hand, $D(N)$ inherits a singular Riemannian metric with singularities on S obtained from the metric g; denote this metric by g as well. Agol, Storm, and Thurston show that this singular metric g can be approximated by smooth Riemannian metrics $\{g_i\}$ with restricted curvature properties. The volumes $\mathrm{vol}(D(N), g_i)$ under the smooth metrics g_i converge to the volume $\mathrm{vol}(D(N), g)$ under its singular metric, with $\mathrm{vol}(D(N), g) = 2\,\mathrm{vol}(N, g) = 2\,\mathrm{vol}(M)$.

First suppose M is closed. Use Ricci flow with surgery to evolve the metric, as in Perelman's proof of the geometrization theorem [**Perelman, 2002, Perelman, 2003**]. The evolution will give $D(\mathrm{guts}(N))$ the hyperbolic metric h. Perelman's techniques imply a monotonicity result, in particular that

$$\mathrm{vol}(D(N), g) \geq \mathrm{vol}(D(\mathrm{guts}(N)), h),$$

with equality if and only if S is totally geodesic in $(D(N), g)$. Then

$$\mathrm{vol}(M) = \frac{1}{2}\,\mathrm{vol}(D(N), g) \geq \mathrm{vol}(\mathrm{guts}(M\backslash\backslash S)),$$

with equality if and only if S is totally geodesic in M.

If M has torus boundary, then similar techniques can be used to approximate the metric on compact sets, giving the same result.

Finally, $(\mathrm{guts}(M\backslash\backslash S), h)$ is a hyperbolic manifold with totally geodesic boundary. Thus by Miyamoto's theorem, theorem 13.16,

$$\mathrm{vol}(\mathrm{guts}(M\backslash\backslash S)) \geq v_{\mathrm{oct}}\chi(\mathrm{guts}(M\backslash\backslash S)). \qquad \square$$

We will be considering the guts of the checkerboard surfaces in an alternating link. By theorem 11.31, the checkerboard surfaces are essential, and thus satisfy the hypotheses of theorem 13.23.

LEMMA 13.24. *Let S be one of the checkerboard surfaces of a link with a twist-reduced alternating diagram K, whose hyperbolic complement we denote by $S^3 - K = M$. Without loss of generality say S is shaded, with W the other checkerboard surface, colored white. Suppose in the polyhedral decomposition* *of M that there are white bigon faces. Then a neighborhood of each white bigon face is part of an I-bundle in $M\backslash\backslash S$, and thus any bigon face of W does not belong to* $\mathrm{guts}(M\backslash\backslash S)$.

PROOF. The I-bundle components of $M\backslash\backslash S$ have the form $Y \times I$, with $Y \times \{0\}$ and $Y \times \{1\}$ subsets of $\widetilde{S} \subset \partial(M\backslash\backslash S)$. Recall that $Y \times \{0\}$ and $Y \times \{1\}$ form the *horizontal boundary* of the I-bundle. The subset $\partial Y \times I$ forms the *vertical boundary*.

A white bigon region of the polyhedral decomposition of M is bounded by two edges and two vertices; recall that edges are crossing arcs, and lie in the intersection $W \cap S$, and the vertices are ideal, forming portions of the parabolic locus. Thus a regular neighborhood of a white bigon face can be visualized as a thickened square, with two sides of its boundary on S and two sides of its boundary on the parabolic locus. Note such a thickened square is a portion of an I-bundle, with horizontal boundary a neighborhood in S of the two crossing arcs of $W \cap S$ that form the boundary of the bigon, and vertical boundary on the parabolic locus. We can complete this thickened square to an I-bundle over a subsurface of S with boundary by attaching a neighborhood of the annulus that forms the parabolic locus. This annulus has boundary components on S, and is parallel to a link component. Its neighborhood can be given the structure of an I-bundle with fibers I parallel to those of the bigon. Thus the union of the neighborhood of the bigon and the neighborhood of this annulus (or possibly two annuli in the case of a link) forms an I-bundle in $M\backslash\backslash S$. □

COROLLARY 13.25. *Let K be a twist-reduced diagram of a hyperbolic alternating link. Let K' be the diagram obtained from K by removing all crossings but one in each twist region of K. Let S denote a checkerboard surface of K, and let S' denote the corresponding checkerboard surface of K'. Then*

$$\mathrm{guts}((S^3 - K)\backslash\backslash S) = \mathrm{guts}((S^3 - K')\backslash\backslash S').$$

13.3.1. Essential annuli. Our method of proving the volume bound on alternating links, theorem 13.17, is to apply the volume bound via guts, theorem 13.23, to the modified diagram K' of K, as in corollary 13.25. We will determine the Euler characteristic of the guts of checkerboard surfaces of K'. Recall that to identify guts, we must first cut along essential annuli. Thus the next step in the proof is to find essential annuli in the cut manifold.

Suppose there is an essential annulus. Then the proof most easily breaks into two cases, depending on whether the annulus is *parabolically compressible* or not, in the sense of the following definition.

DEFINITION 13.26. Let M be a hyperbolic 3-manifold with a properly embedded essential surface S. Let P denote the parabolic locus of $M\backslash\backslash S$. An annulus A, properly embedded in $M\backslash\backslash S$ with $\partial A \subset \widetilde{S}$, is *parabolically compressible* if there exists a disk D with interior disjoint from A, with ∂D meeting A in an essential arc α on A, and with $\beta = \partial D - \alpha$ lying on $\widetilde{S} \cup P$,

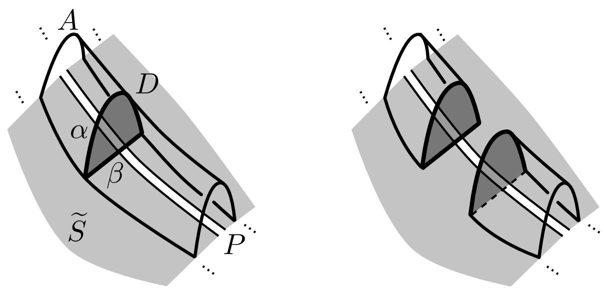

FIGURE 13.3. A portion of a parabolically compressible annulus on the left, and a parabolic compression on the right.

with β meeting P transversely exactly once. We may surger along such a disk; this is called a *parabolic compression*, and it turns the annulus A into a disk meeting P transversely exactly twice, with boundary otherwise on $\widetilde{S}$. See figure 13.3.

DEFINITION 13.27. Let D be a disk properly embedded in $M\backslash\backslash S$ with boundary consisting of two arcs on $\widetilde{S}$ and two arcs on the parabolic locus P. We say D is a *essential product disk (EPD)*.

A proof nearly identical to that of lemma 13.24 shows that EPDs belong to the I-bundle of $M\backslash\backslash S$ (exercise 13.6).

LEMMA 13.28. *Let S be the shaded checkerboard surface of a link with a twist-reduced alternating diagram K, whose hyperbolic complement we denote by $S^3 - K = M$. Suppose there are no white bigon regions, and suppose A is an essential annulus properly embedded in $M\backslash\backslash S$, disjoint from the parabolic locus and not parallel to the parabolic locus, with $\partial A \subset \widetilde{S}$. Then A is not parabolically compressible.*

The proof of lemma 13.28 is completed by considering how a parabolically compressible annulus intersects the polyhedra in the decomposition of an alternating link, similar to several proofs in chapter 11. The following lemma will be useful.

LEMMA 13.29. *Let K be a link with a prime, twist-reduced alternating diagram, with corresponding ideal polyhedral decomposition. Let D_1 and D_2 be normal disks in the polyhedra such that ∂D_1 and ∂D_2 meet exactly four interior edges. Isotope ∂D_1 and ∂D_2 to minimise intersections $\partial D_1 \cap \partial D_2$ in faces. If ∂D_1 intersects ∂D_2, then ∂D_1 intersects ∂D_2 exactly twice, in two faces of the same color.*

PROOF. The boundaries ∂D_1 and ∂D_2 are quadrilaterals, with sides of ∂D_i between intersections with interior edges. Note that ∂D_1 can intersect ∂D_2 at most once in any of its sides by the requirement that the number of intersections be minimal (else isotope through a face). Thus there are at most four intersections of ∂D_1 and ∂D_2. If ∂D_1 meets ∂D_2 four times, then the two quads run through the same faces, both bounding disks, and can be isotoped off each other using the fact that the diagram is prime. Since the quads intersect an even number of times, there are either zero or two intersections. If zero intersections, we are done.

So suppose there are two intersections. Suppose ∂D_1 intersects ∂D_2 exactly twice in faces of the opposite color. Then an arc $\alpha_1 \subset \partial D_1$ has both endpoints on $\partial D_1 \cap \partial D_2$ and meets only one intersection of ∂D_1 with an interior edge of the polyhedron. Similarly, an arc $\alpha_2 \subset \partial D_2$ has both endpoints on $\partial D_1 \cap \partial D_2$ and meets only one intersection of ∂D_2 with an interior edge of the polyhedral decomposition. Then $\alpha_1 \cup \alpha_2$ is a closed curve on F meeting exactly two interior edges. This gives a curve in the diagram of K meeting K exactly twice. Because the diagram is prime, there must be no crossings on one side of the curve. In the polyhedron, this means the arcs α_1 and α_2 are parallel, and we can isotope them to remove the intersections of D_1 and D_2. □

Proof of lemma 13.28. Suppose by way of contradiction that A is an essential, parabolically compressible annulus properly embedded in $M \backslash\backslash S$ with $\partial A \subset \widetilde{S}$. Perform a parabolic compression to obtain an EPD E. Put E into normal form with respect to the polyhedral decomposition of $M \backslash\backslash S$.

Suppose E intersects a white face V of W. Consider the arcs $E \cap V$; such an arc has both endpoints on $\widetilde{S}$. If one cuts off a disk on E that does not meet the parabolic locus, then there will be an innermost such disk. Its boundary consists of an arc in W and and arc in S. We may sketch the boundary of the disk on the diagram of K, since the graph on the polyhedra is identical to the projection graph of the diagram (see theorem 11.6). Thus this innermost disk has boundary intersecting the link diagram exactly twice. Because the diagram of K is prime, this disk bounds a region containing no crossings. But then the original innermost disk in the polyhedron it is not normal: its boundary runs from a single edge back to that edge. This is a contradiction.

So $E \cap W$ consists of arcs running from $\widetilde{S}$ to $\widetilde{S}$, cutting off disks on either side meeting the parabolic locus. Thus the white surface cuts E into normal quadrilaterals $\{E_1, dots, E_n\}$, with $n \geq 2$ by assumption that E intersects a white face. On the end of E, the quadrilateral E_1 has one side on W, two sides on $\widetilde{S}$ and the final side on the parabolic locus (a boundary face). Isotope slightly off the boundary face into the adjacent white face so that E_1 remains normal. Do the same for E_n. Then all quadrilaterals $E_1, \dots, E_n$ have two sides on S and two sides on W.

Superimpose E_1 and E_2 onto the boundary of one of the (identical) polyhedra. An edge of E_1 in a white face V is glued to an edge of E_2 in the same white face, but by a rotation in the face V. Thus when we superimpose, $\partial E_2 \cap V$ is obtained from $\partial E_2 \cap V$ by a rotation in V.

If $E_1 \cap V$ is not parallel to a single boundary edge, then $\partial E_1 \cap V$ must intersect $\partial E_2 \cap V$; see figure 13.4. Then lemma 13.29 implies that ∂E_1 and

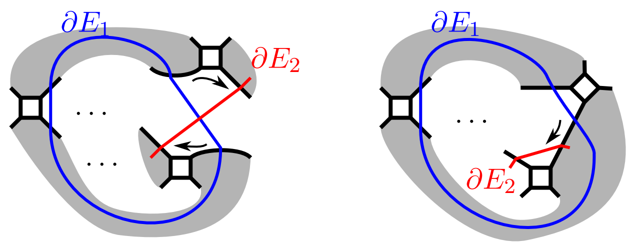

FIGURE 13.4. Left: ∂E_1 is not parallel to a boundary edge in U, hence ∂E_1 meets ∂E_2 in U. Right: ∂E_1 is parallel to a boundary edge. Figure originally appeared in [**Howie and Purcell, 2017**].

∂E_2 also intersect in another white face. But ∂E_1 is parallel to a single boundary edge in its second white face, so ∂E_2 cannot intersect it. This is a contradiction.

So $E_1 \cap V$ is parallel to a single boundary edge (and hence so is $E_2 \cap V$). But then E_1 meets both white faces in arcs parallel to boundary edges. Isotoping ∂E_1 slightly into these boundary faces and transfer the curve to the diagram of the link. This gives a closed curve in the link diagram meeting the projection graph of K in exactly two crossings, running to opposite sides of the crossings. If the crossings are distinct, then because the diagram is twist-reduced, the two crossings must bound white bigons between them, contradicting the fact that there are no white bigon regions in the diagram. So the crossings are not distinct. Returning to the polyhedron, ∂E_1 encircles a single ideal vertex of the polyhedron. Repeating the argument with E_2 and E_3, and so on, we find that each ∂E_i encircles a single ideal vertex. Gluing these together, the original annulus A is parallel to the parabolic locus. This contradicts our assumption on A.

So if there is an EPD E, it cannot meet W. Then it lies completely in a single polyhedron of the decomposition. Its boundary runs through two shaded faces and two boundary faces. Transfer to the link diagram; its boundary defines a curve meeting the link diagram in exactly two crossings, running to opposite sides of the crossings. Because the diagram is twist-reduced, the curve ∂E encloses a string of white bigons. But there are no white bigons, so ∂E must run in and out of the same boundary face. This contradicts the fact that it was normal. □

LEMMA 13.30. *Let K be a hyperbolic alternating link with a prime, twist-reduced diagram and corresponding polyhedral decomposition. Let M denote $S^3 - K$ and let S denote the shaded checkerboard surface. Suppose that there are no white bigons in the polyhedra. Suppose A is an essential annulus embedded in $M \backslash\backslash S$, disjoint from the parabolic locus and not parallel to it, with $\partial A \subset \widetilde{S}$. Then A bounds a Seifert fibered solid torus.*

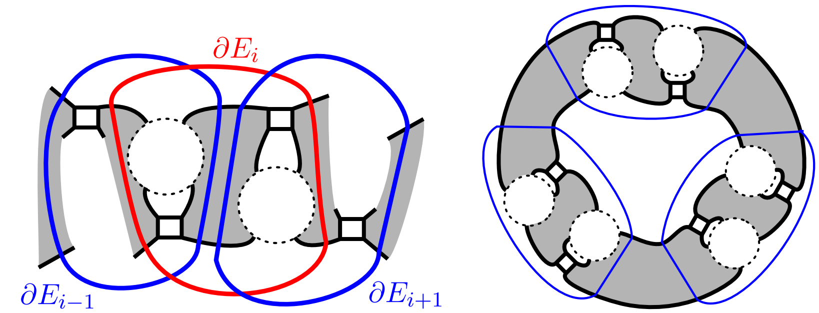

FIGURE 13.5. Left: E_{i-1}, E_i, and E_{i+1} must intersect as shown. Right: cycle of three such tangles.

PROOF. By lemma 13.28 we may assume that A is not parabolically compressible. Put it into normal form with respect to the polyhedral decomposition. Because the Euler characteristic of an annulus is 0, each normal disk making up A must have combinatorial area 0 by the Gauss–Bonnet lemma, lemma 8.35. Because A does not meet the parabolic locus, each such disk must meet exactly four interior edges; see definition 8.30. Thus the white surface W cuts A into squares $E_1, \dots, E_n$. Note that if a component of intersection of $E_i \cap W$ is parallel to a boundary edge, then the disk of W bounded by $E_i \cap W$, the boundary edge, and portions of edges of $\widetilde{S} \cap W$ defines a parabolic compression disk for A, contradicting the fact that A cannot be parabolically compressible. So no component of $E_i \cap W$ is parallel to a boundary edge.

Again superimpose all squares $E_1, \dots, E_n$ on one of the polyhedra. The squares are glued in white faces, and cut off more than a single boundary edge in each white face, so ∂E_i must intersect ∂E_{i+1} in a white face; see again figure 13.4. Then lemma 13.29 implies ∂E_i intersects ∂E_{i+1} in both of the white faces it meets. Similarly, ∂E_i intersects ∂E_{i-1} in both its white faces. Because E_{i-1} and E_{i+1} lie in the same polyhedron, they are disjoint (or $E_{i-1} = E_{i+1}$, but this makes A a Möbius band rather than an annulus; see exercise 13.7). This is possible only if E_{i-1}, E_i, and E_{i+1} line up as in figure 13.5 left, bounding portions of the polyhedron as shown. These transfer to the link diagram to bound tangles; Lackenby calls such tangles *units* in [**Lackenby, 2004**]. Then all E_j form a cycle of such tangles, as in figure 13.5 right.

Observe from figure 13.5 that each disk E_i encircles two units, with a band of shaded surface between E_i and E_{i+2} in the same polyhedron. Then in each polyhedron, these disks of A bound a solid cylinder (a ball) with top and bottom on white faces — one the central region of figure 13.5, right, and one the unbounded region — and sides along disks E_j and shaded faces. The two solid cylinders glue across white faces with a twist, to form a solid torus. As each cylinder can be written as $D^2 \times I$, with $D^2 \times \{0\}$ and $D^2 \times \{1\}$ on white faces, the gluing by a twist in the white face gives the solid torus a Seifert fibering. Thus A bounds a Seifert fibered solid torus. □

THEOREM 13.31. *Let K be a link with a prime, twist-reduced alternating diagram, and corresponding polyhedral decomposition. Let M denote the complement of K, let S and W denote the checkerboard surfaces, and let r_S and r_W denote the number of non-bigon regions of S and W respectively. Then*

$$\chi(\text{guts}(M\backslash\backslash S)) = 2 - r_W, \quad \chi(\text{guts}(M\backslash\backslash W)) = 2 - r_S.$$

PROOF. Suppose first that the diagram has no white bigon regions. Then lemma 13.28 implies there is no embedded essential annulus that is parabolically compressible, and lemma 13.30 implies any parabolically incompressible annulus bounds a Seifert fibered solid torus. It follows that $\chi(\text{guts}(M\backslash\backslash S)) = \chi(M\backslash\backslash S)$. Since $M\backslash\backslash S$ is obtained by gluing two balls along white faces, $\chi(M\backslash\backslash S) = 2 - r_W$.

If the diagram contains white bigon regions, then replace each string of white bigons in the diagram by a single crossing, obtaining a new link K'. Let M' denote $S^3 - K'$ and let S' be the checkerboard surface coming from the same shaded regions as S in K. By corollary 13.25, $\text{guts}(M\backslash\backslash S) = \text{guts}(M'\backslash\backslash S')$. Hence $\chi(\text{guts}(M\backslash\backslash S)) = \chi(\text{guts}(M'\backslash\backslash S')) = 2 - r_W$.

An identical argument applies to $M\backslash\backslash W$, replacing S with W. □

Theorem 13.17 is now almost an immediate consequence of theorem 13.31 and theorem 13.23.

PROOF OF THEOREM 13.17. Let Γ be the 4-regular diagram graph associated to K by replacing each twist-region with a vertex. Let $|v(\Gamma)|$ denote the number of vertices of Γ, and $|f(\Gamma)|$ the number of regions. Because Γ is 4-valent, the number of edges is $2|v(\Gamma)|$, so

$$\chi(S^2) = 2 = -|v(\Gamma)| + |f(\Gamma)| = -\text{tw}(K) + r_S + r_W.$$

Then applying Theorems 13.23 and 13.31 gives

$$\begin{aligned}
\text{vol}(S^3 - K) &\geq -\frac{1}{2}v_{\text{oct}}\chi(\text{guts}(M\backslash\backslash S)) - \frac{1}{2}v_{\text{oct}}\chi(\text{guts}(M\backslash\backslash W)) \\
&= -\frac{1}{2}v_{\text{oct}}(2 - r_S - r_W) \\
&= \frac{1}{2}v_{\text{oct}}(\text{tw}(K) - 2).
\end{aligned}$$
□

13.4. Exercises

EXERCISE 13.1. Show the upper bound of theorem 13.2 is asymptotically sharp, in two steps. First, show there is a sequence of fully augmented links L_i with $t(L_i)$ crossing circles such that $\text{vol}(S^3 - L_i)/t(L_i)$ approaches $10v_{\text{tet}}$ as i goes to infinity. (Hint: take white faces to be regular hexagons.) Then show that there is a sequence of links K_i with twist number $t(K_i)$ such that $\text{vol}(S^3 - K_i)/t(K_i)$ approaches $10v_{\text{tet}}$ as i goes to infinity.

EXERCISE 13.2. Use theorem 9.10 to give an upper bound on the volume of a 2-bridge knot with continued fraction expansion $[0, a_{n-1}, \dots, a_1]$. Find an example of a 2-bridge knot such that your upper bound becomes $2v_{\text{tet}}(\text{tw}(K) - 1)$. Use this to show that the lower bound of theorem 13.4 is asymptotically sharp: The ratio of upper and lower bounds goes to 1 as $\text{tw}(K) \to \infty$.

EXERCISE 13.3. The volume of a regular ideal octahedron is denoted by v_{oct}. In exercise 7.10, it was shown that a fully augmented 2-bridge link decomposes into regular ideal octahedra. Use this to prove that the volume of a 2-bridge link with twist number $\text{tw}(K)$ is at most $2v_{\text{oct}}(\text{tw}(K) - 1)$.

EXERCISE 13.4. Suppose K is a link complement that admits a rotational symmetry about an axis, with order p. That is, suppose there is a curve γ in S^3 such that a rotation of order p about γ preserves K. Show that if $p \geq 7$,

$$\text{vol}(S^3 - K) \geq \left(1 - \frac{4\pi^2}{49}\right)^{3/2} \text{vol}(S^3 - (K \cup \gamma)).$$

EXERCISE 13.5. Prove lemma 13.19.

EXERCISE 13.6. (EPDs lie in the characteristic submanifold) Prove that an essential product disk in $M\backslash\backslash S$ is a subset of the I-bundle of $M\backslash\backslash S$, thus cannot be part of the guts.

EXERCISE 13.7. Let K be a hyperbolic alternating knot with a prime twist-reduced diagram, and corresponding polyhedral decomposition. Let S denote the shaded checkerboard surface, and suppose that A is an essential surface with boundary on $\widetilde{S}$ such that A is the union of exactly two normal squares E_1 and E_2, and such that the sides of ∂E_1 and ∂E_2 in white faces are not parallel to boundary edges of the polyhedra. Then prove that A is a Möbius band.

EXERCISE 13.8. We obtain lower bounds on volumes of 7-highly twisted 2-bridge knots from three theorems in this chapter, namely theorem 13.4, theorem 13.15, and theorem 13.17. Compare the bounds coming from each theorem. Which gives the best volume estimate?

EXERCISE 13.9. How sharp are theorem 13.15 and theorem 13.17? By tracing through the proofs, find conditions that must be satisfied for the lower bound on volume to be sharp.