CHAPTER 6

Completion and Dehn filling

In chapter 3 we considered some incomplete structures on hyperbolic 2-manifolds, particularly the 3-punctured sphere, example 3.18. In this chapter, we examine completions of incomplete hyperbolic structures on 3-manifolds with torus boundary.

6.1. Mostow–Prasad rigidity

We begin by stating a few important results on complete hyperbolic structures on manifolds to set up some context for the rest of the chapter.

Many surfaces admit infinitely many complete hyperbolic structures. For example, in exercises 3.13 and 3.14 you find 2-parameter families of complete hyperbolic structures on the 1-punctured torus and 4-punctured sphere. This flexibility is only possible in two dimensions. In higher dimensions, there is only one complete structure on a finite volume hyperbolic manifold, up to isometry. This result was proved in the case \( M \) is a closed manifold by Mostow [Mostow, 1973], and extended to the case of open manifolds with finite volume by Prasad [Prasad, 1973]. Recall that by theorem 5.25, an open hyperbolic 3-manifold has finite volume if and only if it is the interior of a manifold with torus boundary components.

**Theorem 6.1 (Mostow–Prasad rigidity).** If \( M_1^n \) and \( M_2^n \) are complete hyperbolic n-manifolds with finite volume and \( n \geq 3 \), then any isomorphism of fundamental groups \( \phi : \pi_1(M_1) \to \pi_1(M_2) \) is realized by a unique isometry.

We will not include the proof in this book, as it leads us a little further away from knots and links than we wish to stray. However, the proof of the theorem can be found in the original papers, or in books on hyperbolic geometry including [Benedetti and Petronio, 1992] and [Ratcliffe, 2006].

We also note the following, proved in [Gordon and Luecke, 1989].

**Theorem 6.2 (Knot complement theorem).** If \( K_1 \) and \( K_2 \) are knots with homeomorphic complement, then the knots are isotopic, up to reflection.

Thus isotopic knots have homeomorphic complements. Knots with homeomorphic complements have isomorphic fundamental group. By theorem 6.1,
Mostow–Prasad rigidity, any complete hyperbolic structure on the knot complement is the only complete hyperbolic structure. So the complete hyperbolic structure on a knot complement distinguishes any two knots. This is one reason hyperbolic geometry gives many very nice knots invariants!

### 6.2. Completion of incomplete structures

What about incomplete structures on a manifold $M$ with torus boundary? There are a lot of these. For the figure-8 knot complement, for example, we found a 1-complex parameter family of incomplete structures, parameterized by $w \in \mathbb{C}$ as in figure 4.11. If we take the completion of a hyperbolic structure on a 3-manifold, we obtain surprising topological results.

As a warm up, recall completions of incomplete structure on 2-manifolds. In chapter 3, we saw an example of an incomplete structure on a hyperbolic 3-punctured sphere. Recall that in the developing map for an incomplete structure, ideal polygons approached a limiting line. By selecting a point on a horocycle about infinity, approaching this line, we obtained a Cauchy sequence that did not converge. See figure 3.13. Adjoining a point where each horocycle met the limiting line, we obtained the completion. The completion was given by attaching a geodesic of length $d(v)$, as in figure 3.14.

Now consider an incomplete structure on a 3-manifold $M$ such that $M$ is the interior of a compact manifold with torus boundary. Let $C$ be a cusp torus of $M$. Then the torus $C$ inherits an affine structure from the hyperbolic structure on $M$, and because the structure on $M$ is not complete, the affine structure is not Euclidean (theorem 4.10).

Let $\alpha$ and $\beta$ generate $\pi_1(C) \cong \mathbb{Z} \times \mathbb{Z}$. Corresponding to $\alpha$ and $\beta$ are two holonomy isometries $\rho(\alpha)$ and $\rho(\beta)$. To simplify notation, we will drop the $\rho$, abusing notation slightly, and simply refer to these isometries as $\alpha$ and $\beta$. Assume the action of $\alpha$ and $\beta$ does not induce a Euclidean structure on $C$, so the hyperbolic structure on $M$ is not complete. To form its completion, we remove a small neighborhood $N(C)$ of $C$, take the completion of $N(C)$, and then reattach this neighborhood to $M$. Thus to analyze the completion of $M$, we analyze the completion of neighborhoods of cusp tori.

**Proposition 6.3.** The completion of $N(C)$ is obtained by adjoining some portion of a geodesic to $N(C)$.

**Proof.** Consider the developing map for the affine torus $C$. The image will miss single point (exercise 3.8), for example as in figure 3.3. This image is obtained by considering the action of $\alpha$ and $\beta$ restricted to a horosphere. More precisely, if $C$ has a fundamental domain that is a quadrilateral, then we build its developing image by starting with a copy of that quadrilateral on $C$, which we identify with a horosphere about infinity, and attaching copies of the quadrilateral according to instructions given by the holonomy isometries corresponding to $\alpha$ and $\beta$, acting on the fixed horosphere.

If we shift the original choice of horosphere up, we will see the same image of the developing map. In particular, the developing map will still miss a
single point, with the same complex value for each choice of horosphere. These missed points form a vertical geodesic in \( \mathbb{H}^3 \). We may apply an isometry so that this vertical geodesic runs from 0 to \( \infty \) in \( \mathbb{H}^3 \). Notice that the developing image of the neighborhood \( N(C) \) is obtained by taking developing images of \( C \) on all horospheres about \( \infty \) above some fixed initial height. Thus the developing image \( N(C) \) misses the single geodesic from 0 to \( \infty \) in \( \mathbb{H}^3 \). Hence the completion of \( N(C) \) is obtained by adjoining some portion of this geodesic to \( N(C) \).

As in the case of incomplete 2-manifolds, the length of the portion of adjoined geodesic of proposition 6.3 will be determined by considering the action of the holonomy. Considering this action leads to the following result on the topology of the completion.

**Proposition 6.4.** Let \( N(C) \) be the neighborhood of a cusp torus \( C \) of an incomplete hyperbolic manifold, so \( N(C) \) is homeomorphic to \( C \times (0, 1) \). Then the completion of \( N(C) \) is either homeomorphic to the 1-point compactification of \( N(C) \) obtained by crushing \( C \times \{1\} \) to a point, or it is homeomorphic to the solid torus obtained by attaching a solid torus to \( C \times \{1\} \).

**Proof.** As in the proof of proposition 6.3, consider the developing image of \( N(C) \) and assume it misses the geodesic from 0 to \( \infty \). Note the group \( \langle \alpha, \beta \rangle \) acts on the geodesic from 0 to \( \infty \). Since points in our completion should be identified to their images under the holonomy action, we should identify each point \( z \) on the geodesic from 0 to \( \infty \) with \( \langle \alpha, \beta \rangle \cdot z \). There are two cases.

**Case 1.** The image of \( z \) under the action of \( \alpha \) and \( \beta \) is dense in the line from 0 to \( \infty \). In this case, the completion is the 1-point compactification. It is not a manifold. (Exercise.)

**Case 2.** The image of \( z \) is a discrete set of points on the line, each of some distance \( d(C) \) apart. In this case the completion is obtained by adjoining a geodesic circle of length \( d(C) \) to \( N(C) \). Denote the completion by \( \overline{N(C)} \). We wish to understand the topology of \( \overline{N(C)} \).

We may obtain a manifold homeomorphic to \( N(C) \) by removing a small, closed tubular neighborhood of the geodesic circle adjoined to form \( \overline{N(C)} \). Notice that a tubular neighborhood of a circle is a solid torus. Thus we obtain a manifold homeomorphic to \( \overline{N(C)} \) by attaching a solid torus to the torus \( C \times \{1\} \) of \( N(C) \). □

**Definition 6.5.** Let \( M \) be a manifold with torus boundary component \( T \). Let \( s \) be an isotopy class of simple closed curves on \( T \); \( s \) is called a *slope*. The manifold obtained from \( M \) by attaching a solid torus to \( T \) so that \( s \) bounds a disk in the resulting manifold is called the *Dehn filling of \( M \) along \( s \)* and is denoted \( M(s) \).
By proposition 6.4, the manifold obtained by taking the completion of an incomplete hyperbolic structure on $M$ either fails to be a manifold, or is homeomorphic to a Dehn filling of $M$.

Dehn filling is a very important topological procedure in 3-manifold topology, due to work of Wallace and Lickorish in the 1960s. Independently, they showed the following theorem [Wallace, 1960, Lickorish, 1962]. A nice, highly readable proof can be found in the book [Rolfsen, 1976].

**Theorem 6.6 (Fundamental theorem of Wallace and Lickorish).** Let $M$ be a closed, orientable 3-manifold. Then $M$ is obtained by Dehn filling the complement of a link in $S^3$.

Theorem 6.6 gives a topological result on manifolds. By considering completions of hyperbolic 3-manifolds, we can make Dehn filling a geometric procedure.

**Definition 6.7.** Let $x \in \mathbb{H}^3$ and let $\psi \in \text{PSL}(2, \mathbb{C})$ be an elliptic element fixing $x$, rotating about an axis containing $x$ by some angle $\theta > 0$. Let $B_x$ be an open ball in $\mathbb{H}^3$ centered at $x$. The quotient of $B_x$ under the action of $\psi$ is a hyperbolic cone. The angle $\theta$ is the cone angle of the hyperbolic cone. Note this definition makes sense even when $\theta > 2\pi$.

A 3-dimensional hyperbolic cone manifold is a manifold $M$ in which each point $x$ either has a neighborhood isometric to a ball in $\mathbb{H}^3$, or has a neighborhood isometric to a hyperbolic cone.

In a hyperbolic cone manifold, the set of points that only have neighborhoods of the second kind form a geodesic link in $M$ called the singular locus. The hyperbolic metric on $M$ is smooth everywhere except at points on the singular locus.

**Proposition 6.8.** When the completion $\overline{M}$ of $M$ is topologically a solid torus, obtained by Dehn filling, it has the structure of a cone manifold with singular locus $\Sigma$, the geodesic (link) attached in the completion.

**Proof.** As before, let $C$ be a cusp torus of $M$ with neighborhood $N(C)$, whose developing image misses the geodesic from 0 to $\infty$ in $\mathbb{H}^3$. Let $\zeta \in \pi_1(C)$ generate the kernel of the action of $\pi_1(C) \cong \langle \alpha, \beta \rangle$ on the line from 0 to $\infty$. The isometry $\zeta$ will be a rotation about this line by some angle $\theta$. Then a perpendicular cross section of the circle added to $N(C)$ to form the completion will be a hyperbolic cone, of cone angle $\theta$.

Thus when we attach $\overline{N(C)}$ to $M$, the result $\overline{M}$ is a hyperbolic cone manifold with singular locus along the attached geodesic. \qed

There is one very important case of proposition 6.8. When the cone angle at the singular locus of $\overline{M}$ is actually $2\pi$, then the hyperbolic structure on $\overline{M}$ is smooth everywhere. Thus $\overline{M}$ is a hyperbolic manifold. We conclude:

**Corollary 6.9.** When the holonomy $\rho(\pi_1(C))$ acts on the geodesic omitted from the developing image of $N(C)$ by a fixed translation, and when the generator $\zeta \in \pi_1(C)$ of the kernel has holonomy a rotation by $2\pi$, then
the completion of $M$ is a complete hyperbolic manifold, homeomorphic to the Dehn filled manifold $M(\zeta)$.

6.3. Hyperbolic Dehn filling space

We re-interpret the above section in the language of complex lengths of isometries of $\mathbb{H}^3$.

Anytime $M$ admits a hyperbolic structure, consider a cusp torus $C$ for $M$. The fundamental group of the torus is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, generated by some $\alpha$ and $\beta$. (When $M$ is a knot complement, we often choose $\alpha$ to be the meridian, i.e. a curve bounding a disk in $S^3$, and $\beta$ to be the standard longitude, i.e. the curve homologous to 0 in $M$.) Consider the holonomy elements of $\alpha$ and $\beta$. These are some isometries of $\mathbb{H}^3$. As above, we will continue to abuse notation and denote the holonomy isometries corresponding to $\alpha$ and $\beta$ by $\alpha$ and $\beta$.

Recall the classification of isometries of $\mathbb{H}^3$, from lemma 5.2. Any isometry is one of three types: parabolic, elliptic, or loxodromic. Since $\alpha$ and $\beta$ generate $\mathbb{Z} \times \mathbb{Z}$, they must commute. This is possible only if $\alpha$ and $\beta$ are parabolic, fixing the same point at infinity, or if $\alpha$ and $\beta$ share the same axis (exercise 5.11).

If $\alpha$ and $\beta$ are parabolic, fixing a point at infinity, then they must fix an entire horosphere about infinity. Conjugating to put their fixed point at $\infty$ in $\partial \mathbb{H}^3$, they are of the form $\alpha(z) = z + a$, $\beta(z) = z + b$. Hence they restrict to Euclidean isometries on the horosphere, and the hyperbolic structure is complete.

Now consider incomplete structures. In this case, because $\alpha$ and $\beta$ commute but are not parabolic, they share an axis, and are both given by rotation and/or translation along this axis. In particular, the axis must be exactly the vertical geodesic whose points are omitted from the developing image of $C$ for each horosphere.

**Definition 6.10.** Fix a direction on the axis of $\alpha$ and $\beta$. Any element $\gamma$ of $\pi_1(C)$ translates some signed distance $d$ along the axis, and rotates by total angle $\theta \in \mathbb{R}$, where the sign of $\theta$ is given by the right hand rule. Let $L(\gamma) = d + i\theta$. The value $L(\gamma)$ is called the complex length of $\gamma$. This defines a function $L$ from $\pi_1(C) = H_1(C; \mathbb{Z})$ to $\mathbb{C}$.

Notice that if $\gamma = p\alpha + q\beta$, then $L(\gamma) = pL(\alpha) + qL(\beta)$, so $L$ is a linear map. We may extend it canonically to a linear map $L : H_1(C; \mathbb{R}) \to \mathbb{C}$. The value $L(c)$ for any $c \in H_1(C; \mathbb{R}) \cong \mathbb{R}^2$ will be called the complex length of $c$.

Suppose that the complex length of a simple closed curve $\gamma$ on $C$ equals $2\pi i$. Then in the completion of $M$, $\gamma$ will bound a smooth hyperbolic disk. This implies that the completion of $M$ is a manifold homeomorphic to the Dehn filled manifold $M(\gamma)$, and that $M(\gamma)$ admits a complete hyperbolic structure.

Suppose instead that the complex length of a closed curve $\gamma$ on $C$ equals $\theta i$. Then in the completion of $M$, $\gamma$ will bound a hyperbolic cone, with
cone angle \( \theta \). The completion of \( M \) is still homeomorphic to the Dehn filled manifold \( M(\gamma) \) (since a hyperbolic cone is homeomorphic to a disk). However, the metric on \( M(\gamma) \) inherited from the completion of \( M \) is not smooth. The core of the added solid torus is the singular locus, with cone angle \( \theta \).

Most generally, there will be a unique element \( c \in H_1(C; \mathbb{R}) \) so that \( \mathcal{L}(c) = 2\pi i \).

**Definition 6.11.** We say \( c \in H_1(C; \mathbb{R}) \) such that \( \mathcal{L}(c) = 2\pi i \) is the *Dehn filling coefficient* of the boundary component \( C \).

When \( c \) is of the form \((p, q)\), with \( p \) and \( q \) relatively prime integers, it corresponds to a simple closed curve and the completion is smooth.

We have been looking at a fixed incomplete hyperbolic structure on \( M \), and examining possible completions for this fixed structure. Now we turn our attention to a topological manifold \( X \), homeomorphic to \( M \), and consider all possible hyperbolic structures on \( X \).

**Definition 6.12.** Let \( X \) be a 3-manifold with cusp torus \( C \). The subset of \( H_1(C; \mathbb{R}) \) consisting of Dehn surgery coefficients of hyperbolic structures on \( X \) is called the *hyperbolic Dehn filling space* for \( X \).

If \( X \) admits a complete hyperbolic structure, then we let \( \infty \) correspond to the complete hyperbolic structure on \( X \).

**Theorem 6.13 (Thurston’s hyperbolic Dehn filling theorem).** Suppose \( X \) is a 3-manifold homeomorphic to the interior of a manifold with boundary a single torus \( T \), such that \( X \) admits a complete hyperbolic structure. Then hyperbolic Dehn filling space for \( X \) always contains an open neighborhood of \( \infty \) in \( \mathbb{R}^2 \cup \{\infty\} \cong H_1(C; \mathbb{R}) \cup \{\infty\} \).

More generally, if \( X \) is the interior of a manifold with torus boundary components \( T_1, \ldots, T_n \), and \( X \) admits a complete hyperbolic structure, then the hyperbolic Dehn filling space for \( X \) contains an open neighborhood of \( \infty \) for each \( T_i \).

Theorem 6.13 is an important result, and the result, its proofs, and its extensions continue to have useful consequences. The first proof of theorem 6.13 was sketched in Thurston’s 1979 notes [Thurston, 1979], and uses results on holonomy representations. A proof in the case that \( X \) admits a geometric triangulation was given in [Neumann and Zagier, 1985], presented with expanded details in [Benedetti and Petronio, 1992]. This proof was extended to the case of more general hyperbolic 3-manifolds by Petronio and Porti [Petronio and Porti, 2000]. Precise, universal bounds on the size of the open neighborhood of \( \infty \) provided by the theorem were given by Hodgson and Kerckhoff [Hodgson and Kerckhoff, 2005], about 25 years after theorem 6.13 was proved. All the proofs require work.
In a later chapter (chapter 13), we will present a proof that uses the triangulations of [Neumann and Zagier, 1985], but combines more recent work in [Luo et al., 2008] that allows us to pass to a finite cover, and therefore complete the proof for all manifolds.

In the remainder of this chapter, we consider consequences of the hyperbolic Dehn filling theorem.

**Corollary 6.14.** Let \( X \) be a manifold with torus boundary that admits a complete hyperbolic metric. Then there are at most finitely many Dehn fillings of \( X \) which do not admit a complete hyperbolic metric.

**Corollary 6.15.** Let \( X \) be a manifold with torus boundary components \( T_1, \ldots, T_n \). For each \( T_i \), exclude finitely many Dehn fillings. Remaining Dehn fillings yield a manifold with a complete hyperbolic structure.


Notice that corollary 6.15 does not rule out the fact that a manifold may have infinitely many non-hyperbolic Dehn fillings, as in the following example.

**Example 6.16.** The Whitehead link is the link shown in figure 6.1. We will see that it admits a complete hyperbolic structure (proposition 7.6).

If we erase one of the link components, that action can be seen as attaching a solid torus to the link complement in a trivial way. This is called **trivial Dehn filling**. For this example, perform trivial Dehn filling on the component that clasps itself in figure 6.1, leaving a single unknotted component, a trivial knot in \( S^3 \). Its complement is a solid torus.

**Definition 6.17.** A **lens space** is the 3-manifold obtained by gluing together two solid tori along their common torus boundary components.

Thus any Dehn filling of a trivial knot in \( S^3 \) is a lens space.

**Theorem 6.18.** A lens space cannot admit a hyperbolic structure.

**Proof.** Exercise. \( \square \)

There are infinitely many Dehn fillings on the trivial knot in \( S^3 \) that produce lens spaces. Thus there are infinitely many non-hyperbolic Dehn fillings of the Whitehead link complement.
The fundamental theorem of Wallace and Lickorish, theorem 6.6, implies that any closed orientable 3-manifold is obtained by Dehn filling a link complement in $S^3$. In fact we may take that link complement to be hyperbolic, due to work of Myers [Myers, 1993]. Thus the hyperbolic Dehn filling theorem implies that in some sense, "almost all" 3-manifolds are hyperbolic.

There are still many unanswered questions about hyperbolic Dehn filling space. As of the writing of this book, the following questions are all unknown.

**Question 6.19.** What is the topology of hyperbolic Dehn filling space? For example, is it connected? Is it path connected? That is, if a finite volume manifold $M(s)$ admits a complete hyperbolic structure, and if $M$ also admits a complete hyperbolic structure, is there necessarily a smooth deformation of the hyperbolic structure running from the complete structure on $M$ to the complete structure on $M(s)$? Is it star shaped? That is, if $M(s)$ admits a complete hyperbolic structure, and $M$ admits a complete hyperbolic structure, can we deform the hyperbolic structure on $M$ through cone manifolds with cone angles increasing monotonically from 0 (at the complete structure on $M$) to $2\pi$ (at the complete structure on $M(s)$)?

We do not even know if hyperbolic Dehn filling space is connected for the simplest of examples — the figure-8 knot complement. The following example is discussed in [Cooper et al., 2000].

**Example 6.20** (Dehn filling space for the figure-8 knot). Thurston identified part of the boundary of the neighborhood about infinity separating hyperbolic Dehn fillings from non-hyperbolic ones. This is done on pages 58 through 61 of his notes [Thurston, 1979]. To determine these boundaries, he considers what is happening to the two hyperbolic structures on the tetrahedra as the values of their edge invariants approach the boundaries given by the gluing equations (the boundaries of the region in figure 4.11). When both tetrahedra degenerate, the hyperbolic structure collapses and the limiting manifold is not hyperbolic.

However, when only one tetrahedron degenerates, we still have a hyperbolic structure for a little while. In this case, we will be gluing a positively oriented tetrahedron to a negatively oriented one. We can make sense of this by cutting the negatively oriented tetrahedron into pieces and subtracting them from the positively oriented one, leaving a polyhedron $P$. Faces of $P$ may then be identified to give a hyperbolic structure. No one knows exactly where this stops working, although Hodgson’s 1986 PhD thesis [Hodgson, 1986] gives evidence that the boundary should be as shown in figure 6.2.

In exercise 6.1, you are asked to study how tetrahedra degenerate in the figure-8 knot complement.

**Question 6.21.** What is the hyperbolic Dehn surgery space for the figure-8 knot complement?
DEFINITION 6.22. Dehn fillings that do not yield a hyperbolic manifold are called \textit{exceptional}.

There are many open problems on exceptional Dehn fillings. For example, no hyperbolic manifold can contain an embedded 2-sphere that does not bound a 3-ball (exercise). A manifold that contains such a 2-sphere is called \textit{reducible}. If you start with a hyperbolic 3-manifold, perform Dehn filling, and obtain a reducible manifold, the Dehn filling is called \textit{reducible}.

CONJECTURE 6.23 (The cabling conjecture). No hyperbolic knot complement admits a reducible Dehn filling.

The original wording of the cabling conjecture was that only cables of knots admitted reducible Dehn fillings. The conjecture listed as conjecture 6.23 is the only remaining case to prove. A solution of the problem may involve hyperbolic geometry of knot complements, perhaps including an examination of the surfaces sitting inside the knot complement.

A related open problem concerns lens spaces. Note lens spaces are another class of non-hyperbolic manifolds (exercise). There is no way to put a complete hyperbolic metric on a lens space.

CONJECTURE 6.24 (The Berge conjecture). If Dehn filling on a knot produces a lens space, then the knot is a Berge knot.

Berge knots were studied by John Berge. These knots can be characterized as follows. Give $S^3$ its standard genus-2 Heegaard splitting. That is, a standardly embedded genus 2 surface in $S^3$ splits $S^3$ into two handlebodies. Let $K$ be a curve on the genus-2 surface which is a generator of the fundamental group of both handlebodies. Then $K$ is a Berge knot.
Another open problem concerns the maximal number of exceptional Dehn fillings on a hyperbolic 3-manifold with one cusp. The figure-8 knot complement has 10 exceptional fillings. Gordon conjectured that no manifold admits more than 10 exceptional fillings, and his conjecture was proved in [Lackenby and Meyerhoff, 2013]. The following related conjecture still remains open:

**Conjecture 6.25.** The figure-8 knot is the only hyperbolic 3-manifold which admits the maximal number of exceptional Dehn fillings. That is, there is no hyperbolic 3-manifold besides the figure-8 knot complement which admits 10 exceptional Dehn fillings.

**6.4. Geometry change under Dehn filling**

Theorem 6.13, the hyperbolic Dehn filling theorem, actually gives information on convergence of geometry of spaces. In an appropriate sense, 3-manifolds obtained by hyperbolic Dehn filling a complete hyperbolic manifold \( M \) are “close” geometrically to \( M \).

Convergence of metric spaces was studied by Gromov [Gromov, 1999]. For additional references on convergence in hyperbolic space, we recommend [Canary et al., 2006], [Benedetti and Petronio, 1992, Chapter E], and [Cooper et al., 2000, Chapter 6].

There are actually several different notions of convergence of metric spaces in the literature on hyperbolic 3-manifolds and cone manifolds, and they can be confusing. We will step through some of them here, and mention equivalent notions or other names by which the convergence is known.

**6.4.1. Bilipschitz convergence of spaces.** One way of measuring a “distance” between spaces is via bilipschitz maps and bilipschitz constants, which works particularly well for compact metric spaces.

**Definition 6.26.** Let \( X \) and \( Y \) be metric spaces with distance functions \( d_X \) and \( d_Y \), respectively. For \( K > 0 \), a bijection \( f: X \to Y \) is \( K \)-bilipschitz if for all distinct points \( x, y \in X \),

\[
\frac{1}{K}d_X(x, y) \leq d_Y(f(x), f(y)) \leq Kd_X(x, y).
\]

**Definition 6.27.** Let \( X \) and \( Y \) be compact metric spaces. We define their **bilipschitz distance** to be

\[
\inf\{|\log \text{bilip}(f)| + |\log \text{bilip}(f^{-1})|\}
\]

where the infemum is taken over all bilipschitz mappings \( f \) from \( X \) to \( Y \), and \( \text{bilip}(f) \) denotes the bilipschitz constant, or in other words the minimum value of \( K \) of definition 6.26. If there is no bilipschitz map from \( X \) to \( Y \), the infemum is taken to be infinite.

The bilipschitz distance gives a topology on compact metric spaces, called the **bilipschitz topology**.
Our manifolds are typically not compact, such as knot complements. To deal with convergence of these spaces, we consider compact subsets.

**Definition 6.28.** Let \( \{X_n\} \) be a sequence of locally compact metric spaces with a distinguished basepoint \( x_n \in X_n \) for each \( n \). The sequence \( \{(X_n, x_n)\} \) is said to converge in the **pointed bilipschitz topology** to \( (Y, y) \) if for any \( R > 0 \), the closed neighborhoods of radius \( R \) about \( x_n \) in \( X_n \), denoted \( B_R(x_n) \), converge to the closed neighborhood \( B_R(y) \) about \( y \) in \( Y \) in the bilipschitz topology.

That is, if a sequence of hyperbolic manifolds \( M_n \) converges in the pointed bilipschitz topology to \( M \), then larger and larger compact balls in \( M_n \) become closer and closer to isometric to balls in \( M \).

For many applications, convergence in the pointed bilipschitz topology suffices to show geometric results. In fact, we will also consider a topology on metric that is even finer.

**Definition 6.29.** Let \( \{X_n\} \) be a sequence of locally compact metric spaces with a distinguished basepoint \( x_n \in X_n \) for each \( n \), and a distinguished orthonormal basis \( v_n \) of \( T_{x_n}X_n \). The sequence \( \{(X_n, x_n, v_n)\} \) converges in the **framed pointed bilipschitz topology** to \( (Y, y, v) \) if for sufficiently large \( R > 0 \) and all \( K > 1 \), there exists a number \( n_0 \) such that for \( n \geq n_0 \), there are open neighborhoods \( U_n, U \) of \( B_R(x_n) \) and \( B_R(y) \), and \( K \)-bilipschitz diffeomorphisms \( f_n : (U, v) \rightarrow (U_n, v_n) \) with \( f_n(y) = x_n \) and \( D_yf_n(v) = v_n \).

Framed pointed bilipschitz convergence of spaces is also called geometric convergence of spaces, or convergence in the refined Gromov–Hausdorff topology.

**Example 6.30.** Consider a sequence of ideal tetrahedra \( T_n \) in \( \mathbb{H}^3 \) with vertices at 0, 1, \( \infty \), and \( z_n \), where \( z_n \) is converging to some \( z_0 \in \mathbb{C} \) with \( \Im(z_0) > 0 \). For \( n \) sufficiently large, a point \( y \in \mathbb{H}^3 \) will lie in the interior of all tetrahedra \( T_n \) and the the ideal tetrahedron \( T_0 \) with vertices at 0, 1, \( \infty \), and \( z_0 \). Choose a baseframe at \( y \). A diffeomorphism mapping \( T_0 \) to \( T_n \) will be a bilipschitz map on the intersection of \( T_0, T_n \) with compact balls, with bilipschitz constant getting closer and closer to 1 as \( n \rightarrow \infty \). Taking the basepoints in \( T_n \) to be the images of \( y \), and the baseframes to be the images of the chosen baseframe at \( y \), we obtain a framed pointed bilipschitz map on compact balls.

More generally, if \( M \) is obtained by gluing faces of an ideal polyhedron (possibly with infinite volume) in \( \mathbb{H}^3 \), and \( M_n \) is obtained by the same face-pairings of slightly deformed polyhedra, with the polyhedra of \( M_n \) converging to that of \( M \) in \( \mathbb{H}^3 \) as \( n \rightarrow \infty \), then \( M_n \) converges to \( M \) in the framed pointed bilipschitz topology. This is made precise in [Marden, 2007]; convergence of polyhedra in this manner is called polyhedral convergence, and is shown to be equivalent to framed pointed bilipschitz convergence.

**6.4.2. Convergence of discrete groups.** If \( M \) is a hyperbolic 3-manifold, then \( M \) is homeomorphic to \( \mathbb{H}^3/\Gamma \) where \( \Gamma \) is a discrete, torsion
free subgroup of $\text{PSL}(2, \mathbb{C})$. Given a sequence of manifolds $M_n \cong \mathbb{H}^3/\Gamma_n$, there are different notions of convergence of the groups $\Gamma_n$.

One type of convergence is known as *algebraic convergence*. This is the convergence of groups as in theorem 5.16, in which generators of $\Gamma_n$ converge to generators of $\Gamma$ as Möbius transformations. However, the fact that the discrete subgroups $\Gamma_n$ converge algebraically to $\Gamma$ will not imply that the sequence of hyperbolic spaces $\mathbb{H}^3/\Gamma_n$ converges to $\mathbb{H}^3/\Gamma$. Convergence of spaces requires something stronger.

**Definition 6.31.** Let $\Gamma_n \leq \text{PSL}(2, \mathbb{C})$ be a sequence of discrete groups, and $\Gamma_\infty \leq \text{PSL}(2, \mathbb{C})$ a discrete group. Suppose

1. for any convergent sequence $\{\gamma_{n_j}\} \subset \{\Gamma_n\}$, the limit $\lim_{j} \gamma_{n_j} \in \Gamma_\infty$, and
2. for any $\gamma \in \Gamma_\infty$, there is a sequence $\gamma_n \in \Gamma_n$ such that $\lim \gamma_n = \gamma$.

Then $\Gamma_n$ converges to $\Gamma_\infty$ *geometrically*.

Geometric convergence of groups as in definition 6.31 is also referred to in the literature as convergence in the Chabauty topology. In a paper published in 1950, Chabauty developed a topology on closed subsets of a given set [Chabauty, 1950]. In the case of discrete subgroups of a Lie group, this Chabauty topology agrees with that of definition 6.31.

**Theorem 6.32.** The following are equivalent.

1. Discrete, torsion free groups $\Gamma_n \leq \text{PSL}(2, \mathbb{C})$ converge geometrically to $\Gamma_\infty$.
2. There exist basepoints $x_n \in \mathbb{H}^3/\Gamma_n$ and $x_\infty \in \mathbb{H}^3/\Gamma_\infty$, and oriented frames $v_n$ and $v$ for $T_{x_n}(\mathbb{H}^3/\Gamma_n)$ and $T_{x_\infty}(\mathbb{H}^3/\Gamma_\infty)$ such that $(\mathbb{H}^3/\Gamma_n, x_n, v_n)$ converges to $(\mathbb{H}^3/\Gamma_\infty)$ in the framed pointed bilipschitz topology.

A proof of theorem 6.32 can be found in [Canary et al., 2006]. If either of the two cases holds in theorem 6.32, we say that $\mathbb{H}^3/\Gamma_n$ approaches $\mathbb{H}^3/\Gamma_\infty$ as a *geometric limit*, or $\Gamma_\infty$ is a geometric limit of $\Gamma_n$.

The following theorem is a consequence of the hyperbolic Dehn filling theorem.

**Theorem 6.33.** Let $M$ admit a complete hyperbolic structure with fixed horocusp $C$. Let $s_n$ be a sequence of slopes on $\partial C$ such that the length of (a geodesic representative of) $s_n$, measured in the induced Euclidean metric on $\partial C$, approaches infinity. Then for large enough $n$, the Dehn filled manifolds $M(s_n)$ are hyperbolic and approach $M$ as a geometric limit.

Theorem 6.33 implies that all geometric properties of Dehn fillings of $M$ converge to those of $M$, for example volume converges. There are some additional results giving information on how these quantities converge. For example, Jørgensen proved that volume strictly decreases under Dehn filling.
Theorem 6.34 (Jørgensen’s theorem). Volume strictly decreases under Dehn filling. That is, if $M$ is hyperbolic with cusp $C$, and $s$ a slope on $\partial C$ such that $M(s)$ is hyperbolic, then
\[ \text{vol}(M) > \text{vol}(M(s)). \]

The proof of theorem 6.34 can be found in [Thurston, 1979], and also in [Benedetti and Petronio, 1992].

Note that if $M$ has a complete hyperbolic structure with cusp $C$, then $\partial C$ has a Euclidean structure, and any slope $s \subset \partial C$ is isotopic to a geodesic with well-defined Euclidean length $\ell_{\partial C}(s)$. Provided the length of $s$ is at least $2\pi$, a lower bound on volume under Dehn filling can also be obtained.

Theorem 6.35 ([Futer et al., 2008]). Suppose $M$ is hyperbolic with cusps $C_1, \ldots, C_n$ and slopes $s_1, \ldots, s_n$, once on each $\partial C_i$, such that the minimal length slope $\ell_{\min} = \min\{\ell_{\partial C_j}(s_j)\}$ has length at least $2\pi$. Then the Dehn filled manifold $M(s_1, \ldots, s_n)$ is hyperbolic with volume satisfying
\[ \text{vol}(M(s_1, \ldots, s_n)) \geq \left(1 - \left(\frac{2\pi}{\ell_{\min}}\right)^2\right)^{3/2} \text{vol}(M). \]

6.5. Exercises

Exercise 6.1. (Incomplete structures on the figure-8 knot) Thurston’s notes contain a figure showing all parameterizations of hyperbolic structures on the figure-8 knot [Thurston, 1979, page 52]. For any $w$ in this region, formula 4.3.2 in the notes gives us a corresponding $z$ so that if two tetrahedra with edge invariants $z$ and $w$ are glued, we obtain a (possibly incomplete) hyperbolic structure on the figure-8 knot.

Analyze what happens to the tetrahedra corresponding to $z$ and to $w$ as $w$ approaches a point on the boundary of this region.

More specifically, if $w$ approaches certain points on the boundary of this region, tetrahedra corresponding to both $z$ and $w$ start to become degenerate. Which points are these? Prove that the two tetrahedra are becoming degenerate in this case.

As $w$ approaches other values on the boundary, only one of the tetrahedra degenerates. Which points are these? Prove that only one tetrahedron is degenerating in this case.

Exercise 6.2. We have seen that the completion of an incomplete hyperbolic 3-manifold is no longer homeomorphic to the original hyperbolic 3-manifold. Is this true for completions of incomplete structures on the 3-punctured sphere? What surface do we obtain when we complete an incomplete hyperbolic structure on a 3-punctured sphere? Prove it.

Exercise 6.3. Suppose $M$ is a closed manifold with a complete hyperbolic structure. Prove that $\pi_1(M)$ cannot contain a $\mathbb{Z} \times \mathbb{Z}$ subgroup. Conclude that $M$ cannot contain an embedded torus $T$ such that $\pi_1(T)$
injects into $M$. [Such a torus is called *incompressible*. A Dehn filling resulting in a closed manifold with an embedded incompressible torus is another example of an exceptional filling.]

**Exercise 6.4.** Let $M$ be an orientable 3-manifold with a decomposition into ideal polyhedra, each with a hyperbolic structure, such that the polyhedra induce a hyperbolic structure on $M$. Let $v$ be an ideal vertex of $M$, i.e. an equivalence class of ideal vertices of the polyhedra, where vertices are equivalent if and only if they are identified under the gluing of the polyhedra.

Recall that $\text{link}(v)$ is defined to be the boundary of a neighborhood of $v$ in $M$.

(a) Prove $\text{link}(v)$ always inherits a similarity structure from the hyperbolic structure on $M$. Here a similarity structure is a $(\text{Sim}(\mathbb{E}^2), \mathbb{E}^2)$-structure, where $\text{Sim}(\mathbb{E}^2)$ is a subgroup of the group of affine transformations consisting of elements of the form $x \mapsto Ax + b$, where $A$ is a linear map that rotates and/or scales only. Thus $\text{Sim}(\mathbb{E}^2)$ is formed by rotations, scalings, and translations.

(b) Prove that the only closed, orientable surface which admits a similarity structure is a torus. It follows that $\text{link}(v)$ is always homeomorphic to a torus when $M$ is an orientable manifold with hyperbolic structure (even incomplete).

**Exercise 6.5.** Let $M$ be an orientable 3-manifold that admits an incomplete hyperbolic structure with completion given by attaching the one-point compactification of a cusp neighborhood $N(C)$. Prove that the completion is not a manifold.

**Exercise 6.6.** Prove that a reducible manifold cannot be hyperbolic. That is, it admits no complete hyperbolic structure.

**Exercise 6.7.** Prove that a lens space cannot admit a complete hyperbolic structure.

**Exercise 6.8.** (On complex length of $A$ in $\text{PSL}(2, \mathbb{C})$)

(a) Suppose $A \in \text{PSL}(2, \mathbb{C})$ has axis the geodesic from 0 to $\infty$. Then the matrix of $A$ may be parameterized by a single complex number $\lambda$. What is the form of this matrix?

(b) Denote the trace of a matrix $A$ by $\text{tr}(A)$, and its complex length by $\mathcal{L}(A)$. Prove that $\text{tr}(A) = 2 \cosh(\mathcal{L}(A)/2)$. 
