

DIAGRAM UNIQUENESS FOR HIGHLY TWISTED PLATS

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ABSTRACT. Frequently, knots are enumerated by their crossing number. However, the number of knots with crossing number c grows exponentially with c , and to date computer-assisted proofs can only classify diagrams up to around twenty crossings. Instead, we consider diagrams enumerated by bridge number, following the lead of Schubert who classified 2-bridge knots in the 1950s. We prove a uniqueness result for this enumeration. Using recent developments in geometric topology, including distances in the curve complex and techniques with incompressible surfaces, we show that infinitely many knot and link diagrams have a unique simple m -bridge diagram. Precisely, if m is at least three, if each twist region of the diagram has at least three crossings, and if the length n of the diagram is sufficiently long, i.e., $n > 4m(m-2)$, then such a diagram is unique up to obvious rotations. This projection gives a canonical form for such knots and links, and thus provides a classification of these knots or links.

1. INTRODUCTION

One way of studying knots in S^3 is via their regular projections on 2-spheres in S^3 . Such projections are called *knot diagrams*. Deciding when two diagrams correspond to the same knot is a difficult problem, going all the way back to work of Tait in the 1870s. In 1926, K. Reidemeister proved that any two regular projections are equivalent by a sequence of *Reidemeister moves*; see [22]. However, determining when two diagrams are equivalent by Reidemeister moves is also a very difficult problem, and remains an area of active research, for example see [17, 5].

The earliest attempts to classify knots began in the 1870's by Tait, using the *crossing number* of a knot as the classifying parameter. Today, some one hundred and thirty years later, only knots with less than 17 crossings have been completely classified.¹ For prime knots with 16 crossings, there are 1,701,936 different knots (where knots and their mirror image are not distinguished); see [14]. Thus distinguishing diagrams using crossing number is currently possible only for knots with a small number of crossings.

Another way around the problem of deciding when diagrams are equivalent would be to obtain “canonical” projections for knots. One attempt to do so was to use the

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¹That is, the classification for knots with up to 16 crossings has been published. Others, including Thistlethwaite, have worked on knots with up to 21 crossings.

fact that all knots have $2m$ -plat projections for some $m \in \mathbb{N}$, which are very closely related to m -bridge diagrams. This was used successfully on the class of 4-plats, more commonly known as 2-bridge knots and links. In 1956, H. Schubert gave a complete classification of 2-bridge knots and links. He showed in [23] that these knots are classified by a rational number, and that any two continued fraction expansions of this number correspond to a 4-plat projection. For a discussion of this see [3].

In a somewhat different flavor W. Menasco and M. Thistlethwaite proved in 1993 that any two alternating projections of a knot $\mathcal{K} \subset S^3$ are equivalent by flype moves [19]. Flypes involve the existence of essential 4-punctured spheres in the diagram, and they can be detected in a finite number of steps. However, the sequence of such steps can be arbitrarily long.

In this paper, we prove a *uniqueness of diagrams* statement for an infinite class of plats of arbitrary large width and length. In a sense that can be made precise, these plats are a generic collection in the set of all plats. We show that such plats have a *canonical* form that is unique, and can be read off the diagram immediately, without any need to consider equivalence relations such as Reidemeister moves or flypes.

For a precise definition of a *plat* and some facts about them see Section 2 and references [4] and [15]. The main theorem of this paper is:

Theorem 1.1 (Uniqueness of Diagrams). *Let $\mathcal{K} \subset S^3$ be a knot or link with a $2m'$ -plat projection K' , where $m' \geq 3$ and n' , the length of K' , satisfies $n' > 4m'(m' - 2)$. Suppose also that each twist region of K' contains at least three crossings. Finally, suppose that K is another plat or even plat projection of \mathcal{K} of width m . Then $m \geq m'$, and if $m = m'$ and each twist region of K has at least one crossing, then $K = K'$ up to rotation in a horizontal or vertical axis.*

1.1. The preferred notion of generic. Making mathematical sense of the term “generic” is particularly problematic when considering subsets of countable sets X . In the classic setting, a set $A \subset X$ would be called *generic* if and only if the complement $X \setminus A$ is finite. This definition is very restrictive and, in practice, there are many cases where intuitively one would like to call a set generic, yet it does not satisfy this criterion. A common method to deal with this situation is the following: One considers a family of finite subsets $X_n \subset X$ that exhaust X (typically given by enumerating $X = \{x_1, x_2, \dots\}$, and by defining $X_n = \{x_1, x_2, \dots, x_n\}$). One then computes the ratio $\rho_n = \frac{\#(X_n \cap A)}{\#X_n}$, and defines A to be generic if ρ_n tends to 1.

Of course, the obtained limit ratio depends heavily on the particular chosen enumeration of X .

Another difficulty with statements about “genericity” is that of *double counting*. A typical example here is to count group presentations instead of groups, so that the same group may well be counted several times, or even infinitely often. The problem arises when in the given mathematical context it is impossible to count the objects in question directly, so that double counting becomes unavoidable.

With these difficulties in mind we would like to advocate for the idea of enumerating knot by plats.

Assume Theorem 1.1. Choose $X = \mathbb{Z}$ with the obvious enumeration. Further choose $X_M = [-M, M] \cap \mathbb{Z}$ for some large interval $[-M, M] \subset \mathbb{R}$ and $M \in \mathbb{N}^+$. Choose at random a set $A \subset X_M$ consisting of one integer for each twist region in a $2m$ -plat diagram of length $n > 4m(m-2)$ (approximately $m \cdot n$ integers). Compute the probability ρ_M that A does not contain elements from $\{-2, -1, 0, 1, 2\} \subset X_M$. It is clear that $\rho_M \rightarrow 1$ as $M \rightarrow \infty$. In fact, even better, any such chosen set A determines a *very large* number of knots as the integers in A can be distributed in the twist regions, of the $2m$ -plat, in approximately $(n \cdot m)! > (4m^2(m-2))!$ different ways.

As the diagram is unique (and the knots are enumerated by their diagrams) this enumeration also takes care of the issue of double counting discussed above. Any diagram will be counted exactly once.

Hence the question of which notions of “genericness” one should adopt here is not merely a set theoretic issue, but it should depend on the topological consequence of the enumeration. Furthermore, in the case of the $2m$ -plat, the canonical form has a large number of other nice features:

The canonical form gives information about incompressible surfaces in $S^3 \setminus \mathcal{N}(K)$; see [7], [6] and [26]. In many cases, it gives information about $\pi_1(S^3 \setminus \mathcal{N}(K))$ and its rank, i.e. minimal number of generators [18]. It also gives information about the bridge number for K and Heegaard splittings of $S^3 \setminus \mathcal{N}(K)$ and about manifolds obtained by Dehn surgery on K ; see [18] and [26]. By combining work of Johnson and Moriah [15] and Bachman and Schleimer [1, Corollary 6.2], it follows that such plats are hyperbolic. A subclass was shown to have a property called *semi-adequacy*, with no 2-edge loops in an associated graph [13]. For this subclass, we obtain another proof of hyperbolicity [11], a bound on the volume [9], and the existence of essential spanning surfaces that do not have accidental parabolics [10].

In addition, a knot or link $\mathcal{K} \subset S^3$ with at least C crossings in each twist region, for appropriate C , is known to satisfy several nice geometric properties, whether or not it has the plat projection required by Theorem 1.1. For example, if $C = 6$, then \mathcal{K} and all Dehn fillings of \mathcal{K} are hyperbolic [12]. Additionally, closed embedded essential surfaces are known to be high genus [2]. If $C = 7$, there are known explicit upper and lower bounds on the hyperbolic volume of \mathcal{K} [8]. And if $C = 116$, then the shape of the cusp of \mathcal{K} is bounded [20]. It is conjectured that knots and links with at least 3 crossings per twist region are always hyperbolic.

From a hyperbolic geometry point of view, we argue further that the knots of Theorem 1.1 are generic in the same sense that hyperbolic Dehn filling is generic, following Thurston [24]. In his celebrated hyperbolic Dehn surgery theorem, Thurston showed that Dehn fillings of hyperbolic 3-manifolds yield hyperbolic 3-manifolds “generically”, in the sense that only finitely many Dehn filling slopes need to be excluded per boundary component, and the result is hyperbolic [24]. The links of Theorem 1.1 can also be seen as obtained by Dehn filling. Encircle each twist region by a simple unknot, called a crossing circle. The result is called a fully augmented plat. When one performs $(1, t_{i,j})$ Dehn filling on the $\{i, j\}$ -th crossing circle of a fully augmented plat diagram for each pair $\{i, j\}$, one obtains a regular plat diagram. Details of fully augmented

links and their Dehn fillings can be found in [21]. Links of Theorem 1.1 are obtained when we exclude Dehn fillings with $|t_{i,j}| < 2$.

Thus the links of Theorem 1.1 all lie in a Dehn filling neighborhood of a fully augmented plat. Note this is a very large, explicitly defined neighborhood: The bound of three crossings per twist region excludes far fewer knots than the best current bounds for hyperbolic Dehn filling (e.g. [12] requires six). Moreover, each such knot or link lies in a Dehn filling neighborhood of one of all possible Dehn filling parents of sufficiently long plat diagrams.

For the reasons above, we argue that enumeration of knots by bridge number makes more sense topologically and geometrically than enumeration by crossing number, and that the knots satisfying the conditions of Theorem 1.1 can be viewed as “generic”.

1.2. Even plats. Note that the statement of Theorem 1.1 includes knots and links with an “even plat” projection, which we now describe. Consider a braid with an even number of rows of twist regions on an even number of strings. At the top of the braid, we connect pairs of adjacent strings by bridges, just as in the usual case of a plat. At the bottom of the braid, we connect the second string to the third, the fourth to the fifth, and so on. At the end we connect the $2m$ string to the first one. We obtain a knot or link L depending on the parity of $a_{i,j}$. We will call a projection as above an *even plat*. Technically, L is not a plat in this case but Theorem 1.2 of [15] applies, so that we can compute the bridge distance. As in the case of a plat we can require that the number of crossings in each twist region be $|a_{i,l}| \geq 3$ and that the number of rows of twist regions plus one satisfies $n > 4m(m-2)$, so we have a unique bridge sphere by [25]. We conjecture:

Conjecture 1.2. *Theorem 1.1 holds when K' is an even plat.*

Going back to the question of crossing number, we conjecture that these plat projections also have minimal crossing number, as follows.

Conjecture 1.3. *Knot and link diagrams satisfying the conditions of Theorem 1.1 realize the crossing number of the associated links.*

Theorem 1.1 raises two questions:

Question 1.4. Suppose $\mathcal{K} \subset S^3$ is a knot that has a $2m$ -plat projection satisfying the conditions of Theorem 1.1. Suppose that $\mathcal{K} \subset S^3$ is given by some knot diagram $D(\mathcal{K})$. Is there an “efficient” algorithm to isotope $D(\mathcal{K})$ to its canonical $2m$ -plat projection?

Question 1.5. Given a knot $\mathcal{K} \subset S^3$ can one decide if \mathcal{K} has a $2m$ -plat projection satisfying the conditions of Theorem 1.1?

Remark 1.6. Highly twisted plats are defined carefully in Section 2 in terms of a specific family of words in the braid group. It follows from Theorem 1.1 that the corresponding braids are uniquely determined in the braid group by such a braid word.

1.3. Idea of the proof. Let \mathcal{K} , K' , and K be as in the statement of the theorem, as well as other notation. Recall that plat diagrams determine collections of bridge-spheres (see Definition 2.1) and vertical S^2 -spheres (see Definition 2.9). The main idea is to see how these two collections behave with respect to each other under an ambient isotopy of the knot from K' to K .

The condition that $n' > 4m'(m' - 2)$ implies that the distance of the bridge sphere is greater than $2m'$. Results of Tomova [25], Johnson-Tomova [16], and Johnson-Moriah [15] imply m has to be minimal; thus $m \geq m'$.

Suppose $m = m'$. Consider the ambient isotopy $\varphi : (S^3, K') \rightarrow (S^3, K)$. We show first in Lemma 3.2 that we may take the ambient isotopy to preserve the foliation of (S^3, K') by bridge spheres. Using this, we show in Proposition 3.7 that the image of a vertical 2-sphere (see Definition 2.9) is mapped to a sphere which does not meet any twist region. As a consequence we show in Lemma 3.11 that the image of a vertical 2-sphere can intersect K at most once in each level, and that allows us to obtain Proposition 4.4 which states that the length of the two plats must be the same.

It remains to show that the coefficients in each of the twist regions are the same and that they occur in the same order up to reflection in horizontal and vertical axes.

After showing that vertical 2-spheres are mapped to vertical 2-spheres, we construct for each twist region $t'_{i,j}$ an *isolating sphere* Ω' . These spheres are composed of two disks, each one in a vertical 2-sphere, and each disk intersecting K' in two points. The bounded component of $S^3 \setminus \Omega'$ is a 2-tangle of type $1/a'_{i,j}$. We then show that such tangles must be mapped by φ to tangles of type $1/a_{i,j}$ in K , where $a_{i,j} = a'_{i,j}$. Since tangles $t_{i,j}$ are connected to adjacent tangles, the order of the twist regions is preserved up to reflection in horizontal and vertical axes.

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2. PRELIMINARIES

For any knot or link $\mathcal{K} \subset S^3$ there is an $m \in \mathbb{N}$ so that \mathcal{K} has a $2m$ -plat projection, as indicated in Figure 1. This follows from the classical fact that any knot or link can be presented as a closed braid, and then strands of the braid closure can be pulled across the braid diagram; for example see [4, p. 24].

In Figure 1 a box labeled $a_{i,j}$ indicates a *twist region* with $a_{i,j}$ crossings, where $a_{i,j}$ can be positive, negative, or zero. Note that crossings in a twist region must be alternating. That is, some number of negative crossings cannot be followed by some number of positive crossings, else the diagram could be reduced within the twist regions by cancelling crossings. Thus $a_{i,j}$ indicates the number of crossings of the same sign in the *reduced* twist region. For the example in the figure, $a_{1,1} = a_{2,2} = -3$, and all other $a_{i,j} = -4$. The twist region associated with the box labeled $a_{i,j}$ will be denoted by $t_{i,j}$. We can think of the $t_{i,j}$ twist region as a projection to P , the plane of projection, of a $1/a_{i,j}$ -tangle considering the segments of K as strands contained in the 3-ball ($\text{box} \times I$).

Notice that a $2m$ plat projection K of \mathcal{K} corresponds to an element \mathcal{B}_{2m} in the braid group on $2m - 1$ generators $\{\sigma_1, \dots, \sigma_{2m-1}\}$. This implies that a plat can be given a

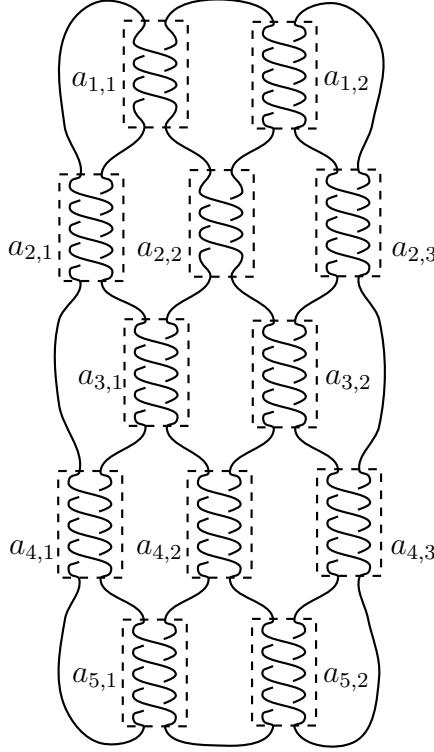


FIGURE 1. A 6-plat projection of a 3-bridge knot.

“standard” form by requiring that the element corresponding to \mathcal{B}_{2m} be written as a concatenation of sub-words $\mathcal{B}_{2m} = b_1 \cdot b_2 \cdot \dots \cdot b_{n-1}$, where b_i has the following properties:

(1) When i is odd, b_i is a product of all σ_j with j even. Namely:

$$b_i = \sigma_2^{a_{i,1}} \cdot \sigma_4^{a_{i,2}} \cdot \dots \cdot \sigma_{2m-2}^{a_{i,m-1}}$$

(2) When i is even, b_i is a product of all σ_j with j odd. Namely:

$$b_i = \sigma_1^{a_{i,1}} \cdot \sigma_3^{a_{i,2}} \cdot \dots \cdot \sigma_{2m-1}^{a_{i,m}}$$

We say m is the *width* of the plat and n , which is *even*, is the *length* of the plat.

In Figure 1, there are five rows, so the length is $n = 6$. A $2m$ -plat projection with properties (1) and (2) that also minimizes the width m will be called *standard*.

A $2m$ -plat will be called *c-highly twisted* if $|a_{i,j}| \geq c$ for some constant c , for all i, j . Similarly, a knot or link that admits a *c-highly twisted* plat projection will be called a *c-highly twisted knot or link*.

Definition 2.1. An m -bridge sphere of a knot or link $\mathcal{K} \subset S^3$ is a 2-sphere which meets \mathcal{K} in $2m$ points and cuts (S^3, \mathcal{K}) into two $2m$ -string trivial tangles (B_1, \mathcal{T}_1) and (B_2, \mathcal{T}_2) . An m -string *trivial tangle* is a pair (B^3, \mathcal{T}) of 3-ball B^3 and a collection \mathcal{T} of m arcs properly embedded in B^3 that are simultaneously isotopic into ∂B^3 fixing ∂B^3 .

Definition 2.2. Two bridge spheres Σ_1 and Σ_2 for a knot or link $\mathcal{K} \subset S^3$ are said to be *isotopic* if there is a homeomorphism of pairs $\psi : (S^3, \mathcal{K}) \rightarrow (S^3, \mathcal{K})$ so that $\psi(\Sigma_1) = \psi(\Sigma_2)$ and ψ is isotopic to the identity on S^3 .

Definition 2.3 (Schubert [23] 1956). The *bridge number* $b(\mathcal{K})$ of a knot or link $\mathcal{K} \subset S^3$ is the minimal number of over-crossing arcs over all regular projections of \mathcal{K} . Equivalently, the bridge number is the minimal value of m such that \mathcal{K} has an m -bridge sphere; see [4, p. 23].

Note that every $2m$ -plat projection defines an m -bridge knot or link with $m \geq b(\mathcal{K})$ and every m -bridge knot or link has a $2m$ -plat projection (see [4, p. 24]). One can also determine a bridge sphere Σ from a $2m$ -plat projection as follows. Arrange the maximum and minimum points of the plat projection to lie on horizontal lines at the top and bottom of the diagram, as in Figure 1. Consider a horizontal line in the projection plane P to lie just below the maximum points, meeting K in exactly $2m$ points. Connect endpoints of the line by a simple arc on the projection plane that does not meet K , forming a closed curve. Then cap this curve by two disks, one above and one below the plane P , to obtain the bridge sphere Σ .

With the diagram arranged in this manner, there is an “obvious” height function on a $2m$ -plat K , where the maximum point on the top bridges corresponds to height $t = 1$ and the minimum point on the bottom bridges corresponds to height $t = 0$. Note the bridge sphere Σ defined above corresponds to some height, $t = 1 - \varepsilon$. More generally, a *horizontal* bridge sphere Σ_t can be constructed for any height $t \in (0, 1)$ in the same manner. Note that all bridge spheres Σ_t for $0 < t < 1$ are isotopic. We can further define horizontal spheres at height $t = 0$ and $t = 1$, but these do not meet K transversely, and meet it in only m points.

Definition 2.4. Let $\Sigma_{g,p}$ be a surface of genus g with p punctures. A simple closed curve $\gamma \subset \Sigma_{g,p}$ is *inessential* in $\Sigma_{g,p}$ if it bounds either a disk or a once punctured disk in $\Sigma_{g,p}$. A simple closed curve in $\Sigma_{g,p}$ is *essential* if it is not inessential. The curve complex $\mathcal{C}(\Sigma)$ is a simplicial complex defined as follows.

Let $[\gamma]$ denote the isotopy class of an essential simple closed curve $\gamma \subset \Sigma$.

- (1) The set of vertices of $\mathcal{C}(\Sigma)$ is $V(\Sigma) = \{[\gamma] \mid \gamma \subset \Sigma \text{ is essential}\}$.
- (2) An n -simplex is an $(n + 1)$ -tuple $\{[\gamma_0], \dots, [\gamma_n]\}$ of vertices that have pairwise disjoint curve representatives.

Define a metric d on the 1-skeleton $\mathcal{C}^1(\Sigma)$ of $\mathcal{C}(\Sigma)$ by setting the length of each 1-simplex to be 1, and taking minimal path length.

Definition 2.5. Given a bridge surface Σ for K , define $\Sigma_K = \Sigma \setminus K$, $V_K = B_1 \setminus \mathcal{T}_1$ and $W_K = B_2 \setminus \mathcal{T}_2$. Let $\mathcal{D}(V_K)$ (resp. $\mathcal{D}(W_K)$) be the set of all essential simple closed curves in Σ_K that bound disks in V_K (resp. W_K). Define the *(bridge) distance* of Σ to be $d(\Sigma_K) = d(\mathcal{D}(V_K), \mathcal{D}(W_K))$ measured in $\mathcal{C}(\Sigma_K)$.

We now refer to two theorems that provide the basis for this paper. The first is a special case of the main theorem of Tomova for knots [25], extended to links by Johnson and Tomova [16, Theorem 4.4].

Theorem 2.6 (Tomova [25] and Johnson-Tomova [16]). *Let $\mathcal{K} \subset S^3$ be a knot or link with two bridge spheres Σ_1 and Σ_2 meeting \mathcal{K} in $2m_1$ and $2m_2$ points, respectively, where m_1 and m_2 are at least 3. Then either Σ_2 is isotopic rel \mathcal{K} to Σ_1 or $d(\Sigma_1) \leq 2m_2$ (or $d(\Sigma_2) \leq 2m_1$).*

Theorem 2.7 (Johnson-Moriah [15]). *Let $\mathcal{K} \subset S^3$ be a 3-highly twisted link with a $2m$ -plat projection K , with $m \geq 3$ and $n - 1$ rows of twist boxes. Then the distance $d(\Sigma)$ of the induced bridge surface Σ is exactly*

$$d(\Sigma) = \lceil n/(2(m-2)) \rceil,$$

where $\lceil x \rceil$ is the smallest integer greater than or equal to x .

Combining these two, we obtain:

Corollary 2.8. *If the 3-highly twisted $2m$ -plat K corresponding to the knot or link $\mathcal{K} \subset S^3$ has length n , where $m \geq 3$ and $n > 4m(m-2)$, then the knot \mathcal{K} is an m -bridge knot or link, and it has a unique m -bridge sphere, up to isotopy.*

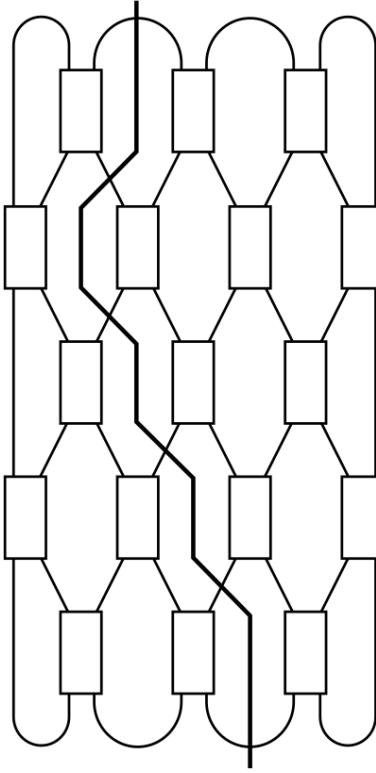
Proof. As $n > 4m(m-2)$, Theorem 2.7 implies $d(\Sigma) > 2m$. Hence if m were not minimal, say \mathcal{K} is an m_1 -bridge knot or link with $m_1 < m$, then by Theorem 2.6, we have $d(\Sigma) < 2m_1 < 2m$, which is a contradiction. Further, any other bridge sphere Σ_1 meeting \mathcal{K} in $2m$ points must be isotopic to Σ . \square

Finally we define the following surfaces first defined in [6]; see also [26].

Definition 2.9. Let K be a $2m$ -plat projection of a knot or link $\mathcal{K} \subset S^3$, with projection plane P . Assume K has length n . Let $\alpha = \alpha(c_1, \dots, c_{n-1})$ be an arc running monotonically from the top of the plat to the bottom so that α is disjoint from all twist regions and there are c_i twist regions to the left of the arc at the i -th row. We further require that there is at least one twist region on each side of α at each level, and that α intersects K in precisely n points. Now connect the endpoints of α by a simple arc $\beta \subset P$ that is disjoint from K , and cap the simple closed curve $\gamma = \alpha \cup \beta$ by two disks, one above P and the other below P . Thus, we have obtained a 2-sphere $S = S(c_1, \dots, c_{n-1})$ called a *vertical 2-sphere*; see Figure 2.

Theorem 2.10 (Finkelstein–Moriah [7] and Wu [26]). *Let $\mathcal{K} \subset S^3$ be a 3-highly twisted link with a $2m$ plat projection K , with $m \geq 3$ and length $n \geq 5$, and let S be a vertical 2-sphere for K . Then S is incompressible and boundary incompressible.*

Remark 2.11. A vertical 2-sphere separates a knot or link \mathcal{K} in a plat projection into two $n/2$ string tangles $(B_1, \mathcal{T}_1), (B_2, \mathcal{T}_2)$. These tangles are non-trivial by Theorem 2.10. We always assume that the projection plane and vertical spheres intersect transversally.

FIGURE 2. A template for a 4-plat with $n = 6$ and an α arc.

Vertical 2-spheres are required to have a twist region on either side on each row of twist regions. We also will need to work with similar 2-spheres that relax this restriction, in the following way.

Definition 2.12. Let $S = S(c_1, \dots, c_{n-1})$ and α be as in Definition 2.9. If there is at least one twist region on each side of α at each level except possibly level i , for i odd and $3 \leq i \leq n-3$, then we say S is an *odd almost vertical 2-sphere*. If there is at least one twist region on each side of α at each level except possibly levels i and $i+2$, where i is odd and $3 \leq i \leq n-5$, then we say S is a *doubly odd almost vertical 2-sphere*. If there is at least one twist region on each side of α at each level except possibly at levels $i-1$, i , and $i+1$, where i is even and $2 < i < n-2$, we say S is an *even almost vertical 2-sphere*. In all three cases, we say S is an *almost vertical 2-sphere*. Note that the set of almost vertical 2-spheres contains the set of vertical 2-spheres.

3. ISOTOPIES

If a knot or link $\mathcal{K} \subset S^3$ has two plat projections K and K' with respect to projection planes P and P' respectively, we can think of K and K' as embedded in an ε neighborhood of P and P' respectively. Because they are projections of the same link,

there is an ambient isotopy $\varphi : (S^3, K') \rightarrow (S^3, K)$ taking K' to K , and taking vertical 2-spheres $S' \subset (S^3, K')$ to spheres $S = \varphi(S') \subset (S^3, K)$.

Remark 3.1. Note that throughout the paper, objects and parameters, such as surfaces and twist coefficients relevant to the pair (S^3, K') of the K' plat will be denoted by a ‘‘’, e.g. $S' \subset (S^3, K')$. Those relevant to (S^3, K) will have no extra mark, e.g. $S \subset (S^3, K)$.

Lemma 3.2. *Let K', K be two $2m$ -plat projections of the same knot or link $\mathcal{K} \subset S^3$, where $m \geq 3$ and n , the length of K' , is greater than $4m(m-2)$. Assume that K' is 3-highly twisted. Let $\varphi : (S^3, K') \rightarrow (S^3, K)$ be an isotopy. Let S' be an almost vertical 2-sphere for K' , and let $S = \varphi(S')$ be its image. Then there exists an isotopy of pairs $\psi : (S^3, K) \rightarrow (S^3, K)$ with the following properties.*

- (a) *The image $\psi(S)$ is a 2-sphere meeting each horizontal bridge sphere of K in a single essential curve, so that any such curve meets K at most once.*
- (b) *For each horizontal bridge sphere Σ'_t for K' , the image $(\psi \circ \varphi)(\Sigma'_t)$ is a horizontal bridge sphere for K .*

Proof. The lemma follows from Corollary 2.8, which states that the bridge sphere is unique up to isotopy, as follows:

The link space (S^3, K') is foliated by horizontal bridge spheres. That is, the link K' is contained in a region $S^2 \times [0, 1] \subset S^3$ in which each $S^2 \times \{t\}$ is a bridge sphere Σ'_t , with the exception of $t = 0$ and $t = 1$, where the spheres meet the diagram in m points. By the definition of an almost vertical 2-sphere, each horizontal sphere Σ'_t intersects S' in a single essential curve, and no such curve meets K' more than once.

Now the isotopy $\varphi : (S^3, K') \rightarrow (S^3, K)$ takes this foliation of K' to a foliation of K by bridge spheres. Note that images $\varphi(S')$ intersect images $\varphi(\Sigma'_t)$ in a single essential curve. Because the bridge sphere is unique up to isotopy, each level $\varphi(\Sigma'_t)$ is isotopic to a horizontal bridge sphere for K , under an isotopy fixing K . The images of $\varphi(S')$ and $\varphi(\Sigma'_t)$ will still meet in single curves under this isotopy. Thus we need to ensure that the bridge spheres $\varphi(\Sigma'_t)$ can be simultaneously isotoped to the horizontal foliation of K . We do this by an open-closed argument:

Let $I \subset (0, 1)$ be the sub-interval containing $1/2$ for which we may simultaneously isotope Σ'_t to Σ_t for any $t \in I$. Then I is nonempty by Corollary 2.8: we may isotope $\Sigma'_{1/2}$ to $\Sigma_{1/2}$.

The interval I is open, because if $t \in I$, then for small enough $\varepsilon > 0$, the image $\varphi(\Sigma' \times (t - \varepsilon, t + \varepsilon))$ is contained in some $\Sigma \times (t - \delta, t + \delta)$, taking Σ'_t to Σ_t . There is an isotopy of the ball with boundary Σ_t keeping Σ_t fixed and taking the small neighborhood $\varphi(\Sigma' \times [t, t + \varepsilon])$ of Σ_t to a small fibered neighborhood. By stretching or contracting, we may assume $\varphi(\Sigma' \times \{s\})$ goes to $\varphi(\Sigma \times \{s\})$. Similarly for $\Sigma' \times (t - \varepsilon, t]$. So I is open.

Moreover, I is closed, for if t_n is a sequence in I with limit t , then $\varphi(\Sigma'_{t_n})$ is contained in Σ_{t_n} ; by continuity $\varphi(\Sigma'_t)$ lies on Σ_t . So $I = (0, 1)$.

Thus, $\varphi(S^2 \times (0, 1))$ can be isotoped to be horizontal under an isotopy of pairs $\psi : (S^3, K) \rightarrow (S^3, K)$. Now consider the images of $\varphi(S^2 \times \{0\})$ and $\varphi(S^2 \times \{1\})$ under this isotopy. They each meet K in m points. Moreover, K is contained in the image $(\psi \circ \varphi)(S^2 \times [0, 1])$. Thus $(\psi \circ \varphi)(S^2 \times \{0\})$ and $(\psi \circ \varphi)(S^2 \times \{1\})$ must also be horizontal. Under this isotopy, the image of S' is taken to S , satisfying the conclusions of the lemma. \square

Let $S_{i,j}$ denote a 4-punctured sphere embedded in $S^3 \setminus K$ that encloses the twist region $t_{i,j}$. Each $S_{i,j}$ is chosen so that the collection of all such 4-punctured spheres over all pairs (i, j) is disjoint. Let $B_{i,j}$ denote the bounded closed ball in S^3 whose boundary contains $S_{i,j}$. Thus, $B_{i,j} \subset S^3$ contains the $1/a_{i,j}$ tangle determined by the $t_{i,j}$ twist region of K .

Lemma 3.3. *With the same hypotheses on K' and K from Lemma 3.2, there exists an isotopy $\psi_2 : (S^3, K) \rightarrow (S^3, K)$ that preserves properties (a) and (b) of that lemma, and in addition does the following.*

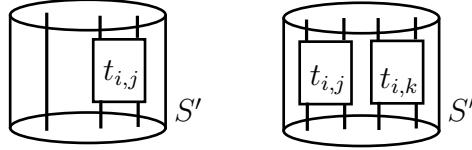
(c) *It takes S to a surface either disjoint from $B_{i,j}$ or meeting $B_{i,j}$ in disjoint disks, each separating the two strands of K within $B_{i,j}$, and each disjoint from K . Note this implies all points of $S \cap K$ lie outside of twist regions.*

The proof of the lemma uses incompressibility and boundary incompressibility to slide surfaces past each other. However, the argument is not quite standard, because we need to take care to ensure that isotopies preserve horizontal bridge spheres at every level. Additionally, while we know that vertical 2-spheres are incompressible and boundary incompressible by Theorem 2.10, this is not true for 2-spheres which are almost vertical. In order to prove the lemma in the case of almost vertical 2-spheres, we will need the following definition and lemmas.

Definition 3.4. A *vertical boundary compression disk* for an almost vertical 2-sphere $S' \subset S^3 \setminus \mathcal{N}(K')$ is a boundary compression disk arranged so that its intersection with any horizontal bridge sphere is either empty, or a simple arc from K' to S' , or a single point on $\mathcal{N}(K') \cap S'$.

Lemma 3.5. *Suppose that a portion of the surface S' forms a vertical annulus between two heights, bounding regions of K' as shown in the left or right parts of Figure 3. That is, either the annulus bounds three strands, one parallel to the annulus and the other two forming twist region $t_{i,j}$, or the annulus bounds four strands, one pair forming twist region $t_{i,j}$ and the other forming twist region $t_{i,k}$. Then the region bounded by the annulus does not contain a vertical boundary compression disk that meets each horizontal level in an arc with one endpoint on a segment of K' in a twist region.*

Proof. The region bounded by the vertical annulus in S^3 is a solid cylinder. In either case, the solid cylinder, together with the strands of K' intersecting it, form either a 3 or 4-tangle. There is a separating compressing disk E for the tangle, separating a

FIGURE 3. Two ways that S' can enclose strands of K'

twist region on one side and either a twist region or a strand of K' on the other. We view E as a rectangle in the solid cylinder with top and bottom on the disks bounding the solid cylinder and sides on the vertical annulus. Note E separates the 3 or 4-tangle into a 2-tangle and either a 1 or 2-tangle.

Now suppose by way of contradiction that D is a vertical boundary compression disk meeting K' in a twist region. Then D meets K' in a 2-tangle separated off by E ; denote this 2-tangle by (B_1, T_1) and denote the other tangle separated by E by (B_2, T_2) . Because D meets K' in the tangle (B_1, T_1) , its boundary spirals around the twist region, and hence it meets the boundary of the tangle in a spiral. Therefore, D will intersect E in a series of (at least three) arcs running from one side of E on the annulus to the opposite side of E on the annulus, and we may ensure that each arc of $E \cap D$ meets any horizontal level in a single point on E .

Consider an arc of intersection $E \cap D$. This arc splits D into two disks: one lying in (B_1, T_1) and the other, call it D' , in (B_2, T_2) . The disk D' must meet each horizontal level in an arc with an endpoint on E and an endpoint on S' . The union of all horizontal levels meeting this component of $E \cap D$ is then D' . Thus D' is a disk embedded in the tangle (B_2, T_2) with boundary that does not meet the top or bottom of the tangle. This is impossible: such a disk would intersect one of the strands of K' running from the top to bottom of (B_2, T_2) . \square

Lemma 3.6. *There is no vertical boundary compression disk for an almost vertical 2-sphere S' in $S^3 \setminus \mathcal{N}(K')$.*

Proof. We may assume S' is not a vertical 2-sphere, because those are known to be boundary incompressible by Theorem 2.10.

Suppose there is a vertical boundary compressing disk D' . Every almost vertical 2-sphere has two sides, one containing several twist regions in each row, and the other side containing no twist regions in one or more rows. Call the latter the *small side*. Without loss of generality, we will assume that the small side is to the left of S' . First we claim that D' lies in the small side. If it lies in the large side, then by changing the plat diagram by adding twist regions to the small side, increasing its width, the almost vertical 2-sphere becomes a vertical 2-sphere. The disk D' becomes a boundary compressing disk for the vertical 2-sphere, which contradicts Theorem 2.10.

We first treat the cases that the vertical 2-sphere S' is odd or doubly odd. Assume first that $S' = S'(c_1, \dots, 1, 0, 1, \dots, c_{n-1})$ is an odd almost vertical 2-sphere, hence $c_i = 0$ for some i odd, $3 \leq i \leq n-3$. Then there is another vertical 2-sphere $S'' = S''(c_1, \dots, 1, 1, 1, \dots, c_{n-1})$ such that $S' \setminus S''$ is a single disk Δ' in the i -th level

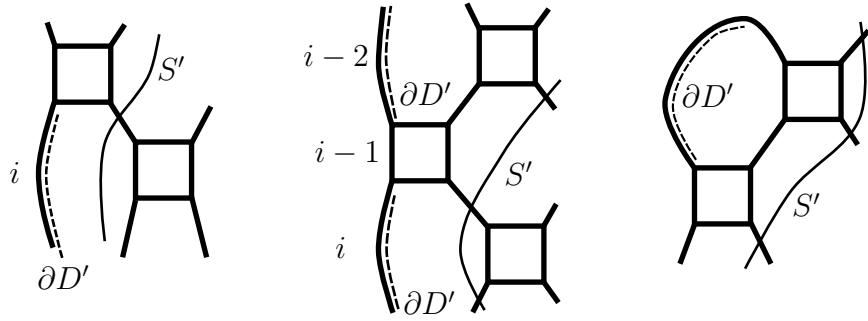


FIGURE 4. Left: In the i -th row, $\partial D' \cap K'$ lies on the strand of K' disjoint S' . Middle: In the $(i-2)$ -th row, $\partial D' \cap K'$ must lie on the leftmost strand of K' . Right: $\partial D' \cap K'$ will not meet S' at the top of the diagram if it lies on the leftmost strand of K' in this row.

with two punctures on K' . The boundary of D' must run through this disk Δ' , else it would be a boundary compressing disk for S'' , which is a contradiction.

Similarly, in the case that $S' = S'(c_1, \dots, 1, 0, 1, 0, 1, \dots, c_{n-1})$ is doubly odd, there is a vertical 2-sphere $S'' = S''(c_1, \dots, 1, 1, 1, 1, 1, \dots, c_{n-1})$ such that $S' \setminus S''$ consists of two disks Δ'_i and Δ'_{i+2} . Again the boundary of D' must meet at least one of these. If it meets Δ'_i , set $\Delta' = \Delta'_i$. If it does not meet Δ'_i , then rotate the diagram through a horizontal axis to obtain a plat projection with S' doubly odd, so that Δ'_{i+2} goes to row j with j odd, $j \geq 3$, and Δ'_i goes to row $j+2$. Now set $\Delta' = \Delta'_j$.

In any case, odd or doubly odd, we obtain an almost vertical 2-sphere S' that bounds no twist regions on its small side in some row that we will denote by i , where i odd and $i \geq 3$, and that agrees with a vertical 2-sphere S'' for all rows ℓ with $1 \leq \ell < i$.

Moreover, the boundary of D' meets Δ' in the i -th row. We will analyze how the disk D' meets the diagram in rows i , $i-1$, $i-2$, etc.

In row i , note there is only one segment of K' on the small side of S' . Hence, since $\partial D'$ meets Δ' , $\partial D'$ must have a sub-arc on this segment of K' by definition of vertical boundary compression disk. There must also be a component of $\partial D' \cap S'$ on Δ' . The sub-arc of $\partial D'$ on S' and the one on K' must eventually meet, as they form $\partial D'$. However, notice they cannot meet in the i -th row or just above it, since the strand of K' meeting $\partial D'$ is disjoint from S' in this row; see Figure 4, left.

Now consider the intersection of D' with the $(i-1)$ -st row of twist regions. Between the top and bottom of the $(i-1)$ -st row of twist region, S' forms a cylinder bounding a single twist region on the small side. Since $\partial D'$ has an arc on K' within this twist region, by continuity, it must have an arc on the outside on the cylinder, and the disk D' in this region must spin around the knot, following the twist region. Note again that arcs of $\partial D' \cap K'$ and $\partial D' \cap S'$ cannot meet in the $(i-1)$ -st row or just above it, because S' does not meet either of the strands of K' in that twist region at the horizontal levels within the row. And going up, the arc $\alpha(c_1, \dots, c_{n-1})$ defining S' turns right on the $(i-2)$ level, still avoiding those two strands of K' . See Figure 4, middle.

In the $(i-2)$ -nd level, the small side of S' forms a cylinder bounding a twist region of K' and an unknotted vertical segment of K' , as on the left of Figure 3. By Lemma 3.5,

$\partial D'$ cannot have an arc on one of the segments of K' in the twist region. Hence its boundary runs along the vertical segment of K' , shown in Figure 4. There are now two cases:

If $i = 3$, then this is the top of the diagram. One boundary component of $\partial D'$ lies on the segment of K' to the far left, on the left-most bridge. But the surface S' meets the second bridge to the left. These segments do not agree, so arcs of $\partial D'$ cannot meet up, and therefore the vertical compressing disk D' cannot exist in this case. See Figure 4.

If $i > 3$, since S' does not meet the vertical segment of K' in level $(i - 2)$, D' must continue to a still higher level. In this $(i - 3)$ -rd level, S' could go one of two ways. Its small side could either bound two full twist regions on the same level, or a single twist region. By Lemma 3.5, the former cannot happen, since the vertical compression disk cannot lie in a cylinder bounding two twist regions. So S' bounds a cylinder containing a single twist region, just as in level $i - 1$.

We now repeat the argument and find that in odd levels, D' runs along the outermost segment of K' , and in even levels, S' has exactly one twist region on its small side. Eventually we must reach the top of the diagram. At the top, D' has an arc on the outermost segment of K' , which meets the outermost bridge. However, S' must meet the next bridge over. This is a contradiction. We conclude that there cannot be a vertical boundary compressing disk for an odd or doubly odd almost vertical 2-sphere.

Now consider an even almost vertical 2-sphere $S' = S'(c_1, \dots, 0, 0, 0, \dots, c_{n-1})$, where the zeros are in rows $i - 1$, i , and $i + 1$ for i even, $4 \leq i \leq n - 4$. Suppose there is a boundary compressing disk D' . Again D' lies on the small side of S' , and if D' meets a horizontal level, then it meets K' in that level. Note that in the i -th level, the small side contains no segments of K' . So D' either lies entirely above or entirely below the i -th row. Then D' is a vertical boundary compressing disk for the doubly odd almost vertical 2-sphere $S'(c_1, \dots, 0, 1, 0, \dots, c_{n-1})$. This is a contradiction. \square

Proof of Lemma 3.3. Assume we have already isotoped (S^3, K) as in Lemma 3.2, and denote the composition of isotopies from (S^3, K') to (S^3, K) to (S^3, K) by φ .

Recall that $S_{i,j} = \partial B_{i,j}$. Consider the intersection of spheres $S \cap S_{i,j}$. The components are disjoint simple closed curves embedded both in S and in $S_{i,j}$. Each $S_{i,j}$ meets K four times. By a general position argument, we may assume each component of $S \cap S_{i,j}$ is disjoint from K . Then on $S_{i,j}$, such a component bounds disks on either side.

Step 1. Suppose a component of $S \cap S_{i,j}$ bounds a disk on $S_{i,j}$ that is disjoint from K . Then an innermost such component bounds an embedded disk $E \subset S_{i,j}$ disjoint from S and K . Because S is incompressible, ∂E must also bound an embedded disk in S ; call it D . The union of E and D is a sphere in S^3 disjoint from K , hence it bounds a ball B . We will use the ball to isotope S through $S_{i,j}$, removing the intersection; however we must take some care to preserve the conclusion of Lemma 3.2.

If ∂E lies on a single horizontal level, i.e. $\partial E \subset \Sigma_t$ for some t , then the curve ∂E must be exactly the curve $S \cap \Sigma_t$, since S meets any level in a single simple closed curve by Lemma 3.2. Then the disk $D \subset S$ must also meet each level in a single simple closed curve or a single point. We may isotope D by pushing vertically to E , removing all

intersections with levels above (or below) Σ_t , and then finally pushing off E to remove the intersection $S \cap S_{i,j}$, preserving the conclusions of Lemma 3.2.

So suppose ∂E does not lie in a single level, but lies in a minimal range $\Sigma \times [a, b]$. Then for $t \in (a, b)$, $S \cap \Sigma_t$ is a single simple closed curve meeting ∂E in two points. These points separate $S \cap \Sigma_t$ into an arc lying in D and an arc lying outside of D . The arc lying in D along with the arc $E \cap \Sigma_t$ together bound a disk: a slice of the ball B bounded by $D \cup E$. Consider levels Σ_a and Σ_b . Note $E \cap \Sigma_a$ is a single point. By continuity, either $D \cap \Sigma_a$ is also a single point, or it is a simple closed curve bounding a disk in B . In the latter case, an argument similar to that of the previous paragraph implies we may isotope S , preserving conclusions of Lemma 3.2, to be disjoint from levels below Σ_a . Similarly, we may isotope S so that D is disjoint from levels above Σ_b . Thus we may assume that D lies in the range $\Sigma \times [a, b]$. But then the ball B also lies in this range, and we may isotope S horizontally through the ball, preserving levels, to remove the intersection $S \cap S_{i,j}$ while preserving the conclusions of Lemma 3.2. Repeating this a finite number of times, we remove all intersections $S \cap S_{i,j}$ that bound a disk in $S_{i,j}$ disjoint from K .

Step 2. Suppose now a component of intersection of $S \cap S_{i,j}$ bounds a disk in $S_{i,j}$ that meets K exactly once. Again we will argue that we can isotope the intersection away, preserving the conclusions of Lemma 3.2. For this argument, we need to use the fact that a vertical 2-sphere is *meridionally incompressible*. That is, if a disk E is embedded in S^3 with $\partial E \subset S$ and E meets K in a single meridional curve on K , then ∂E bounds a disk on S with the same property: it meets K exactly once in a meridional curve. This result cannot be found in the literature, however its proof is nearly identical to Wu's proof that vertical 2-spheres are essential [26]. We include it in the appendix, Section 5.

Assuming this fact, the proof in this step proceeds as above. If $E \subset S_{i,j}$ is a disk meeting K in a single puncture, then the fact that S is meridionally incompressible implies that ∂E also is the boundary of a disk $D \subset S$ meeting K in a single puncture. Then $D \cup E$ bounds a ball in S^3 . Consider the intersection of the ball with horizontal levels. An argument similar to the previous step implies that we may isotope through horizontal levels, preserving the conclusions of Lemma 3.2.

Step 3. At this stage, we may assume that all intersections in $S \cap S_{i,j}$ are curves on $S_{i,j}$ bounding disks with exactly two punctures on either side. Because S is embedded, if there are multiple components of $S \cap S_{i,j}$ then all such intersection curves must be parallel on $S_{i,j}$. It follows that $S \cap B_{i,j}$ is either empty, all of S , or a collection of disks and annuli in S .

First note $S \cap B_{i,j} \neq S$, for in this case, S is a punctured 2-sphere contained entirely in the tangle $B_{i,j}$. There is a disk embedded vertically in the tangle that separates the two strands of K ; denote this disk by D . If we cut along D , we obtain two trivial tangles, each containing a single strand of K , which we will denote by B_1 and B_2 . The disk D must intersect S in simple closed curves. An innermost component of intersection bounds a disk on D , hence it bounds a disk on S . Thus it can be isotoped away in a level-preserving manner as above; we conclude that S is disjoint from D .

But then S is an essential sphere with n punctures embedded in a trivial tangle B_i with a single knot strand. This is impossible.

Now we show that we can rule out annulus components of $S \cap B_{i,j}$. For suppose A is an annulus of $S \cap B_{i,j}$. Then ∂A bounds an annulus A' on $S_{i,j}$, and together A and A' bound a solid torus V in S^3 . If V is disjoint from K , then we may isotope A through V in a level-preserving manner (as above), to A' and slightly past, removing two components of intersection $S \cap S_{i,j}$. So K must meet A .

Now consider intersections of the disk D and tangles B_1 and B_2 with the annuli and solid torus. If D is disjoint from A and A' , then $A \cup A'$ and the solid torus V lie entirely in a single trivial tangle, say B_1 . We are assuming A meets K . Then it must meet K at least twice in B_1 . But K is unknotted in B_1 , so between some two points of entry and exit of K , we have a vertical boundary compressing disk for A , hence S . Pulling back via φ^{-1} to K' gives a vertical boundary compressing disk for S' , contradicting Lemma 3.6.

If A' is disjoint from D , but A meets D , then $\partial A'$ must be parallel to ∂D , hence A meets D only in simple closed curves. An innermost curve on D bounds a disk in D . By incompressibility of S , it bounds a disk in S as well, and as in Step 1 we may isotope it away.

So suppose A' meets D . Then ∂D intersects A' nontrivially. A component of intersection of $\partial D \cap A'$ cannot be an arc from one boundary component of A' to another, because such an arc bounds a disk in A' ; we may isotope away an innermost such arc in a level-preserving manner. A component of $\partial D \cap A'$ could be an essential simple closed curve in A' . Then, after isotopy, we may assume D meets A in an essential simple closed curve. But an innermost such curve bounds a disk on D , and as above we may isotope it away. So a component of $\partial D \cap A'$ must be an essential arc from one side of $\partial A'$ to the other. Then D cuts A' into disks, and after level-preserving isotopy, cuts the solid torus into balls, each embedded in a trivial one-tangle. Again intersections with K give boundary compression disks for S' , a contradiction.

Step 4. So the only remaining case is that $S \cap B_{i,j}$ consists of disk components, each with boundary encircling two punctures on K on $S_{i,j}$.

Let E be a disk component of $S \cap B_{i,j}$. If E is disjoint from K , then it must separate the two strands of K and we are done. So suppose E is not disjoint from K . Then consider its intersections with the disk D that divides $B_{i,j}$ into one-tangles B_1 and B_2 . If E is disjoint from D , then E lies entirely in one of the tangles B_1 or B_2 , say B_1 . Because ∂E encircles two punctures of K on $S_{i,j}$, it must run parallel to ∂D . Then if K intersects E , it must do so in two points in B_1 . Then we may form a vertical boundary compressing disk for S by connecting adjacent points of intersection. Pulling back via φ^{-1} to K' gives a vertical boundary compressing disk for S' , and this contradicts Lemma 3.6. So E cannot be disjoint from D .

Now $D \cap E$ consists of arcs and closed curves of intersection. Suppose first there is a closed curve of intersection $D \cap E$. Then there is one which is innermost in D . This curve bounds a disk $D' \subset D$, and hence by incompressibility of S bounds a disk in S . Again as in Step 1 we may isotope away this intersection in a level-preserving manner. So we may assume there are no closed curves of intersection, and thus $D \cap E$ consists

only of arcs with both endpoints on $S_{i,j}$. Then intersections of S with B_1 and B_2 are disks whose boundary consist of arcs of $D \cap E$ and arcs on $S_{i,j}$. The disks cut B_1 and B_2 into balls. Any such ball must contain an arc of K , else we can use it to form an isotopy removing an intersection of $S \cap D$, and as above we may ensure the isotopy is horizontal with respect to bridge spheres, i.e. preserves the conclusions of Lemma 3.2.

Again, no disk of $E \cap B_1$ meets K more than once, or as before we can obtain a vertical compression disk for S' , contradicting Lemma 3.6. Similarly for $E \cap B_2$. Thus, for each $\ell = 1, 2$, each disk of intersection of $E \cap B_\ell$ meets K exactly once, with boundary consisting of an arc on D and an arc on $S_{i,j}$. It follows that E is a disk meeting K exactly twice, meeting D in a single arc, and $S_{i,j} \cap E$ separates the four punctures of $S_{i,j}$ into pairs.

As before, we will slide intersections of S with $B_{i,j}$ up or down, reducing the number of such intersections. However, we need to take some care to ensure this can be done while preserving the conclusions of Lemma 3.2. Assume the row of twist regions runs from height a to b , i.e. the horizontal bridge sphere Σ_a lies just below the twist regions, and the horizontal bridge sphere Σ_b lies just above.

Let $t \in [a, b]$ be the minimal value such that $\Sigma_t \cap S \cap S_{i,j} \neq \emptyset$. If $t \neq a$, then by stretching and compressing vertically, there is an isotopy of (S^3, K) taking the sub-manifold $\Sigma \times (a, t)$ to the sub-manifold $\Sigma \times (a - \varepsilon, a)$, keeping Σ_b fixed. The isotopy preserves the conclusions of Lemma 3.2, while taking the lowest intersection point $S \cap S_{i,j}$ to a point at the very bottom of some ball $B_{i,j}$. Then $S \cap B_{i,j}$ contains a disk E' with boundary an arc on $S_{i,j}$ and an arc on D , meeting Σ_a . The arc on D bounds a disk D' with boundary on $S_{i,j}$ and on D meeting Σ_a . The union $D' \cup E'$ is a disk with boundary on $S_{i,j}$; it bounds a disk F on $S_{i,j}$ meeting a single puncture of $S_{i,j}$. Then $F \cup D' \cup E'$ is a sphere, bounding a ball B in S^3 with the property that $B \cap \Sigma_t$ is either empty, or is a disk whose boundary contains a single arc α on S , and an arc β on $S_{i,j}$ and D . Use this ball to isotope S horizontally to D , and then slightly past, removing an arc of intersection with D . Now we are in previous Step 2, and the result follows. \square

We now use the previous lemmas to show that $\varphi(S')$ does not meet twist regions.

Proposition 3.7. *Let K' and K be two $2m$ -plat projections of the same knot or link $\mathcal{K} \subset S^3$, where $m \geq 3$ and the length of K' is greater than $4m(m-2)$. Assume K' is 3-highly twisted. Then there exists an isotopy $\psi : (S^3, K) \rightarrow (S^3, K)$ so that if S' is any almost vertical 2-sphere for K' its image $S = (\psi \circ \varphi)(S')$ does not meet any twist region.*

Proof. Let $\varphi : (S^3, K') \rightarrow (S^3, K)$ be an isotopy satisfying the conclusions of Lemmas 3.2 and 3.3. Then $S = \varphi(S')$ meets each horizontal level Σ_t in K in a single simple closed curve, S intersects K outside of twist regions, and if S does intersect a twist region, it runs between the two strands of the twist region.

Suppose that S runs through a twist region. Let Σ_c and Σ_d be horizontal bridge spheres just below and above the twist region, respectively. The region between Σ_c and Σ_d is homeomorphic to $\Sigma \times [c, d]$, where Σ is a $2m$ -punctured disk. Because S does

not meet K in twist regions, we can choose c and d so that S does not meet K in the region $\Sigma \times [c, d]$. There is an isotopy ζ from Σ_c to Σ_d given by sweeping through the horizontal bridge spheres. The isotopy can be arranged to be the identity away from disks bounding points of K in a twist region. It is a nontrivial Dehn twist, twisting an amount equal to the number of crossings in the twist region, inside a disk bounding points of K in a twist region. In particular, the isotopy ζ takes the simple closed curve $\gamma = S \cap \Sigma_c$ in $\Sigma \times \{c\}$ to a distinct curve $\tau(\zeta)$ in $\Sigma \times \{d\}$, given by a Dehn twist.

Now consider the effect of φ^{-1} on $\Sigma \times [c, d]$. The isotopy takes this to a region $\Sigma' \times [a, b]$, with Σ'_a the image of Σ_c , and Σ'_b the image of Σ_d . The portion of the surface S between levels c and d is taken to a portion of the surface S' between levels a and b . Note that the surface S' does not meet any twist regions, and moreover, since S avoids K in this region, S' must avoid K' in this region.

But again, there is an isotopy ζ' from Σ'_a to Σ'_b given by sweeping through horizontal bridge spheres, and again this isotopy can be taken to be the identity away from twist regions. Because S' does not meet any twist regions, it must take the curve $\gamma' = S' \cap \Sigma'_a$ on $\Sigma' \times \{a\}$ to the same curve $\zeta'(\gamma') = \gamma'$ on $\Sigma' \times \{b\}$.

On the other hand, we know

$$\tau(\gamma) = \varphi \circ \zeta' \circ \varphi^{-1}(\gamma) = \varphi \circ \zeta'(\gamma') = \varphi(\gamma') = \gamma.$$

But this implies that the Dehn twist τ is trivial. This contradicts the fact that K is 1-highly twisted. \square

Remark 3.8. Note that the argument in the previous lemma uses in a crucial way the fact that the diagram of K' is fixed, and the twist regions of the diagram are all reduced. In general, an isotopy between bridge surfaces in an arbitrary diagram could first spin around a twist region, then spin in the opposite direction, undoing the twists in that twist region. However, this would result in a twist region that is not reduced. The map ζ' above is completely determined by the reduced diagram, and cannot create crossings that can be removed in a different level.

Remark 3.9. In the previous lemmas, we have obtained isotopies giving nice properties for K and S . From now on, we will compose our initial isotopy $\varphi : (S^3, K') \rightarrow (S^3, K)$ with these isotopies, and denote the composition by φ . Thus we assume that φ satisfies the conclusions of previous lemmas.

Corollary 3.10. *If S' is a vertical sphere, and $S = \varphi(S')$, then the intersection of S with the projection plane P contains precisely one simple closed curve δ so that $\delta \cap K \neq \emptyset$ and contains all n points of intersection with K .*

Proof. We can assume that the knot or link K is contained in an $\varepsilon/2$ neighborhood of P , i.e. $K \subset P \times [-\varepsilon/2, \varepsilon/2]$, and that S intersects P transversally. If there is more than one simple closed curve in the intersection $S \cap P$ which meets K then consider the annuli $S \cap (P \times [-\varepsilon, \varepsilon])$. These annuli can be capped off by disks at the $P \times -\varepsilon$ and $P \times \varepsilon$ level which do not meet K . This fact is preserved by the inverse isotopy φ^{-1} which sends the disks to disks in $S^3 \setminus \mathcal{N}(K')$. One of these disks must be essential

since it separates intersection points with K . Thus it is a compressing disk for S' , in contradiction to Theorem 2.10. \square

Lemma 3.11. *Let K', K be two $2m$ -plat projections of the same knot or link $\mathcal{K} \subset S^3$, where $m > 3$ and the length n of K' is greater than $4m(m-2)$. Moreover, assume that K' is 3-highly twisted and K is 1-highly twisted. Let $\varphi : (S^3, K') \rightarrow (S^3, K)$ be an isotopy (satisfying conclusions of isotopies above). Let S' be an almost vertical 2-sphere for K' , and let $S = \varphi(S')$ be its image. Then S meets the diagonal segments of K in any level between twist regions at most once. In particular, S cannot meet a segment of K in a point on the outside of an odd level when it has an additional intersection point on either of the two levels that the outside arc spans.*

Proof. First, by Lemma 3.2, we may assume that each horizontal bridge sphere for K meets S in a single essential curve, and each such curve meets K at most once. Also by that lemma, we may assume that the isotopy φ takes horizontal bridge spheres for K' to horizontal bridge spheres for K .

Consider now the diagonal segments of K between the i -th and $(i+1)$ -st row of twist regions, and assume in contradiction that S intersects K more than once in this region. Let p_1 and p_2 be two points of intersection. Then p_1 lies on a horizontal bridge sphere Σ_u , and p_2 lies on a horizontal bridge sphere Σ_w . We may assume $u > w$, and without loss of generality, we may assume that the difference $|u - w|$ is minimal among all such points and bridge spheres. That is, assume there are no points of intersection of $S \cap K$ between Σ_u and Σ_w .

Now consider the preimage of the region between Σ_u and Σ_w under the map φ . Both Σ_u and Σ_w are mapped to horizontal bridge spheres under φ^{-1} , and the region $\Sigma \times [u, w]$ between them is mapped to a region $\varphi^{-1}(\Sigma \times [u, w])$ foliated by horizontal bridge spheres. Since $\varphi^{-1}(\Sigma_u)$ and $\varphi^{-1}(\Sigma_w)$ meet $S' \cap K'$, they must lie on different levels, separated by twist regions. Then the isotopy given by the horizontal foliation of $\varphi^{-1}(\Sigma \times [u, w])$, coming from the fixed, reduced diagram of K' , gives a nontrivial homeomorphism from $\varphi^{-1}(\Sigma_u)$ to $\varphi^{-1}(\Sigma_w)$; it is nontrivial because moving through the twist regions performs Dehn twists on the bridge sphere. Also note that sweeping above or below the interval $[u, v]$ does not cancel these Dehn twists, as the diagram of K' is reduced.

On the other hand, the isotopy given by the horizontal foliation of $\Sigma \times [u, w]$ in K , coming from the diagram of K , just gives the identity homeomorphism between Σ_u and Σ_w , since p_1 and p_2 lie in the same region, so there are no twist regions between Σ_u and Σ_w . But this is a contradiction: the homeomorphism must be the same in both $S^3 - K$ and $S^3 - K'$.

Finally, note that if S meets a segment of K on the outside of an odd level, then we may isotope S vertically, preserving the conclusions of Lemmas 3.2 and 3.3, to meet either the diagonal segment above the odd row or the one below, so the above argument again gives a contradiction if S meets either of these levels elsewhere. \square

4. IMAGES OF PLATS

4.1. Images of vertical 2-spheres. In this subsection prove the following theorem, which allows us to control the images of vertical 2-spheres.

Theorem 4.1. *Let K', K be $2m$ -plat projections of the same knot or link \mathcal{K} . Assume $m \geq 3$, the length n' of K' satisfies $n' > 4m(m-2)$, and K' is 3-highly twisted, then the isotopy map $\varphi : (S^3, K') \rightarrow (S^3, K)$ takes all vertical 2-spheres $S' \subset (S^3, K')$ to vertical 2-spheres $S \subset (S^3, K)$ simultaneously.*

In order to prove the theorem, we will be discussing different regions of the plat diagram, and how arcs meet these regions. For ease of reference, we give regions of the diagram the following labels.

Definition 4.2. Let $K \subset S^3$ be a knot or link with a $2m$ -plat projection. Enclose every twist region in a box which we also denote by $t_{i,j} = (\text{twist region}) \times [-\varepsilon, \varepsilon]$. Then $P \setminus (K \cup (\bigcup_{i,j} t_{i,j}))$ is a collection of regions, as in Figure 2.

The regions of the graph come in four different types up to symmetry:

- (1) A unique unbounded region, denoted by \mathcal{U} .
- (2) Regions with four edges that are segments of K . Such a region will be called a *generic region*, denoted by \mathcal{Q} .
- (3) Regions with three edges that are segments of K . They appear on the top or bottom or on the left-most or right-most sides of the plat, but not in a corner. These regions will be called triangular and denoted by \mathcal{T}^t , \mathcal{T}^b , \mathcal{T}^r , and \mathcal{T}^l respectively.
- (4) Four bigon regions which appear in the corners and are denoted $\mathcal{B}^{t,l}$, $\mathcal{B}^{t,r}$, $\mathcal{B}^{b,l}$ and $\mathcal{B}^{b,r}$.

Let Δ be the portion of the projection plane P outside the unbounded region. That is, $\Delta = P \setminus \mathcal{U}$.

Lemma 4.3. *The image S of a vertical 2-sphere S' must intersect each row of twist regions so that it has at least one twist region on each side in that row.*

Proof. By Lemma 3.2, S intersects horizontal bridge spheres in each row of twist regions. By Proposition 3.7, it avoids all twist regions. By Corollary 3.10, S meets P in a single simple closed curve which, by Lemma 3.11, meets diagonals on each level no more than once. Note this implies that S consists of exactly one arc in \mathcal{U} and one arc in Δ , for otherwise S would exit Δ by crossing an arc of the diagram on the outside of an odd level, as well as one of the two diagonal levels spanned by that outside arc, contradicting Lemma 3.11.

Assume that the image S does not have one twist region on both sides of the vertical arc $\alpha = S \cap \Delta$ on some level. Without loss of generality we can assume that there is no

twist region to the right side of the arc α on that level. If the level is even, then there is a disk with boundary on S contained in the unbounded face. If such a disk exists, then its image under φ^{-1} is a disk on a horizontal bridge sphere Σ'_t for some $t \in [0, 1]$, and boundary on S' . But for all $t \in [0, 1]$, $S' \cap \Sigma'_t$ bounds points of K' on both sides. So this cannot happen.

It remains to consider an odd level such that the arc α has no twist regions on the right side in this level. The only way this can happen is if the arc α passes through some region \mathcal{T}^r , or one of the four \mathcal{B} 's, as in Definition 4.2. Suppose first that α meets \mathcal{B} . Then α must meet the segment of K forming a bridge, and the segment of K between twist regions. But we may isotope S along the bridge, preserving the conclusions of Lemmas 3.2 and 3.3 and Proposition 3.7, moving the intersection on the bridge to lie on the level between the first and second rows of twist regions. Then we obtain a contradiction to Lemma 3.11. So α meets a region \mathcal{T}^r .

Let $\beta \subset K$ denote the rightmost arc in \mathcal{T}^r . Let T_r denote the tangle in the component of $S^3 \setminus S$ which contains β . The tangle T_r contains a *cutting point* c : a point c on the sub-arc β so that there is a disk D , meeting c , embedded in T_r that meets K in a single point. Moreover, we can choose the disk D to lie in a horizontal bridge sphere for K , so that the boundary of D is the intersection of that horizontal bridge sphere with S .

Now consider the preimage S' of the sphere S . Since horizontal bridge spheres go to horizontal bridge spheres, the preimage D' of D is a disk on a horizontal bridge sphere for K' with $\partial D'$ on S' . There is no such disk which intersects K' in a single point. \square

Proposition 4.4. *Let K' and K be two $2m$ -plat projections of the same knot or link $\mathcal{K} \subset S^3$ such that $m \geq 3$ and K' is 3-highly twisted, and K is 1-highly twisted. If n, n' are the respective lengths of the plats K and K' , with n' satisfying $n' > 4m(m-2)$, then $n = n'$.*

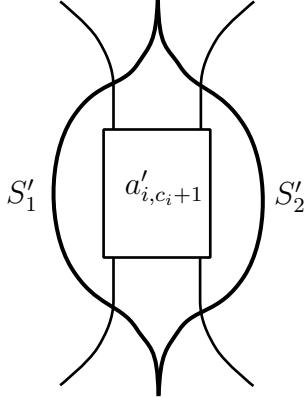
Proof. Let $\varphi : (S^3, K') \rightarrow (S^3, K)$ be an isotopy between K' and K . Let $S' \subset K'$ be a vertical 2-sphere for K' . Let $S = \varphi(S')$ be its image under φ . Note that S intersects K in n' points.

If $n < n'$ then by the pigeonhole principle, there is some level where S must intersect K more than once. This contradicts Lemma 3.11. So $n \geq n'$.

On the other hand, if $n > n'$, consider a vertical 2-sphere S' in K' . By Lemma 4.3, its image S has at least one twist region on both sides on each level. However, $n > n'$ means there are levels it cannot reach. Thus $n = n'$. \square

Proof of Theorem 4.1. Denote the collection of vertical spheres in K' by $\{\mathcal{S}'\}$. It follows from Corollary 3.10 that for each vertical sphere $S'_u \subset \mathcal{S}'$ the intersection $\varphi(S'_u)$ with the projection plane P is a single simple closed curve. Proposition 3.7 shows that the n points of intersection of $\varphi(S'_u)$ and K are contained only in the arcs of $K \setminus (\bigcup_{i,j} \{t_{i,j}\})$.

Lemma 4.3 shows that there is at least one twist region on each side of $\varphi(S'_u)$ at each row. Hence $\{\varphi(S')\}$ is a collection of vertical 2-spheres in (S^3, K) . \square

FIGURE 5. An isolating 2-sphere Ω'

4.2. Preserving the order. As in Section 3, let $S'_1, S'_2 \subset (S^3, K')$ be two vertical spheres which differ by one twist region on the i -th row. Namely, without loss of generality $S'_1 = S'_1(c_1, \dots, c_i, \dots, c_{n-1})$ and $S'_2 = S'_2(c_1, \dots, c_i+1, \dots, c_{n-1})$. We assume that $1 < c_i < m$ when i is even and $1 < c_i < m-1$ when i is odd so that the spheres are well defined. Thus the bounded component B' of $S^3 \setminus (S'_1 \cup S'_2)$ is a rational 2-tangle as in Figure 5 containing the t'_{i,c_i+1} twist region only.

We first start with notation. Let S'_1 and S'_2 be as above. Then $S'_1 \cap S'_2$ is a disk Δ' , and $S'_1 \setminus \Delta'$ is a disk Δ'_1 meeting K' in two points. Similarly, $S'_2 \setminus \Delta'$ is a disk Δ'_2 , and the union $\Delta'_1 \cup \Delta'_2$ is a 2-sphere Ω' meeting K' four times as in Figure 5. We call Ω' an *isolating 2-sphere*.

Definition 4.5. Twist regions that can be isolated by $\Omega' = \Delta'_1 \cup \Delta'_2$ for Δ'_1, Δ'_2 coming from vertical 2-spheres will be called *allowable twist regions*. Such twist regions are inside the diagram, not on the far left or far right. Twist regions on the far left or far right but not in one of the corners will be called *almost allowable twist regions*. That is, almost allowable twist regions are those $t'_{i,j} \in K'$ such that either i is odd with $3 \leq i \leq n-3$ and $j \in \{1, m-1\}$ or i is even with $4 \leq i \leq n-4$ and $j \in \{1, m\}$.

The only twist regions which are not allowable or almost allowable are the extreme right and left twist regions in rows 1, 2, $n-2$ and $n-3$. These twist regions are called *extreme twist regions*.

We first treat allowable twist regions.

Lemma 4.6. *Let $t'_{i,j} \in K'$ be an allowable twist region and let S'_1 and S'_2 be its corresponding vertical 2-spheres. The bounded component of $S^3 \setminus (\varphi(S'_1) \cup \varphi(S'_2))$ in (S^3, K) is a rational 2-tangle as in Figure 5 containing a single twist region $t_{r,s}$.*

Proof. By Theorem 4.1 the images $S_1 = \varphi(S'_1)$ and $S_2 = \varphi(S'_2)$ are vertical 2-spheres, and by Corollary 3.10 each meets the projection plane P in a single simple closed curve component δ_ℓ , for $\ell = 1, 2$ respectively. Since S'_1 and S'_2 are identical except in a

neighborhood of the i -th row, it follows that S_2 is identical to S_1 everywhere except in a neighborhood of some r -th row.

Let Δ be the disk in S_1 and S_2 on which S_1 and S_2 agree. Let Δ_1 be $S_1 \setminus \Delta$, and let Δ_2 be $S_2 \setminus \Delta$. Then $\Delta_1 \cup \Delta_2$ is a 2-sphere Ω . Note that the union of the arcs $\delta_1 \setminus (\delta_1 \cap \delta_2)$ and $\delta_2 \setminus (\delta_1 \cap \delta_2)$ is the curve $\Omega \cap P$. Thus the intersection $\Omega \cap P$ contains a single curve which intersects K in exactly 4 points, with the top of $\Omega \cap P$ just above the r -th row and the bottom just below the r -th row. The only 2-sphere Ω with this property is as in Figure 5. That is, Ω bounds a twist region in K . So

$$S_2(c_1, \dots, c_r, \dots, c_{n-1}) = S_1(c_1, \dots, c_r \pm 1, \dots, c_{n-1}).$$

Thus $S_1 = \varphi(S'_1)$ and $S_2 = \varphi(S'_2)$ are vertical 2-spheres which differ by the image under φ of the 2-tangle corresponding to the twist region. \square

Lemma 4.7. *If $t'_{i,j}$ is an allowable twist region in (S^3, K') that is mapped to $t_{r,s}$ in (S^3, K) then the number of crossings in the twist regions agree: $a_{r,s} = a'_{i,j}$.*

Proof. Since φ is an isotopy, it will map a rational tangle $1/a'_{i,j}$ corresponding to $t'_{i,j}$ to a rational tangle $1/a_{r,s}$ corresponding to $t_{r,s}$. The only way that the two tangles might disagree is if the two strings of the tangle are twisted about each other during the isotopy. However such a twisting will induce a homeomorphism of the four punctured sphere Ω' that fixes the punctures. Such a homeomorphism will force the images $\varphi(\Delta'_1)$ and $\varphi(\Delta'_2)$ to intersect P in more than in one component, and thus S_1 and S_2 will not be vertical spheres, in contradiction to Theorem 4.1. \square

Lemma 4.8. *Let $t'_{i,j}$ be an allowable twist region in K' . Then the allowable twist regions adjacent to $t'_{i,j}$ in K' are mapped to twist regions adjacent to $\varphi(t'_{i,j}) = t_{r,s}$ in (S^3, K) , in the same order, up to rotation along a vertical or horizontal axis through the $t_{r,s}$ twist region.*

Proof. By the definition of an allowable twist region, there are two vertical 2-spheres which isolate the twist region $t'_{i,j}$, as in Lemma 4.6. We know by Lemma 4.7 that the twist region $t_{r,s}$ in K has the same number of crossings as $t'_{i,j}$, and is isolated by images of the two vertical 2-spheres. Given $t'_{i,j} \in K'$, each allowable adjacent twist region (at most four in the general case) shares a segment of K' . This segment intersects only one of the vertical 2-spheres used in the isolation of $t'_{i,j}$.

This 2-sphere is shared by the twist region adjacent to $t'_{i,j}$, so it must be mapped to the 2-sphere separating the images of these adjacent twist regions in (S^3, K) . Thus adjacent allowable twist regions are mapped to adjacent allowable twist regions. Since twist regions are symmetric with respect to reflections in the vertical and horizontal axis, so is the relative position of the corresponding twist regions in K ; see Figure 6. \square

4.3. Almost allowable twist regions. Consider now almost allowable twist regions in row i . In this case at least one of the spheres that will be used to define the isolating sphere for the twist region will not be a vertical 2-sphere. However, the spheres will be almost vertical. Since we know that almost vertical 2-spheres do not go through twist

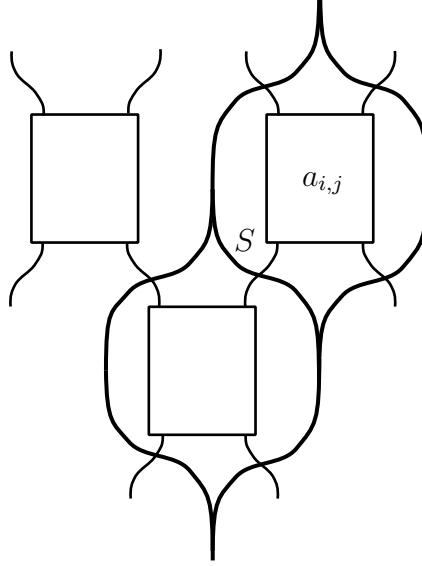


FIGURE 6. Two adjacent twist regions with a shared vertical 2-sphere S

regions by Proposition 3.7, the argument is similar to that of allowable twist regions, although there is a difference between the cases when i is odd or even.

For ease of notation, we will consider almost allowable twist regions on the left side of the diagram in this subsection. By symmetry, the results also hold on the right.

When i is odd and $3 \leq i \leq n - 3$, consider a vertical 2-sphere

$$S'_\ell = S'(1, \dots, 1, 1, 1, \dots, 1),$$

and an almost vertical 2-sphere W'_i defined exactly as the vertical sphere, but with coefficients $(1, 0, 1)$ in the $(i - 1)$ -st, i -th, and $(i + 1)$ -st locations. That is, write

$$W'_i = S'(1, \dots, 1, 0, 1, \dots, 1),$$

where the coefficients of W'_i and S'_ℓ are identical aside from the i -th coefficient. Note that W'_i is an odd almost vertical 2-sphere as in Definition 2.9.

As before the intersection $S'_\ell \cap W'_i$ is a disk Δ' and the union of disks $\Delta'_1 = S'_\ell \setminus \text{int}(\Delta')$ and $\Delta'_2 = W'_i \setminus \text{int}(\Delta')$ is a 2-sphere $\Omega'_i = \Delta'_1 \cup \Delta'_2$ that intersects K' in four points; see Figure 7. This is an isolating 2-sphere for the twist region $t'_{i,1}$. As $\partial\Delta' \cap K = \emptyset$, the image

$$\varphi(\Omega'_i) = \varphi(\Delta'_1 \cup \Delta'_2) = \varphi(\Delta'_1) \cup \varphi(\Delta'_2) := \Omega_i$$

is a 2-sphere which intersects K in four points and bounds a 2-tangle in $S^3 \setminus \mathcal{N}(K)$. Choose Ω_i to minimize intersections with the projection plane up to isotopy rel K .

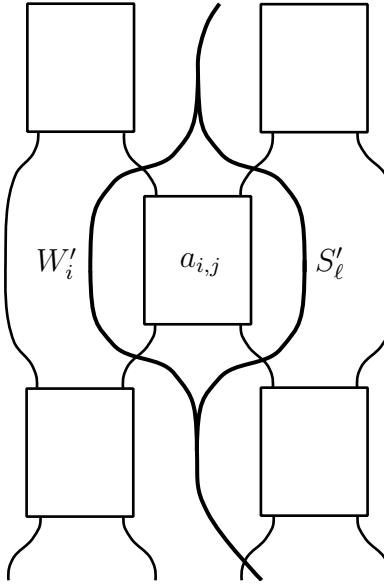
When i is even and $4 \leq i \leq n - 4$, consider the doubly odd almost vertical 2-sphere

$$W_i'^1 = S'(1, \dots, 1, 0, 1, 0, 1, \dots, 1),$$

and the even almost vertical 2-sphere

$$W_i'^2 = S'(1, \dots, 1, 0, 0, 0, 1, \dots, 1).$$

Thus the coefficients of $W_i'^1$ and $W_i'^2$ are identical except in the i -th position.

FIGURE 7. An almost allowable twist region with S'_ℓ

The intersection $W'_i \cap W''_i$ is a disk Δ' and the union of the disks $\Delta'_1 = W'_i \setminus \text{int}(\Delta')$ and $\Delta'_2 = W''_i \setminus \text{int}(\Delta')$ is a 2-sphere $\Omega'_i = \Delta'_1 \cup \Delta'_2$ that intersects K' in four points. This is an isolating 2-sphere for the twist region $t'_{i,1}$. Let Ω_i denote the image $\varphi(\Omega'_i)$; it is a 2-sphere which intersects K in four points. Choose Ω_i to minimize the intersection with the projection plane in its isotopy class rel K .

We have thus defined an isolating 2-sphere Ω'_i in K' , and its image Ω_i in K for i even and odd, for $3 \leq i \leq n-3$. In both cases, we want to show that Ω_i is an isolating 2-sphere for the twist region $t_{r,1}$.

Lemma 4.9. *Let $\Omega'_i \subset (S^3, K')$ be an isolating sphere for an almost allowable twist region on the left, $t'_{i,1} \subset K'$. Then the bounded component of $S^3 \setminus \Omega_i \subset (S^3, K)$ is a rational 2-tangle, as in Figure 5, containing only the twist region $t_{r,1}$ so that $a_{r,1} = a'_{i,1}$. A similar statement holds for almost allowable twist regions on the right. Moreover, adjacent twist regions in K' , both allowable and almost allowable, are mapped to adjacent twist regions in K .*

Proof. Recall from Lemmas 3.2, 3.3, Proposition 3.7, and Lemma 3.11 that W'_i , W''_i , and W'_i meet each horizontal bridge sphere in a simple closed curve, avoid all twist regions of K , and meet each level between twist regions at most once.

The proof is separated into two cases depending on whether i is odd or even.

Case (1): i is odd. Since S'_ℓ is a vertical 2-sphere its image $\varphi(S'_\ell)$ is also a vertical 2-sphere. Hence the image of Δ'_1 is a disk Δ_1 , and the intersection of Δ_1 with the projection plane P is an arc η that contains the two points of intersection of Δ_1 with K on P .

Furthermore, as $\partial\varphi(\Delta'_1) = \partial\varphi(\Delta'_2)$, the image of the disk Δ'_2 is a disk Δ_2 such that $\partial\Delta_2 = \partial\Delta_1$, and Δ_2 meets K twice. Since we know that Δ_2 avoids twist regions, as above we can conclude that Ω is an isolating 2-sphere for the twist region $t_{r,1}$ in K .

The same arguments as in Lemmas 4.7 and 4.8 show that $a_{i,1} = a'_{i,1}$ and that odd almost allowable twist regions in K appear in the same order relative to the rest of the plat as those of K' . Then it follows that for i odd with $3 \leq i \leq n-3$, the image $\varphi(W'_i)$ is isotopic to $S(1, \dots, 1, c_i = 0, 1, \dots, 1)$ in K .

Case (2): i is even. For even i with $4 \leq i \leq n-4$, we know that $\varphi(W'_{i-1})$ is isotopic to $S(1, \dots, 1, c_{i-1} = 0, 1, \dots, 1)$ and $\varphi(W'_{i+1})$ is isotopic to $S(1, \dots, 1, c_{i+1} = 0, 1, \dots, 1)$. Notice that this implies that the image of

$$W_i^{1'} = S'(1, \dots, 1, c_{i-1} = 0, 1, c_{i+1} = 0, 1, \dots, 1)$$

is isotopic to

$$W_i^1 = S(1, \dots, 1, c_{i-1} = 0, 1, c_{i+1} = 0, 1, \dots, 1).$$

Again we determine the image of the spheres $W_i^{1'} = S'(1, \dots, 1, 0, c_i = 0, 0, 1, \dots, 1)$. We know the disk $\Delta' = W_i^{1'} \cap W_i^{1'}$ is mapped to a disk Δ that agrees with W_i^1 everywhere except on the disk $\varphi(\Delta'_2) = \Delta_2$ that meets K twice, but as before $\partial\Delta_2$ lies on W_i^1 . Also, Δ contains all the other $n-2$ intersection points of $\varphi(W_i^{1'})$ with K in consecutive order, aside from skipping two intersections at the levels of $\varphi(\Delta'_1)$ (where recall $\Delta'_1 = W_i^{1'} \setminus \Delta'$).

Because Δ_2 meets no twist regions, the isolating sphere Ω'_i has image an isolating sphere Ω_i , isolating the twist region $t_{i,1}$ for i even and $4 \leq i \leq n-4$. The same arguments as in Lemma 4.7 and Lemma 4.8 show that $a_{i,1} = a'_{i,1}$ and that almost allowable twist regions in K appear in the same order relative to the rest of the plat as those of K' . \square

4.4. Extreme twist regions. We still need to show that the extreme twist regions are fixed as well. We will define an isolating sphere for an extreme twist region on the top left corner of the K' plat and the others are defined by symmetry. Consider the intersection of $S'(1, 1, \dots, 1) \cap P'$ with the second top external region, i.e. the left-most region \mathcal{T}^t , and the generic region \mathcal{Q} on the second row counted from the left, where recall definitions of regions are in Definition 4.2. A sub-arc of this arc of intersection can be completed to a simple closed curve γ' circling both the $t'_{1,1}$ and $t'_{2,1}$ twist regions. The curve γ' is indicated in Figure 8. Cap γ' by disks above and below to obtain a 2-sphere $S_1^{\ell'}$ bounding the rational 2-tangle corresponding to the $1/(a'_{1,1} + 1/a'_{2,1})$ continued fraction expansion.

Lemma 4.10. *The 2-sphere $\varphi(S_1^{\ell'})$ bounds a rational 2-tangle corresponding to the $1/(a_{1,1} + 1/a_{2,1})$ continued fraction expansion. Thus, up to rotation, $a_{1,1} = a'_{1,1}$ and $a_{2,1} = a'_{2,1}$, and similarly for the other three corners.*

Proof. Since a sub-arc of γ' is also a sub-arc of $S'(1, \dots, 1)$, the image $\varphi(S_1^{\ell'})$ shares a sub-disk with $\varphi(S'(1, \dots, 1))$, which by Lemma 4.8 is equal to $S(1, \dots, 1)$.

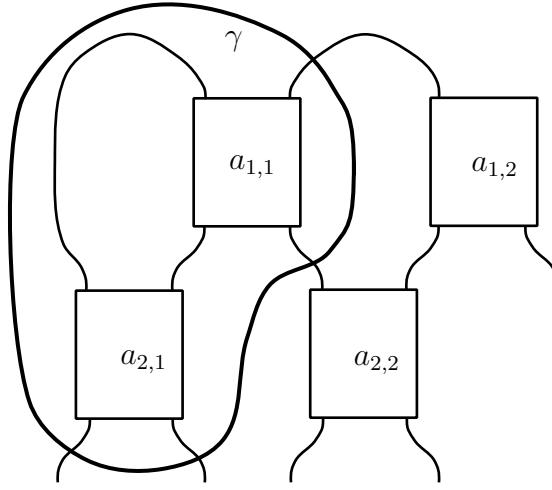


FIGURE 8. The γ curve for an isolating sphere for the extreme twist regions

It now follows from the same argument as in Lemma 4.6 that $\varphi(S_1'^\ell) \cap P$ is a single simple closed curve γ and $\varphi(S_1'^\ell) = S_1^\ell$. By Lemma 4.8 and the proof of Lemma 4.9 the points $K \cap \varphi(S_1'^\ell)$ are fixed. Thus $a_{1,1} = a'_{1,1}$ and $a_{2,1} = a'_{2,1}$, and similarly for the extreme twist regions in the other corners of the plat. \square

4.5. Proof of Theorem 1.1. We are now ready to prove the main theorem.

Proof of Theorem 1.1. Let $\mathcal{K} \subset S^3$ be a knot or link with a $2m'$ -plat projection K' that is 3-highly twisted, and length $n' > 4m'(m' - 2)$. Assume \mathcal{K} has a $2m$ -plat projection or even plat projection K as well. Then Corollary 2.8 implies $m \geq m'$. If $m' = m$, and K is 1-highly twisted, then Proposition 4.4 implies that the length n of K equals the length n' . (So in fact, K is not an even plat projection.)

It then follows from Lemmas 4.6 through 4.10 that the twist regions in both plats are equal, namely $t_{i,j} = t'_{i,j}$ up to reflection by a vertical or a horizontal axis, and $t_{i,j}$ and $t'_{i,j}$ contain the same number of (signed) crossings, i.e. $a_{i,j} = a'_{i,j}$. Thus up to reflection by a vertical or a horizontal axis, $K = K'$ and the plat is unique. \square

5. APPENDIX

In this section, we prove that vertical 2-spheres are meridionally incompressible. The proof follows very closely the proof of Wu that vertical 2-spheres are essential [26].

Lemma 5.1. *Suppose (B, T) is a rational tangle consisting of a single twist region with q crossings (i.e. a $1/q$ tangle). Let C be a circle on ∂B of slope ∞ , $X = B \setminus \mathcal{N}(T)$, and P a component of $(\partial B \cap X) \setminus C$. If $q \geq 3$ then P is meridionally incompressible in X .*

Proof. Suppose D is a meridionally compressing disk for P in X . There exists a disk E embedded in X with boundary on ∂B , separating B into two trivial one-tangles. Consider intersections of D and E . Suppose there is a closed curve component of

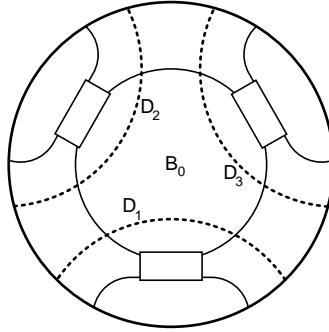


FIGURE 9. Tangle to the left of $S(1, \dots, 1)$ is equivalent to this tangle when $k = 3$; figure from [26].

$D \cap E$. Then there is an innermost such component γ on D , bounding a disk $D' \subset D$. Note that γ also bounds a disk E' on E thus $D' \cup E'$ is a sphere in B which bounds a 3-ball B' . If B' is disjoint from the tangle T , we may isotope D through B' to remove the component of intersection γ . If T is not disjoint from B' , then T must meet $\partial B' = D' \cup E'$. But E is disjoint from T , and D' can meet T only in a single meridional puncture. But then T can puncture the 2-sphere $\partial B'$ at most once. This is a contradiction. So after a finite number of isotopies, we may assume there are no closed curve components of $E \cap D$.

Now suppose $E \cap D$ consists of arc components. An outermost arc α on D bounds a disk $D' \subset D$. It also bounds a disk $E' \subset E$. Then consider $\partial D' \setminus \alpha$ and $\partial E' \setminus \alpha$. These are arcs on ∂B disjoint from T whose endpoints agree; thus they bound a disk $F' \subset \partial B$, and the union $D' \cup E' \cup F'$ is a 2-sphere in B bounding a 3-ball B' . Again if B' is disjoint from T , we may isotope E' through B' , with an arc of its boundary remaining in F' , until we isotope away the arc of intersection. So suppose B' does meet T . Since E is disjoint from T , and D' meets T at most once in a meridional puncture, it follows that T must meet F' exactly once. Because B' lies in a trivial one-tangle, T must run through B' in an unknotted arc from F' to D' . But then D' can be isotoped through B' , with an arc of its boundary remaining in E' , to be parallel to F' , removing the intersection component α . Repeat this procedure until all other arcs of intersection are removed. It follows that D is parallel to a disk on ∂B which meets exactly one puncture of T ; hence P is meridionally incompressible. \square

Lemma 5.2 (See Lemma 5 of [26]). *The vertical 2-sphere $S(1, \dots, 1)$ is meridionally incompressible.*

Proof. Consider the tangle (B, T) to the left of $S(1, \dots, 1)$, and let X denote $B \setminus \mathcal{N}(T)$. By an isotopy of (B, T) , we can untwist the twist regions of T which lie in the even rows, so the tangle (B, T) is equivalent to the one shown in Figure 9, where each twist box corresponds to the first box on an odd row of the diagram. There are k such boxes, where $k = n/2$ and n denotes the height of the plat. (Recall that n is even.)

Let D_1, \dots, D_k be disks represented by the dotted lines in Figure 9, which cut (B, T) into $k + 1$ sub-tangles $(B_0, T_0), \dots, (B_k, T_k)$, where (B_0, T_0) is the tangle in the middle,

which intersects all the D_i . Let $P_i = D_i \cap X$ be the twice punctured disk corresponding to D_i .

Now suppose D is a meridional compressing disk for $S(1, \dots, 1)$; we may assume the meridian meeting D lies completely in some B_i . Consider components $P_i \cap D$, and first suppose $i \geq 1$. Any innermost curve or outermost arc of $D \cap P_i$ in D bounds a disk D' in D . If D' is disjoint from the tangle, then because P_i is incompressible and boundary incompressible in $B_i - \mathcal{N}(T_i)$ (by [26, Lemma 3]) and in $B_0 - \mathcal{N}(T_0)$ (by [26, Lemma 5]), such a curve of intersection can be slid off. Repeat a finite number of times to remove all such intersections. Any remaining innermost disk or outermost arc of intersection with P_i must bound a meridional disk in D . But Lemma 5.1 implies P_i is meridionally incompressible in X_i , hence the portion of D in B_i can be isotoped through B_i to the boundary, and we may remove this intersection.

It follows that D must have its boundary completely contained in $\partial B_0 - \cup D_i = Q$. But since D meets T_0 in a single meridian, ∂D must encircle one component of T_0 on one side and $2k+n-1$ components, an odd number, on the other side. Since the disks D_i meet endpoints of T_0 in pairs, this is impossible. \square

Lemma 5.3 (See Lemma 6 of [26]). *Any vertical 2-sphere $S(c_1, \dots, c_{n-1})$ is meridionally incompressible.*

Proof. There is a sequence of vertical spheres S_1, S_2, \dots, S_{r+1} so that $S_1 = S(1, \dots, 1)$, $S_{r+1} = S(c_1, \dots, c_{n-1})$, and S_i and S_{i+1} differ by a single twist region in the diagram. That is, $S_i \cup S_{i+1} - \text{int}(S_i \cap S_{i+1}) = \partial B_i$ for some tangle (B_i, T_i) bounding a single twist region with $a \neq 0$ twists. By Lemma 5.2, we know S_1 is meridionally incompressible. The proof will be by induction on the length of the sequence. Assume, therefore, that S_r is meridionally incompressible.

Now suppose that D is a meridian compressing disk for S_{r+1} . The disk D must meet the tangle (B_r, T_r) , else it would be a meridionally compressing disk for S_r , which is impossible. Let $P_r = \partial B_r \cap S_r$; this is a twice punctured disk on ∂B_r . Consider curves and arcs of intersection of D and P_r . Innermost curves and outermost arcs bounding disks in D can be isotoped away, using the incompressibility and boundary incompressibility of P_r in B_r , and incompressibility of S_r .

Any innermost curve or arc of intersection therefore bounds a disk of D meeting the meridian of the link. But P_r is meridionally incompressible by Lemma 5.1, hence this intersection can be isotoped outside of P_r . It follows that D is disjoint from P_r , but meets the tangle (B_r, T_r) . This is impossible unless D is completely contained in (B_r, T_r) . But in that case, D is a meridional compressing disk for $P_{r+1} = \partial B_r \setminus P_r$, contradicting Lemma 5.1. \square

REFERENCES

1. David Bachman and Saul Schleimer, *Distance and bridge position*, Pacific J. Math. **219** (2005), no. 2, 221–235.
2. Ryan Blair, David Futer, and Maggy Tomova, *Essential surfaces in highly twisted link complements*, Algebr. Geom. Topol. **15** (2015), no. 3, 1501–1523.

3. Steven A. Bleiler and Yoav Moriah, *Heegaard splittings and branched coverings of B^3* , Math. Ann. **281** (1988), no. 4, 531–543.
4. Gerhard Burde and Heiner Zieschang, *Knots*, de Gruyter Studies in Mathematics, vol. 5, Walter de Gruyter & Co., Berlin, 1985.
5. Alexander Coward and Marc Lackenby, *An upper bound on Reidemeister moves*, Amer. J. Math. **136** (2014), no. 4, 1023–1066.
6. Elizabeth Finkelstein and Yoav Moriah, *Tubed incompressible surfaces in knot and link complements*, Topology Appl. **96** (1999), no. 2, 153–170.
7. ———, *Closed incompressible surfaces in knot complements*, Trans. Amer. Math. Soc. **352** (2000), no. 2, 655–677.
8. David Futer, Efstratia Kalfagianni, and Jessica S. Purcell, *Dehn filling, volume, and the Jones polynomial*, J. Differential Geom. **78** (2008), no. 3, 429–464.
9. ———, *Guts of surfaces and the colored Jones polynomial*, Lecture Notes in Mathematics, vol. 2069, Springer, Heidelberg, 2013.
10. ———, *Quasifuchsian state surfaces*, Trans. Amer. Math. Soc. **366** (2014), no. 8, 4323–4343.
11. ———, *Hyperbolic semi-adequate links*, Comm. Anal. Geom. **23** (2015), no. 5, 993–1030.
12. David Futer and Jessica S. Purcell, *Links with no exceptional surgeries*, Comment. Math. Helv. **82** (2007), no. 3, 629–664.
13. Adam Giambrone, *Combinatorics of link diagrams and volume*, J. Knot Theory Ramifications **24** (2015), no. 1, 1550001, 21.
14. Jim Hoste, Morwen Thistlethwaite, and Jeff Weeks, *The first 1,701,936 knots*, Math. Intelligencer **20** (1998), no. 4, 33–48.
15. J. Johnson and Y. Moriah, *Bridge distance and plat presentations*, arXiv:1312.7093 [math.GT].
16. Jesse Johnson and Maggy Tomova, *Flipping bridge surfaces and bounds on the stable bridge number*, Algebr. Geom. Topol. **11** (2011), no. 4, 1987–2005.
17. Marc Lackenby, *A polynomial upper bound on Reidemeister moves*, Ann. of Math. (2) **182** (2015), no. 2, 491–564.
18. Martin Lustig and Yoav Moriah, *Generalized Montesinos knots, tunnels and \mathcal{N} -torsion*, Math. Ann. **295** (1993), no. 1, 167–189.
19. William Menasco and Morwen Thistlethwaite, *The classification of alternating links*, Ann. of Math. (2) **138** (1993), no. 1, 113–171.
20. Jessica S. Purcell, *Cusp shapes under cone deformation*, J. Differential Geom. **80** (2008), no. 3, 453–500.
21. ———, *An introduction to fully augmented links*, Interactions between hyperbolic geometry, quantum topology and number theory, Contemp. Math., vol. 541, Amer. Math. Soc., Providence, RI, 2011, pp. 205–220.
22. Kurt Reidemeister, *Elementare Begründung der Knotentheorie*, Abh. Math. Sem. Univ. Hamburg **5** (1927), no. 1, 24–32.
23. Horst Schubert, *Knoten mit zwei Brücken*, Math. Z. **65** (1956), 133–170.
24. William P. Thurston, *The geometry and topology of three-manifolds*, Princeton Univ. Math. Dept. Notes, 1979, Available at <http://www.msri.org/communications/books/gt3m>.
25. Maggy Tomova, *Multiple bridge surfaces restrict knot distance*, Algebr. Geom. Topol. **7** (2007), 957–1006.
26. Ying-Qing Wu, *Incompressible surfaces in link complements*, Proc. Amer. Math. Soc. **129** (2001), no. 11, 3417–3423 (electronic).

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