

My research is in low dimensional topology and geometry. I am especially interested in 3-manifolds, hyperbolic geometry, and knot theory, particularly the overlap of these areas.

OVERVIEW

A 3-manifold is often described combinatorially, for example by a knot diagram, or by gluing a collection of polyhedra, or by attaching handles to a given manifold. Figure 1 shows a few examples.

Just over a decade ago, Perelman posted the outlines of a proof of the Geometrization Theorem on the ArXiv [56, 57], and with his work and that of others it is now known that all 3-manifolds decompose into pieces that admit a geometric structure (see, for example [7, 14, 40, 51]). In an appropriate sense, the most common geometric structure is hyperbolic, i.e. admitting a metric with constant sectional curvature  $-1$  [67]. For closed or finite volume 3-manifolds, the hyperbolic structure is unique [53, 58]. Therefore, it is known that a combinatorial description of a hyperbolic 3-manifold uniquely determines the geometry of that manifold.

However, what is still unknown in general is, given a combinatorial description of a hyperbolic 3-manifold, how does one determine geometric information about that 3-manifold? Or given other invariants of a 3-manifold, including quantum invariants, how are they related to the hyperbolic structure? Much of my research has focused on these types of questions. In recent years, the investigation of these problems has been called “Effective Geometrization,” or “WYSIWYG Topology: What you see is what you get,” indicating that salient properties of the combinatorics of a manifold seem to have relevance to the geometry as well. These types of problems are of major importance in the field. While our knowledge of 3-manifold geometry, quantum topology, and combinatorial 3-manifolds has progressed rapidly in recent years, we are still often unable to apply resulting theorems to related problems encountered in other settings. What is needed is a dictionary between the geometry and other invariants.

**Organization.** In this statement, I highlight some of the major research questions that I have investigated, with an emphasis on more recent projects, as well as problems that I am currently investigating. I have grouped the discussion into three main sections, ordered as follows:

- (1) Geometry of knots and links
- (2) Relations between quantum and topological invariants
- (3) Geodesics, Heegaard splittings and infinite volume manifolds

1. GEOMETRY OF KNOTS AND LINKS

Knots and links are excellent examples of 3-manifolds with simple description, and yet which often have complicated geometric properties. Their geometry provides a window into the geometry of an extremely broad range of hyperbolic 3-manifolds, in a very concrete sense. For example,

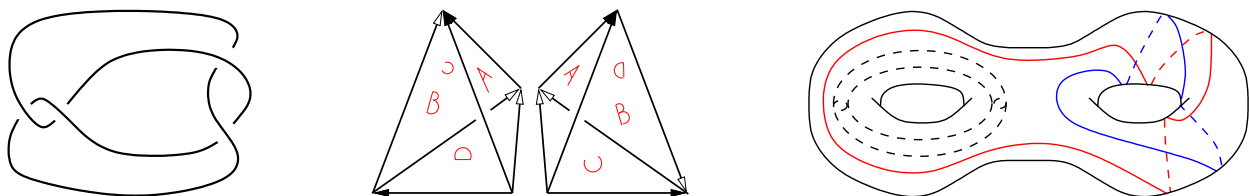


FIGURE 1. Examples of combinatorial descriptions of 3-manifolds. Left to right: A knot diagram determines the knot complement  $S^3 - K$ . Gluing two tetrahedra along faces shown, with vertices removed, gives a description of the same manifold. A Heegaard diagram gives another manifold with torus boundary.



FIGURE 2. Left to right: a twist region, a generalized twist region, the infinite alternating weave.

any closed orientable 3-manifold is obtained by Dehn filling the complement of a link in  $S^3$  (see Definition 1.3), due to work Wallace and Lickorish in the 1960's [69, 46]. Some significant work I have done has concerned geometry change under Dehn filling, which I discuss below.

In addition to giving information on closed 3-manifolds, knot and link complements also can be used to approximate the geometry of very large classes of infinite volume hyperbolic manifolds. This is due to my work Souto in [65]. Our main theorem is the following.

**Theorem 1.1.** *Let  $N$  be a complete hyperbolic 3-manifold with finitely generated fundamental group and a single topological end. If  $N$  is homeomorphic to a submanifold of  $S^3$ , then  $N$  is a geometric limit of a sequence of hyperbolic knot complements in  $S^3$ .*

The proof of the above theorem uses a great deal of modern hyperbolic geometry. In particular, it uses the Ending Lamination Theorem [49, 10], the Tameness Theorem [3, 13], and the Density Theorem [55, 11], all proved since the year 2000.

While this result underlines the importance of the study of the geometry of knots in hyperbolic geometry, it also leaves many open questions. For example, we did show that certain infinite volume manifolds cannot be the geometric limit of any sequence of knot complements. However, our proof extends to show that these manifolds are the geometric limit of a sequence of link complements in some closed 3-manifold. Can they be the geometric limit of link complements in  $S^3$ ? Also, our result is not at all constructive. Given any particular example of a hyperbolic 3-manifold as in the theorem, can one find an explicit sequence of knot complements that approach the manifold? Such questions are discussed in [65].

**1.1. Diagrams and link volume.** Classically, knots and links have been described by a diagram: a 4-valent planar graph with over-under crossing information at each vertex, as in Figure 1. It has been known since the early 1980's, due to work of Thurston [68], that all knots except torus and satellite knots have a complement admitting a hyperbolic structure. By Mostow-Prasad rigidity, the hyperbolic structure is unique [53, 58]. However, relating geometric properties of the hyperbolic 3-manifold  $(S^3 - K)$  to the graph theoretic properties of the diagram of  $K$  is often difficult. A significant amount of my research addresses this problem, including ongoing research.

One area of investigation is the volume of links. A hyperbolic structure on a knot or link complement produces a manifold with finite hyperbolic volume, and volume is a knot invariant. Lackenby was the first to bound volumes of a class of knots in terms of its diagrams: for alternating knots, he showed that the volume was bounded above and below by linear functions of the *twist number* of the diagram [42].

**Definition 1.2.** A *twist region* in a knot diagram is a collection bigons in the diagram arranged end-to-end; the collection is maximal in the sense that there are no additional adjacent bigons, and the knot diagram is alternating between them. See Figure 2, left. A single crossing adjacent to no bigons is also a twist region. The *twist number* of the diagram is the number of twist regions in a reduced diagram.

One of my first published papers extends Lackenby's work on volumes of alternating links to the class of links called highly twisted links, with high numbers of crossings in each twist region

[59]. Not long after this result was published, with Futer and Kalfagianni, I improved it [24], and then extended it to broader classes of diagrams, including those with symmetry [25], those with generalized twist regions (see Figure 2) [27], and 3–braids [26]. With other collaborators I have found volume bounds on additional important classes of knots and links, including twisted torus links [15] and chain links (with undergraduate students) [37].

One tool used to prove many of these results is a geometric result bounding the change in volume under Dehn filling.

**Definition 1.3.** Let  $M$  be a 3–manifold with torus boundary components  $T_1, T_2, \dots, T_n$ . Let  $s_1, s_2, \dots, s_n$  be a collection of *slopes*, or isotopy classes of simple closed curves  $s_i \subset T_i$ . Then the *Dehn filling of  $M$  along  $s_1, \dots, s_n$* , denoted  $M(s_1, \dots, s_n)$ , is the manifold obtained from  $M$  by attaching a collection of solid tori to the  $T_i$  such that the slope  $s_i$  bounds a disk in the solid torus.

When  $M$  admits a hyperbolic structure, its interior is homeomorphic to the quotient of hyperbolic 3–space  $\mathbb{H}^3$  under the action of a discrete, faithful group  $\Gamma$  of isometries of  $\mathbb{H}^3$ , with  $\Gamma$  isometric to the fundamental group  $\pi_1(M)$ . The torus boundary components of  $M$  will be realized as *cusps*. A cusp is homeomorphic to  $T^2 \times \mathbb{R}$ , where  $T^2$  is a 2–torus, and isometric to the quotient of a horoball under a rank–2 parabolic subgroup of  $\Gamma$ . On the boundary of a horoball, a slope can be represented by a Euclidean geodesic, and thus it inherits a length.

Using analytic tools, Futer, Kalfagianni and I constructed an explicit metric on the Dehn filling of a hyperbolic 3–manifold, provided each slope  $s$  had length at least  $2\pi$ . The resulting metric was not hyperbolic, merely negatively curved. However, we were able to use volume bounds for manifolds with negatively curved metrics to show the following, in [24].

**Theorem 1.4.** *Let  $M$  be a complete, finite–volume hyperbolic manifold homeomorphic to the interior of a manifold with cusps  $C_1, \dots, C_k$ . Let  $s_1, \dots, s_k$  be slopes on  $\partial C_1, \dots, \partial C_k$ , each with length greater than  $\ell_{\min} > 2\pi$ . Then  $M(s_1, \dots, s_k)$  is a hyperbolic manifold with volume*

$$\text{vol}(M(s_1, \dots, s_k)) \geq \left(1 - \left(\frac{2\pi}{\ell_{\min}}\right)^2\right)^{3/2} \text{vol}(M)$$

The above theorem not only leads to volume bounds on link complements, but also has applications elsewhere, both geometric and topological. Most notably, it was used by Gabai, Meyerhoff, and Milley in their proof that the Weeks manifold has smallest volume [33, 48]. We have used the explicit metric we constructed in other Dehn filling applications, such as [18] and [32].

**Volume and other invariants.** As we have learned more about the volume of links, and techniques to bound volume, we have found several interesting relationships between volume and other invariants of a knot. In particular, with Futer and Kalfagianni, we have found that the volume of a knot is often closely related to coefficients of its Jones polynomials. Indeed, many of the results above lead to such relations. This is discussed in more detail in Section 2.

In another direction, with Champanerkar and Kofman, I am currently working on a project to investigate relations between volumes and other invariants, especially ratios of the invariants in a limit. D. Thurston showed that the volume of a link is bounded by the product of its crossing number and the constant  $v_8$ , which is the volume of a regular ideal octahedron. We have been investigating sequences of knots and links for which the ratio volume per crossing number is as large as possible. The links we have considered approach the infinite alternating weave  $W$ , shown in Figure 2. One result from our paper [17] is the following.

**Theorem 1.5.** *Suppose  $K_n$  is a sequence of links with prime, alternating, twist–reduced diagrams that contain no bigons and no cycle of tangles, such that*

- (1) *there are subgraphs  $G_n$  of the diagram graphs of  $K_n$  that form a regular Følner sequence for the graph of  $W$ , and*

$$(2) \lim_{n \rightarrow \infty} |G_n|/c(K_n) = 1.$$

Then

$$\lim_{n \rightarrow \infty} \frac{\text{vol}(K_n)}{c(K_n)} = v_8.$$

Interestingly, very similar conditions on convergence give a similar result for the determinant of the knot. In particular, if  $K_n$  is a sequence of alternating links satisfying (1) and (2) above, then we also show

$$\lim_{n \rightarrow \infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = v_8.$$

These results lead to many interesting open questions concerning asymptotics of knots, their volumes, their determinants, and other invariants. For full details, see [17].

**Ongoing project.** The bounds on volumes of knots that we have found so far apply to a wide variety of classes of knots and links, but not yet to all knots and links. Recall that any knot can be represented by a closed braid. Futer and I have a plan to bound volumes of closed braids. The first step in our project is to remove the braid axis, and find a bound on the volume of the resulting fibered manifold. Then bounding the cusp area, and applying Theorem 1.4 above, we will be able to bound the volume of the original knot. The missing step is an effective bound on volumes of fibered manifolds. Work of Brock [8] gives bounds on volume in terms of the monodromy of a fibered manifold, and Futer and I can show that in the case of braids, monodromy is related to generalized twist regions, as in Figure 2. However, Brock’s result is not effective.

With Schleimer, Futer and I have been working to make Brock’s result effective. One necessary ingredient in our work has been the application of cone deformation tools, initially developed by Hodgson and Kerckhoff [36, 35], which I have used in a significant way in the past, e.g. in [59, 60]. Using cone deformations and new bounds on thin parts of manifolds, Futer, Schleimer, and I have been able to make effective some portions of the Drilling Theorem of Brock and Bromberg [9]. We expect that these results will lead to additional interesting consequences.

1.1.1. *Hyperbolicity.* We note very briefly that the above work concerns hyperbolic knots and links. By work of Thurston [68], a link complement will be hyperbolic if and only if it contains no embedded essential tori, spheres, or annuli. However, for a generic diagram, it is often difficult to rule out such surfaces. I have several results, individually and with coauthors, giving diagrammatical conditions that guarantee that a link complement will be hyperbolic, or not, including [31], [64], [63], [62], and [23].

1.2. **Cusp shapes.** To apply the Dehn filling results mentioned above, for example Theorem 1.4, requires knowing the length of a slope on a cusp of a hyperbolic manifold. Recall that the cusp of a hyperbolic manifold  $\mathbb{H}^3/\Gamma$  is the quotient of a standard horoball in  $\mathbb{H}^3$  by a rank-2 parabolic subgroup of  $\Gamma$ . The boundary of this cusp will inherit a Euclidean metric. The Euclidean similarity structure is the *cuspidal shape* of the manifold. For a knot, the maximum possible area of the boundary of an embedded cusp is called the *cuspidal area*.

Again, it is difficult in general to determine cusp shapes and cusp areas, and there are several open conjectures concerning them. In light of theorems such as Theorem 1.4, and others (for example [4, 41, 36]), it would be very useful to be able to determine slope lengths and cusp shapes more generally.

I have some results along these lines. In the papers [60], [61], and [64], and also in [31] with Futer, I bound cusp shapes or slope lengths in terms of a diagram for highly twisted links, for links with high numbers of twists in generalized twist regions, and for a class of links called fully augmented links.

One interesting class of knots for which cusp shape is not yet known is that of alternating knots. Computer investigation has led to several open conjectures, including a conjecture of Thistlethwaite

that crossing arcs in a reduced diagram correspond to short geodesics. If this conjecture is true, it would imply that the cusp areas of these knots are bounded below in terms of their twist numbers. With Futer and Kalfaginni, I proved this conjecture for 2-bridge knots [26].

More recently, with Lackenby, we have been able to make considerable progress on this conjecture. Namely, we have been able to show that the cusp area of an alternating knot is bounded below by a universal constant times its twist number [43]. We do this by considering special immersed essential surfaces in the alternating link complement [44], and showing that these surfaces must contain embedded geodesics of bounded length. All our estimates can be made explicit. The main result of [43] is the following.

**Theorem 1.6.** *There exists a constant  $A$  such that if  $D$  is a prime, twist reduced alternating diagram of a hyperbolic knot  $K$ , with twist number  $\text{tw}(D)$ , and if  $C$  denotes the maximal cusp of the hyperbolic 3-manifold  $S^3 - K$ , then*

$$A(\text{tw}(D) - 2) \leq \text{Area}(\partial C) < 10\sqrt{3}(\text{tw}(D) - 1).$$

Moreover,  $A$  is at least  $2.436 \times 10^{-16}$ .

While we do not believe that our lower bound on  $A$  is optimal (or even close to optimal), the fact that we obtain an explicit lower bound, universal over all alternating knots, is noteworthy. Finding explicit bounds is often difficult.

## 2. RELATIONS BETWEEN QUANTUM AND TOPOLOGICAL INVARIANTS

This section describes ongoing work with Futer and Kalfagianni to determine relations between hyperbolic geometry and quantum invariants, particularly the Jones and colored Jones polynomials.

The Volume Conjecture, formulated by Kashaev [38] and H. Murakami and J. Murakami [54], states that the volume of a hyperbolic knot is determined by the asymptotics of its  $n$ -th colored Jones polynomials. This conjecture has been difficult to verify, except for a small number of examples. However, if it is true, it gives a nice relationship between hyperbolic geometry and quantum topology. If it is true, we would expect there to be relations between volumes of hyperbolic links and coefficients of the colored Jones polynomials for large  $n$ . With Futer and Kalfagianni, I have found many such relations, which have led us to the following question.

**Question 2.1** (Coarse Volume Conjecture). *Does there exist a function  $B(K)$  of the coefficients of the colored Jones polynomials of a knot  $K$  such that for hyperbolic knots,  $B(K)$  is coarsely related to hyperbolic volume? That is, do there exist universal constants  $C_1 \geq 1$  and  $C_2 \geq 0$  such that*

$$C_1^{-1}B(K) - C_2 \leq \text{vol}(S^3 - K) \leq C_1B(K) + C_2 \quad \text{for all knots } K?$$

Evidence for the Coarse Volume Conjecture includes computational results due to Champanerkar, Kofman, and Patterson [16], and work of Dasbach and Lin on alternating knots [21]. It also includes many results of myself with Futer and Kalfagianni, including [24, 25, 26, 27, 22]. Many of these results are indirect, relating volume to twist number and coefficients of the Jones polynomials to twist number.

In the research monograph [22], we develop a more direct way of investigating these relationships. Namely, we decompose the knot complement along incompressible *state surfaces*, and use these surfaces to bound volumes by appealing to work of Agol, Storm, and Thurston [5], which gives volume bounds in terms of the guts of an incompressible surface.

To state our results, we briefly recall some definitions. Any crossing of a knot diagram can be resolved in one of two ways, and a choice of resolution gives a *state*. Applying a state to all crossings, we obtain a collection of disjoint, embedded circles in the plane. We build a state surface by requiring these circles to bound disks below the projection plane, and attaching twisted bands to pairs of disks at crossings. See Figure 3.

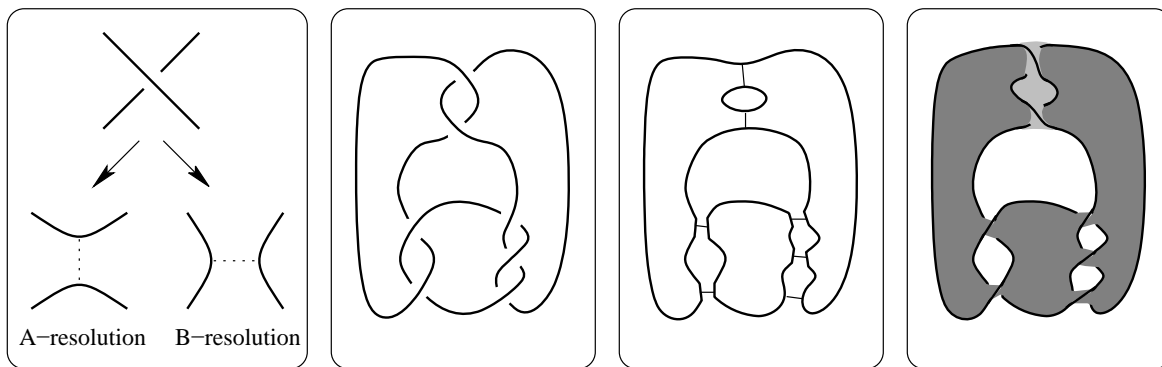


FIGURE 3. Left to right: A crossing and its resolution. A link diagram. Result of taking all- $A$  resolutions. The corresponding state surface.

Given a state  $\sigma$ , we obtain a state graph  $\mathbb{G}_\sigma$  as follows. First, after applying the state to the diagram, add an edge for each removed crossing to the collection of state circles, as in the panel second from right in Figure 3. Then collapse each state circle to a vertex. This is the state graph  $\mathbb{G}_\sigma$ . If  $\mathbb{G}_\sigma$  contains no edge-loops, then the link is said to be  $\sigma$ -adequate. For the all- $A$  or all- $B$  state, Lickorish and Thistlethwaite showed that  $A$  and  $B$ -adequate knots have nontrivial Jones polynomial [47]. Such links include all alternating links, and many other classes of links.

On the other hand, Kauffman’s formulation of the Jones polynomial gives the polynomial as a sum over all possible states of the knot diagram [39]. Therefore, it is not too surprising that for many classes of knots, in particular  $A$  and  $B$ -adequate ones, the coefficients of the Jones polynomial are known to have relations to appropriate state surfaces. See, for example [20].

We find relations of the coefficients of the Jones polynomial to the volume. One sample of results we obtain is the following theorem, from [22].

**Theorem 2.2.** *Suppose  $K$  is an  $A$ -adequate link with  $\beta'_n$  the penultimate coefficient of the colored Jones coefficient. Suppose that all 2-edge loops in the state graph come from twist regions. Then*

$$\text{vol}(S^3 - K) \geq v_8(|\beta'_n| - 1).$$

In [22], we decomposed the state surface complement into polyhedra; a survey of the decomposition is written in [29]. The properties of the polyhedra can be read off of a directed graph related to the link diagram. This decomposition has allowed us to use techniques such as normal surface theory and graph theory to obtain related results, including those in [30, 23], and recently in [6], which is a joint paper I wrote with undergraduate students.

**2.1. Jones coefficients and fibers.** In [22], we discovered that for  $A$ -adequate links, the penultimate coefficient of the Jones polynomial detects whether the all- $A$  state surface is a fiber. When it is not, we showed that the surface is actually quasifuchsian, that is, it has no accidental parabolics and no finite cover in which it is a fiber [30]. One main result is the following.

**Theorem 2.3.** *Let  $K$  be a hyperbolic link with a prime,  $A$ -adequate diagram. Then the penultimate coefficient of the colored Jones polynomial  $\beta'_n$  determines the geometric type of the all- $A$  surface  $S_A$ , as follows.*

- (1) *If  $\beta'_n = 0$ , then  $S_A$  is a fiber in  $S^3 - K$ .*
- (2) *If  $\beta'_n \neq 0$ , then  $S_A$  is quasifuchsian.*

**Ongoing project.** In fact, we show more than Theorem 2.3 in [30]. For a class of links called *homogeneously adequate*, we show that a particular state surface is quasifuchsian if and only if the corresponding state graph is a tree, and a fiber otherwise. For these link complements, again we

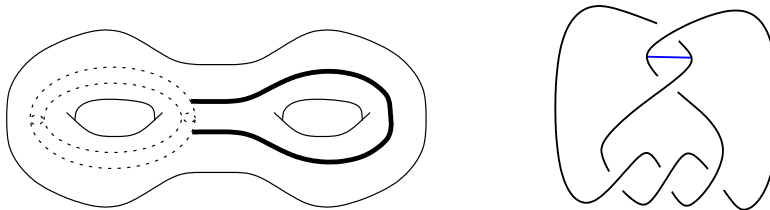


FIGURE 4. Left: a compression body with core tunnel. Right: an unknotting tunnel

have a decomposition into ideal polyhedra, and again techniques from normal surface theory apply. What is missing in extending Theorem 2.3 to homogeneously adequate links is the connection between the state graph and the Jones polynomial, in this case.

Similarly, in [28] we proved a conjecture of Garoufalidis for  $A$ -adequate links: every cluster point of the highest and lowest degrees of the  $n$ -th colored Jones polynomials, divided by  $n^2$ , is a boundary slope of an essential surface [34]. We also expect that this result extends to homogeneously adequate links, and we expect that the essential state surface will play a role here. This problem is of particular interest because the class of homogeneously adequate links is very broad; it is still unknown whether every knot admits a homogeneously adequate diagram. See [30] and [6] for more details on these links.

### 3. GEODESICS, HEEGAARD SPLITTINGS AND INFINITE VOLUME STRUCTURES

In addition to descriptions by link diagrams, by Dehn filling, and by gluing polyhedra, 3-manifolds are also commonly presented by the topological/combinatorial description of a Heegaard splitting, developed in Poul Heegaard's 1898 thesis. In this section, we describe some of our results relating to geometry and Heegaard splittings.

A genus  $g$  handlebody is a solid genus  $g$  surface. That is, start with a ball and attach to it  $g$  1-handles, homeomorphic to neighborhoods of an interval in  $\mathbb{R}^3$ . Any closed orientable 3-manifold is obtained by gluing two handlebodies of the same genus along their common boundary; this is a classical theorem following from work of Moise [50]. When a 3-manifold has boundary, rather than gluing handlebodies, one may obtain the manifold by gluing *compression bodies*.

**Definition 3.1.** Let  $S$  be a closed, oriented (possibly disconnected) surface. A *compression body*  $C$  is obtained by taking the product  $S \times [0, 1]$  and attaching 1-handles to  $S \times \{1\}$  such that the result is connected. The boundary  $S \times \{0\}$  is called the *interior boundary*, denoted  $\partial_- C$ . The *exterior boundary*, denoted  $\partial_+ C$ , is  $(\partial C - \partial_- C)$ . We also say that a handlebody is a compression body, with  $\partial_- C = \emptyset$ .

A *Heegaard splitting* of a 3-manifold  $M$  is a decomposition of the manifold into two compression bodies. We obtain  $M$  by gluing the compression bodies along their common exterior boundary. Every compact orientable 3-manifold admits a Heegaard splitting.

An example of a simple compression body is shown in Figure 4. In this figure,  $\partial_- C$  is a torus  $T^2$ , and  $\partial_+ C$  is a genus-2 surface. To build this compression body, a 1-handle was attached to  $T^2 \times [0, 1]$ . The core of that 1-handle is shown by the thick line. We call this the *core tunnel* of the compression body.

Heegaard splittings of knot complements give rise to the notion of *unknotting tunnels*. When a knot complement has a Heegaard splitting into a genus 2 handlebody and a compression body of the form in Figure 4, we say that the core tunnel is an *unknotting tunnel* for the knot. A knot that admits a single unknotting tunnel is called *tunnel number 1*. For example, all 2-bridge knots are tunnel number 1. One example is shown on the right of Figure 4.

An unknotting tunnel has a topological description. As before, when the manifold is given a geometric structure, it is interesting to determine how the topology and geometry are related.

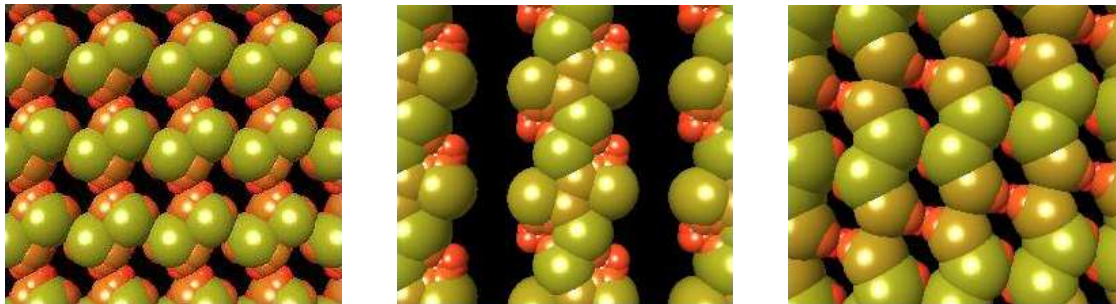


FIGURE 5. Ford domains on compression bodies, from a computer program I wrote to visualize these structures. Figure from [45].

Adams asked in the 1990’s whether all unknotting tunnels are isotopic to geodesics in a hyperbolic structure [1]. Adams and Reid showed this was the case for 2–bridge knots [2]. I find this problem to be a very intriguing mix of topology and geometry, and I have worked on it with several coauthors. By considering the geometry of unknotting tunnels and related problems, we have discovered much interesting mathematics, in both finite and infinite volume manifolds.

For example, with Cooper and Futer, I showed that unknotting tunnels are geodesic “generically,” in an appropriate sense [18]. The following is one result of that paper.

**Theorem 3.2.** *Let  $X$  be an orientable hyperbolic 3–manifold with two cusps and unknotting tunnel  $\sigma$ . Choose a generic Dehn filling slope  $\mu$  on one cusp of  $X$ , and let  $\tau \subset X(\mu)$  be the tunnel associated to  $\sigma$ , i.e.  $\tau$  is obtained by following  $\sigma$  from the unfilled cusp to the filled cusp, traversing the core of the Dehn filling solid torus, and then following  $\sigma$  in reverse. Then  $\tau$  is an unknotting tunnel for  $M(\mu)$ , and  $\tau$  is isotopic to a geodesic in the hyperbolic metric on  $X(\mu)$ .*

The proof uses Dehn filling techniques, including the negatively curved metric built to prove Theorem 1.4. One other tool we needed was explicit information on which Dehn filling slopes leave the Heegaard genus unchanged, i.e. an explicit version of theorems by Moriah and Rubinstein [52] and Rieck and Sedgwick [66]. Futer and I found these conditions in [32]. Namely, we give explicit conditions on Dehn filling slopes that guarantee that a Heegaard surface in the unfilled manifold remains a Heegaard surface in the filled manifold.

Another question that Adams asked in [1] was whether unknotting tunnels had universally bounded length. Cooper, Lackenby and I proved that this was not the case, but that for any  $L$ , there exists a hyperbolic 3–manifold with an unknotting tunnel of length greater than  $L$  [19].

The investigation of this problem led Lackenby and me to study geometric structures on compression bodies, especially those in Figure 4. Unlike the case of finite–volume manifolds, there are uncountably many hyperbolic structures on this compression body, parametrized by conformal structures on the exterior boundary. I wrote a computer program to visualize these structures using Ford domains. A few screenshots from my program are shown in Figure 5. The algorithm for the program was developed in joint work with Lackenby [45].

Given computer evidence, as well as additional results using techniques from Kleinian groups, Lackenby and I conjecture that in a geometrically finite hyperbolic structure on a compression body, the core tunnel is always isotopic to a geodesic [45].

However, the most obvious generalization of this conjecture is not true. S. Burton and I showed that there exist hyperbolic structures on compression bodies with tunnel number  $n \geq 2$ , and a choice of tunnel system, for which the geodesics in the isotopy classes of the tunnels self–intersect [12]. The tunnels Burton and I found, however, were not as closely related to geometric properties of the hyperbolic structure as other choices of tunnels. In fact, we did find a tunnel system for



these manifolds for which all tunnels are isotopic to geodesics. It is an open question as to whether there is always a choice of a tunnel system for which tunnels are isotopic to geodesics.

There is still much to discover concerning hyperbolic structures even for simple compression bodies. Because compression bodies are building blocks of all 3-manifolds, I believe that this is a very important area of investigation.

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