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Monash University

SAMPLE EXAM

FACULTY OF SCIENCE

EXAM CODE	**ENG1091**
TITLE OF PAPER	**Mathematics for Engineering**
EXAM DURATION	THREE hours writing time
READING TIME	TEN minutes

THIS PAPER IS FOR STUDENTS STUDYING AT

- [ ] Berwick
- [x] Clayton
- [x] Malaysia
- [ ] Distributed Learning
- [ ] Open Learning
- [ ] Caulfield
- [ ] Gippsland
- [ ] Peninsula
- [ ] Enhancement Studies
- [ ] Sth Africa
- [ ] Pharmacy
- [ ] Other

INSTRUCTIONS TO CANDIDATES

During an exam, you must not have in your possession, a book, notes, paper, calculator, pencil case, mobile phone or other material/item which has not been authorised for the exam or specifically permitted as noted below. Any material or item on your desk, chair or person will be deemed to be in your possession. You are reminded that possession of unauthorised materials in an exam is a discipline offence under Monash Statute 4.1.

Students should ONLY enter their ID number and desk number on the examination script book, NOT their name. Please take care to ensure that the ID number and desk number are correct and are written legibly.

No examination papers are to be removed from the room.

1. All questions may be attempted.
2. Marks for each question are given on the paper.
3. Calculators are not allowed. Leave all answers in exact form.
4. Answer all questions in the spaces provided.
5. Formulae sheets are provided in a separate booklet.

AUTHORISED MATERIALS

CALCULATORS	NO
OPEN BOOK	NO
SPECIFICALLY PERMITTED ITEMS	NO

Candidates must complete this section if required to answer in this paper

STUDENT ID __ __ __ __ __ __ __ __ DESK NUMBER __ __ __ __

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Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11

1. (a) Given $\underset{\sim}{v} = 7\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$ and $\underset{\sim}{w} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k}$ compute

(i) $\underset{\sim}{v} \times \underset{\sim}{w}$, (ii) $\underset{\sim}{v} \cdot (\underset{\sim}{v} \times \underset{\sim}{w})$, (iii) $(2\underset{\sim}{w} - 3\underset{\sim}{v}) \cdot (\underset{\sim}{w} \times \underset{\sim}{v})$

(iv) The vector projection of $\underset{\sim}{v}$ in the direction of $\underset{\sim}{w}$.

i) $\underset{\sim}{v} \times \underset{\sim}{w} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & k \\ 7 & 2 & -3 \\ 2 & -3 & 5 \end{vmatrix} = \underset{\sim}{i} - 41\underset{\sim}{j} - 25\underset{\sim}{k}$

ii) $\underset{\sim}{v} \cdot (\underset{\sim}{v} \times \underset{\sim}{w}) = 0$ (iii) $(2\underset{\sim}{w} - 3\underset{\sim}{v}) \cdot (\underset{\sim}{w} \times \underset{\sim}{v}) = 0$

iv) vec. projection $= \hat{\underset{\sim}{w}}\,(\underset{\sim}{v} \cdot \hat{\underset{\sim}{w}}) = \underset{\sim}{w}\left(\dfrac{\underset{\sim}{v} \cdot \underset{\sim}{w}}{\underset{\sim}{v} \cdot \underset{\sim}{w}}\right)$

$\underset{\sim}{v} \cdot \underset{\sim}{w} = 14 - 6 - 15 = -7$ $\underset{\sim}{w} \cdot \underset{\sim}{w} = 4 + 9 + 25 = 38$

$\Rightarrow$ vec. projection $= -\frac{7}{38}\,(2\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k})$

3 marks

(b) Construct a parametric equation for the straight line that passes through the two points $(-2, 1, 4)$ and $(3, 1, 2)$. Construct a second parametric equation for a second straight line for the points $(-7, 1, 6)$ and $(8, 1, 0)$.

1st line: $\underset{\sim}{r}_1(t) = (-2,1,4) + t\left((3,1,2) - (-2,1,4)\right)$

$= (-2,1,4) + t\,(5,0,-2)$

2nd line: $\underset{\sim}{r}_2(s) = (-7,1,6) + s\left((8,1,0) - (-7,1,6)\right)$

$= (-7,1,6) + s\,(15,0,-6)$

3 marks

(c) Show that the two lines defined in part (b) coincide (i.e. that they are one and the same line).

If the lines coincide then we must have

$$\underset{\sim}{r}_1(t) = \underset{\sim}{r}_2(s)$$

$\Rightarrow \quad (-2,1,4) + t(5,0,-2) = (-7,1,6) + s(15,0,-6)$

$x: \quad -2+5t = -7+15s \quad \Rightarrow t = -1+3s$

$y: \quad 1 = 1$

$z: \quad 4-2t = 6-6s \quad \Rightarrow \quad t = -1+3s$

i.e. no constraints on t or s

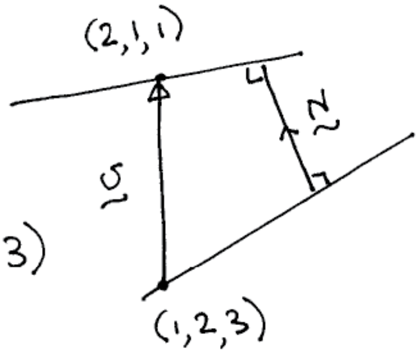
5 marks

(d) Find the shortest distance between the pair of lines defined by

Line 1 : $x(t) = 1 + 7t$, $y(t) = 2 + 2t$, $z(t) = 3 - 3t$

Line 2 : $x(s) = 2 + 2s$, $y(s) = 1 - 3s$, $z(s) = 1 + 5s$

L = scalar projection of $\underset{\sim}{v}$ in $\underset{\sim}{N}$



$= \dfrac{\underset{\sim}{v} \cdot \underset{\sim}{N}}{|N|}$ $\quad \underset{\sim}{v} = (2,1,1) - (1,2,3)$

$= (1,-1,-2)$

$\underset{\sim}{N} = (1,-41,25)$ from part (a).

$\Rightarrow L = (1,-1,-2) \cdot \dfrac{(1,-41,-25)}{|(1,-41,25)|} = \dfrac{92}{(1+41^2+25^2)^{1/2}} = \dfrac{92}{\sqrt{2307}}$

5 marks

2. (a) Use Gaussian elimination with back-substitution to find all solutions of the following system of equations. Be sure to record the details of each row-operation (for example, as a note on each row of the form $2(2) - 3(1) \to (2')$.)

$$\begin{array}{rcrcrcr} 2x & - & 3y & + & 2z & = & 6 \\ x & + & 4y & - & z & = & 0 \\ -x & + & 2y & - & z & = & -2 \end{array}$$

$2x - 3y + 2z = 6$ (1)

$x + 4y - z = 0$ (2) $\leftarrow 2(2) - (1)$

$-x + 2y - z = -2$ (3) $\leftarrow 2(3) + (1)$

$2x - 3y + 2z = 6 \Rightarrow x = -1$

$11y - 4z = -6 \Rightarrow z = 7$

$y = 2 \Rightarrow y = 2$

$x = -1, \quad y = 2, \quad z = 7$

8 marks

(b) Evaluate each of the following

(i) $\begin{bmatrix} 3 & 4 \\ 1 & 7 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix}$, (ii) $\begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix}$, (iii) $\det \begin{bmatrix} 2 & 4 & 1 \\ 3 & 2 & 3 \\ 2 & 5 & 6 \end{bmatrix}$.

i) $$\begin{bmatrix} 3 & 4 \\ 1 & 7 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 15+8 & 6+16 \\ 5+14 & 2+28 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 22 \\ 19 & 30 \\ 2 & 4 \end{bmatrix}.$$

ii) $$\begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1+6 & 2+0 & 3+9 \\ 2+2 & 4+0 & 6+3 \\ 3+4 & 6+0 & 9+6 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 2 & 12 \\ 4 & 4 & 9 \\ 7 & 6 & 15 \end{bmatrix}$$

iii) $$\det = 2 \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} - 4 \begin{vmatrix} 3 & 3 \\ 2 & 6 \end{vmatrix} + 1 \begin{vmatrix} 3 & 2 \\ 2 & 5 \end{vmatrix}$$

$$= 2(-3) - 4(12) + (15-4)$$

$$= -6 - 48 + 11$$

$$= -43$$

4 marks

(c) Let

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}.$$

Find numbers a and b such that

$$A^2 + aA + bI = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

where I is the 2 by 2 identity matrix.

$$A^2 = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 4+15 & 6+3 \\ 10+5 & 15+1 \end{bmatrix} = \begin{bmatrix} 19 & 9 \\ 15 & 16 \end{bmatrix}$$

$$A^2 + aA + bI = 0$$

$$\Rightarrow \begin{bmatrix} 19 & 9 \\ 15 & 16 \end{bmatrix} + a\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad 19 + 2a + b = 0 \quad (1)$$

$$9 + 3a = 0 \quad (2)$$

$$15 + 5a = 0 \quad (3)$$

$$\left.\begin{matrix} (2) \\ (3) \end{matrix}\right\} \Rightarrow a = -3$$

$$16 + a + b = 0 \quad (4) \quad \Rightarrow b = -13$$

check (1) : $19 + 2(-3) + (-13) = 0$ ✓

So $a = -3$ and $b = -13$

4 marks

(d) Use the result of the part (c) to compute the inverse of A.

$$A^2 - 3A - 13I = 0$$

$$\Rightarrow \quad A - 3I - 13A^{-1} = 0$$

$$\Rightarrow \quad A^{-1} = \frac{1}{13}(A - 3I) = \frac{1}{13}\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} - \frac{1}{13}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{13}\begin{bmatrix} -1 & 3 \\ 5 & -2 \end{bmatrix}$$

4 marks

3. **(a)** Use integration by parts to show that, for $n > 0$,

$$\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$$

$$I_n = \int \cos^n x\,dx = \int \cos x \cdot \cos^{n-1} x\,dx$$

$$= \sin x \cos^{n-1} x + (n-1)\int \sin^2 x \cos^{n-2} x\,dx$$

$$= \sin x \cos^{n-1} x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\,dx$$

$$\Rightarrow I_n = \sin x \cos^{n-1} x + (n-1)(I_{n-2} - I_n)$$

$$\Rightarrow I_n = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n} I_{n-2} \qquad n \neq 0.$$

10 marks

(b) Use the previous result to show that, for $n > 0$,

$$\int_0^{\pi/2} \cos^{2n}(x)\, dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \frac{\pi}{2}$$

From part a we have

$$\int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx = \left[\sin x \cos^{2n-1} x \right]_0^{\frac{\pi}{2}} + \frac{2n-1}{2n} \int_0^{\frac{\pi}{2}} \cos^{2n-2} x \, dx$$

0

for $n > 0$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx = \frac{2n-1}{2n} \int_0^{\frac{\pi}{2}} \cos^{2n-2} x \, dx$$

$$= \left(\frac{2n-1}{2n}\right)\left(\frac{2n-3}{2n-2}\right) \cdots \left(\frac{1}{2}\right) \int_0^{\frac{\pi}{2}} \cos^0 x \, dx$$

$$= \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} \cdot \frac{\pi}{2}$$

10 marks

4. Evaluate each of the following integrals

(a) $\displaystyle\int \theta \sin\theta \, d\theta$

$I = \int \theta \sin\theta \, d\theta$ by parts.

$= -\theta\cos\theta + \int \cos\theta \, d\theta$

$= -\theta\cos\theta + \sin\theta + C$

5 marks

(b) $\displaystyle\int x \log_e(x) \, dx$

$I = \int x \log x \, dx$ by parts

$= \frac{1}{2}x^2 \log x - \int \frac{1}{2}x^2 \frac{1}{x} \, dx$

$= \frac{1}{2}x^2 \log x - \frac{1}{4}x^2 + C$

5 marks

(c) $\int y\sqrt{2+y}\,dy$

$I = \int y\,(2+y)^{1/2}\,dy \qquad$ by parts

$= \frac{2}{3} y\,(2+y)^{3/2} - \frac{2}{3}\int (2+y)^{3/2}\,dy$

$= \frac{2}{3} y\,(2+y)^{3/2} - \frac{4}{15}(2+y)^{5/2} + C$

or Put $u = 2+y \Rightarrow du = dy$

$\Rightarrow I = \int (u-2)\sqrt{u}\,du$

$= \int u^{3/2} - 2u^{1/2}\,du$

$= \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + C$

$= \frac{2}{5}(2+y)^{5/2} - \frac{4}{3}(2+y)^{3/2} + C$

5 marks

5. Determine which of the following improper integrals converge and which diverge.

(a) $I = \displaystyle\int_0^1 \frac{1}{1-x}\, dx$

Put

$$I(\varepsilon) = \int_0^{\varepsilon} \frac{1}{1-x}\, dx \quad \text{with} \quad \varepsilon < 1$$

$$\Rightarrow \quad I(\varepsilon) = -\left[\log(1-x)\right]_0^{\varepsilon} = -\log(1-\varepsilon)$$

$$\Rightarrow \quad \lim_{\varepsilon \to 1} I(\varepsilon) = -\lim_{\varepsilon \to 1} \log(1-\varepsilon) = -\infty$$

Thus the integral is a divergent improper integral.

5 marks

(b) $I = \displaystyle\int_1^{\infty} \sin^2(x) e^{-2x}\, dx$

Note that

$$0 < \sin^2(x) e^{-2x} < e^{-2x} \quad \text{for } 1 < x < \infty$$

$$\Rightarrow \quad 0 < \lim_{\varepsilon \to \infty} \int_1^{\varepsilon} \sin^2 x\, e^{-2x} < \underbrace{\lim_{\varepsilon \to \infty} \int_1^{\varepsilon} e^{-2x}\, dx}_{= \frac{1}{2} e^{-2}}$$

$$\Rightarrow \quad 0 < \underbrace{\int_1^{\infty} \sin^2 x\, e^{-2x}\, dx}_{\text{convergent}} < \frac{1}{2} e^{-2}$$

5 marks

(c) $I = \displaystyle\int_1^{\infty} \frac{e^x}{\sqrt{1+e^x}}\, dx$

Note that

$$0 < \frac{e^x}{\sqrt{e^x + e^x}} < \frac{e^x}{\sqrt{1+e^x}} \quad \text{for } 1 < x$$

$$\Rightarrow \quad 0 < \underbrace{\lim_{\varepsilon \to \infty} \int_1^{\varepsilon} \frac{1}{\sqrt{2}} e^{x/2}\, dx}_{= \infty} < \lim_{\varepsilon \to \infty} \int_1^{\varepsilon} \frac{e^x}{\sqrt{1+e^x}}\, dx$$

$$\Rightarrow \quad \infty < \underbrace{\lim_{\varepsilon \to \infty} \int_1^{\varepsilon} \frac{e^x}{\sqrt{1+e^x}}\, dx}_{\text{divergent.}}$$

//

OR

Since $\displaystyle\lim_{x \to \infty} \frac{e^x}{\sqrt{1+e^x}} = 1 \neq 0$

the integral must diverge.

10 marks

(d) $I = \displaystyle\int_1^\infty \left(\frac{\sin x}{x}\right)^2 \mathrm{d}x.$

$$I = \int_1^\infty \left(\frac{\sin x}{x}\right)^2 dx$$

Put

$$I_\varepsilon = \int_1^\varepsilon \left(\frac{\sin x}{x}\right)^2 dx \, , \quad \varepsilon > 1.$$

Note that

$$0 < \left(\frac{\sin x}{x}\right)^2 < \frac{1}{x^2} \quad \text{for } x > 1$$

$$\Rightarrow \quad 0 < \int_1^\varepsilon \left(\frac{\sin x}{x}\right)^2 dx < \int_1^\varepsilon \frac{1}{x^2} \, dx$$

$$= \left[-\frac{1}{x}\right]_1^\varepsilon = 1 - \frac{1}{\varepsilon}$$

Now take limits as $\varepsilon \to \infty$.

$$\Rightarrow \quad 0 < \lim_{\varepsilon \to \infty} \int_1^\varepsilon \left(\frac{\sin x}{x}\right)^2 dx < 1$$

Thus we conclude that the given integral I converges.

5 marks

6. Use a suitable test to find which of the following infinite series are convergent. Be sure to state which test you used and to show your working.

(a) $S = \dfrac{1}{\log 2} - \dfrac{1}{\log 3} + \dfrac{1}{\log 4} - \dfrac{1}{\log 5} + \cdots + (-1)^n \dfrac{1}{\log n} + \cdots$

Alternating series.

$a_n = \frac{1}{\log n}$ clearly i) $\lim_{n\to\infty} \frac{1}{\log n} = 0$

ii) $\frac{1}{\log n+1} < \frac{1}{\log n}$ $n > 1$

$\Rightarrow$ Series converges by the Alt. series test.

6 ~~marks~~

(b) $S = \displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^2+1}$

$0 < \frac{\cos^2 n}{n^2+1} < \frac{1}{n^2+1} < \frac{1}{n^2}$ and $\int_1^\infty \frac{dx}{x^2}$ is finite.

$\Rightarrow$ the series converges by comparison with $\Sigma \frac{1}{n^2}$.

6 marks

(c) $S = \displaystyle\sum_{n=2}^{\infty} \frac{1}{n \log n}$ (Hint : first compute $d\log(\log(x))/dx$)

$\frac{d}{dx}\left(\log(\log(x))\right) = \frac{1}{x}\cdot\frac{1}{\log x}$

$\Rightarrow \int_2^\infty \frac{1}{x\log x} = \left[\log(\log(x))\right]_2^\infty = \infty$

thus the series diverges by the integral test.

8 marks

(d) $S = \sum_{n=1}^{\infty} \frac{n^2}{3n^5 + 5n^2 - 4}$

Note that when $n \geq 1$ we have

$$5n^2 - 4 > 0$$

$$\Rightarrow 3n^5 + 5n^2 - 4 > 3n^5 \quad \text{for } n \geq 1$$

$$\Rightarrow 0 < \frac{n^2}{3n^5 + 5n^2 - 4} < \frac{n^2}{3n^5} = \frac{1}{3n^3} \quad n \geq 1$$

Thus the given Series converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^3}$

5 marks

(e) $S = \sum_{n=1}^{\infty} \frac{1}{n^2 + \sin^2 n}$

Note that $0 < \frac{1}{n^2 + \sin^2 n} < \frac{1}{n^2}$

$$\Rightarrow \quad 0 < \sum_{n=1}^{\infty} \frac{1}{n^2 + \sin^2 n} < \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\text{convergent.}}$$

Thus the infinite series <u>converges</u>.

5 marks

7. (a) Compute the Taylor series for $\sqrt{x}$ around the point $x = 1$.

$f, f', f'' \cdots$	$x=1$	$n!\,a_n$
$x^{\frac{1}{2}}$	1	$= 0!\,a_0$
$\frac{1}{2} x^{-\frac{1}{2}}$	$\frac{1}{2}$	$= 1!\,a_1$
$\frac{1}{2}\left(-\frac{1}{2}\right) x^{-\frac{3}{2}}$	$\frac{1}{2}\left(-\frac{1}{2}\right)$	$= 2!\,a_2$
$\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) x^{-\frac{5}{2}}$	$\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)$	$= 3!\,a_3$
$\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right) x^{-\frac{7}{2}}$	$\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)$	$= 4!\,a_4$

$$a_n = (-1)^{n+1} \frac{1}{2^n} \frac{1}{n!} (1 \cdot 3 \cdot 5 \cdots 2n-3)$$

$$n \geq 2$$

$$a_0 = 1 \qquad a_1 = \frac{1}{2}$$

$$\sqrt{x} = 1 + \frac{1}{2}(x-1) - \frac{1}{2^2}\frac{1}{2!}(x-1)^2 + \cdots + a_n (x-1)^n + \cdots$$

10 marks

(b) Without doing any further calculations, write down the Taylor series for $\sqrt{1-u^2}$ around the point $u = 0$.

Put $x = 1 - u^2 \Rightarrow x - 1 = u^2$

$$\Rightarrow \sqrt{1-u^2} = 1 - \frac{1}{2}u^2 - \frac{1}{2^2}\frac{1}{2!}u^4 + \cdots + \sigma_n u^{2n} + \cdots$$

with $\sigma_n = -\frac{1}{2^n} n! \, (1\cdot 3\cdot 5 \cdots 2n-3)$

$n \geqslant 2$

5 marks

(c) Use the result of part (b) to obtain an infinite series expansion for the function $s(x)$ defined by

$$s(x) = \int_0^x \sqrt{1-u^2}\, du \qquad 0 < x < 1$$

$$S(x) = \int_0^x \left(1 - \frac{1}{2}u^2 - \frac{1}{2^2}\frac{1}{2!}u^4 + \cdots + \sigma_n u^{2n} \cdots\right) du$$

$$= x - \frac{1}{2}\frac{x^3}{3} - \frac{1}{2^2}\frac{1}{2!}\frac{x^5}{5} + \cdots + \sigma_n \frac{x^{2n+1}}{2n+1} + \cdots$$

$$\sigma_n = -\frac{1}{2^n}\frac{1}{n!}(1\cdot 3\cdot 5 \cdots 2n-3)$$

$n \geqslant 2$

5 marks

8. Use l'Hopital's rule to evaluate each of the following limits.

(a) $L = \lim_{x \to -2} \dfrac{x^2 - 4}{x + 2}$

$$L = \lim_{x \to -2} \frac{x^2 - 4}{x + 2}$$

$$= \lim_{x \to -2} \frac{2x}{1} = -4$$

5 marks

(b) $L = \lim_{x \to 1} \dfrac{\log(x)}{\sin(2\pi x)}$

$$L = \lim_{x \to 1} \frac{\log(x)}{\sin(2\pi x)}$$

$$= \lim_{x \to 1} \frac{\frac{1}{x}}{2\pi \cos(2\pi x)}$$

$$= \frac{1}{2\pi}$$

5 marks

(c) $L = \lim_{x \to \infty} e^{-2x} \log(x + 3)$

$$L = \lim_{x \to \infty} e^{-2x} \log(x+3)$$

$$= \lim_{x \to \infty} \frac{\log(x+3)}{e^{2x}}$$

$$= \lim_{x \to \infty} \frac{\frac{1}{x+3}}{2e^{2x}} = 0.$$

5 marks

(d) Prove that for any $n > 0$

$$0 = \lim_{x \to \infty} x^{-n} \log(x)$$

$$L = \lim_{x \to \infty} x^{-n} \log(x)$$

$$= \lim_{x \to \infty} \frac{\log(x)}{x^n} = \lim_{x \to \infty} \frac{\frac{1}{x}}{n x^{n-1}}$$

$$= \frac{1}{n} \lim_{x \to \infty} \frac{1}{x^n} = 0 \text{ since } n > 0.$$

5 marks

9. **(a)** State what is meant by the phrase *a separable first order ordinary differential equation.*

The ODE is such that it may be re-written as

$$\frac{dy}{f(y)} = \frac{dx}{g(x)}$$

4 marks

(b) Find all functions $y(x)$ that are solutions of

$$\frac{dy}{dx} + y = \sin(2x)$$

i) homogenous equation : $\frac{dy}{dx} + y = 0 \Rightarrow y_h(x) = Ae^{-x}$

ii) particular solution : try $y_p(x) = B\sin 2x + C\cos 2x$

$\Rightarrow y' = 2B\cos 2x + (-2)C\sin 2x$

$y' + y = (C+2B)\cos 2x + (B-2C)\sin 2x = \sin 2x$

$\Rightarrow C + 2B = 0$ and $B - 2C = 1$

$\Rightarrow C = -2B \Rightarrow B = 1/5$ and $C = -2/5$

Full solution : $y(x) = Ae^{-x} + \frac{1}{5}\sin 2x - \frac{2}{5}\cos 2x$

10 marks

(c) Use your result from part (b) to compute all functions $u(x)$ that are solutions of

$$\frac{d^2u}{dx^2} + \frac{du}{dx} = \sin(2x)$$

Note : $y = \frac{du}{dx} \Rightarrow \frac{du}{dx} = Ae^{-x} + \frac{1}{5}\sin 2x - \frac{2}{5}\cos 2x$

$\Rightarrow u(x) = C - Ae^{-x} - \frac{1}{10}\cos 2x - \frac{1}{5}\sin 2x$

6 marks

10. Consider the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 15y = 45x - 36 + 8e^{3x}$$

(a) Write down the homogeneous version of the above differential equation.

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 15y = 0.$$

1 marks

(b) Find the general solution of the homogeneous differential equation.

$$\lambda^2 + 2\lambda - 15 = 0. \quad \Rightarrow \lambda_1 = 3, \lambda_2 = -5$$

$$\Rightarrow y_h(x) = Ae^{3x} + Be^{-5x}$$

5 marks

(c) Find any particular solution of the full differential equation.

Guess $y_p(x) = Ax + B + Cxe^{3x}$

$\Rightarrow \quad y_p' = A + Ce^{3x} + 3Cxe^{3x}$

$\Rightarrow \quad y_p'' = 6Ce^{3x} + 9Cxe^{3x}$

$\Rightarrow \quad y_p'' + 2y_p' - 15y_p = 6Ce^{3x} + 2A + 2Ce^{3x} - 15Ax - 15B$

$\Rightarrow \quad -15A = 45, \quad 2A - 15B = -36, \quad 8C = 8$

$\Rightarrow \quad A = -3, \quad B = 2, \quad C = 1$

So $y_p(x) = -3x + 2 + xe^{3x}$

6 marks

(d) Hence write down the full general solution of the differential equation.

$$y(x) = y_p(x) + y_h(x)$$
$$= -3x + 2 + xe^{3x} + Ae^{3x} + Be^{-5x}$$

1 marks

(e) Find the particular solution such that at $x = 0$ we have $y = 0$ and $dy/dx = 1$.

$y(0) = 0 \quad \Rightarrow \quad 0 = 2 + A + B$

$y'(0) = 1 \quad \Rightarrow \quad 1 = -3 + 1 + 3A - 5B$

So

$$A + B = -2$$
$$3A - 5B = 3$$

$\Rightarrow \quad 8A = -7 \quad \Rightarrow A = -\frac{7}{8}$ and $B = -2 + \frac{7}{8} = -\frac{9}{8}$

So the particular solution is

$$y(x) = -3x + 2 + xe^{3x} - \frac{7}{8}e^{3x} - \frac{9}{8}e^{-5x}$$

7 marks

11. **(a)** Given the function $f(x,y) = (x-y)/(x+y)$ evaluate each of the following

(i) $\dfrac{\partial f}{\partial x}$, (ii) $\dfrac{\partial f}{\partial y}$, (iii) $\dfrac{\partial^2 f}{\partial y \partial x}$, (iv) $\dfrac{\partial^2 f}{\partial x \partial y}$.

$$f(x,y) = \frac{x-y}{x+y}$$

i) $$\frac{\partial f}{\partial x} = \frac{1}{x+y} - \frac{x-y}{(x+y)^2} = \frac{2y}{(x+y)^2}$$

ii) $$\frac{\partial f}{\partial y} = \frac{-1}{x+y} - \frac{x-y}{(x+y)^2} = \frac{-2x}{(x+y)^2}$$

iii) $$\frac{\partial^2 f}{\partial y \partial x} = -\frac{1}{(x+y)^2} + \frac{1}{(x+y)^2} + \frac{2(x-y)}{(x+y)^3}$$

$$= \frac{2(x-y)}{(x+y)^3}$$

iv) $$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{2(x-y)}{(x+y)^3}$$

5 marks

(b) Given the curve

$$x(s) = 2s + 3s^2, \qquad y(s) = 4s^2 - 2s, \qquad -\infty < s < \infty$$

and the function

$$f(x, y) = \sin(5x + y) - 2e^y$$

compute df/ds at $s = -1$.

$$\frac{dx}{ds} = 2 + 6S = 2 - 6 = -4 \quad \text{at } S = -1$$

$$\frac{dy}{ds} = 8S - 2 = -8 - 2 = -10 \quad \text{at } S = -1$$

$$x(-1) = -2 + 3 = 1 \qquad y(-1) = 4 + 2 = 6$$

$$\frac{\partial f}{\partial x} = 5\cos(5x+y) = 5\cos 11 \quad \text{at } S = -1$$

$$\frac{\partial f}{\partial y} = \cos(5x+y) - 2e^y = \cos 11 - 2e^6 \quad \text{at } S = -1$$

$$\Rightarrow \left(\frac{df}{ds}\right)_{S=-1} = \left(\frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}\right)_{S=-1}$$

$$= (5\cos 11)(-4) + (\cos 11 - 2e^6)(-10)$$

$$= -30\cos 11 + 20e^6$$

5 marks

(c) Compute the tangent plane to the function $f(x, y) = \sqrt{4x^2 + y^2}$ at $(2, 3, 5)$.

$$z(x,y) = f(2,3) + (x-2)\frac{\partial f}{\partial x}\bigg|_{(2,3)} + (y-3)\frac{\partial f}{\partial y}\bigg|_{(2,3)}$$

$f(2,3) = 5$

$$\frac{\partial f}{\partial x} = \frac{1}{2}(4x^2+y^2)^{-\frac{1}{2}}(8x) = \frac{8}{5}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}(4x^2+y^2)^{-\frac{1}{2}}(2y) = \frac{3}{5}$$

$$\Rightarrow z(x,y) = 5 + \frac{8}{5}(x-2) + \frac{3}{5}(y-3)$$

5 marks

(d) Use linear approximation and the result of part(c) to estimate $\sqrt{4(1.97)^2 + (3.03)^2}$

$$z(1.97, 3.03) = 5 + \frac{8}{5}(-0.03) + \frac{3}{5}(0.03)$$

$$= 4.97$$

$$\Rightarrow f(1.97, 3.03) \simeq 4.97$$

5 marks