A tourist's guide to intersection theory on moduli spaces of curves

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In the past few decades, moduli spaces of curves have attained notoriety amongst mathematicians for their incredible structure. In fact, the study of moduli spaces is at the centre of a rich confluence of rather disparate areas such as geometry, topology, combinatorics, integrable systems, matrix models and string theory. Starting from baby principles, I will describe exactly what a moduli space is and motivate the study of its intersection theory. This scenic tour will guide us towards the pinnacle of the talk, a new proof of Kontsevich's combinatorial formula. There should be something in the talk for everyone, whether they are a seasoned traveller or a new tourist to the fascinating world of moduli spaces of curves.

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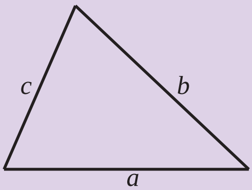
What is a moduli space?

- A moduli space parametrises a family of geometric objects.
- Different points in a moduli space represent different geometric objects and nearby points represent objects with similar structure.

Toy example: The moduli space of triangles

Consider a triangle with side lengths a, b and c.

$$\mathcal{M}_\triangle = \left\{(a,b,c) \in \mathbb{R}^3_+ \mid a+b>c,\ b+c>a,\ \text{and } c+a>b\right\}$$



Baby enumerative geometry

Baby question

How many triangles

- are isosceles;
- have at least one side of length 5; and
- have at least one side of length 7?

Define $X_{iso} \subseteq \mathcal{M}_{\triangle}$, the locus of isosceles triangles.
Define $X_5 \subseteq \mathcal{M}_{\triangle}$, the locus of triangles with one side of length 5.
Define $X_7 \subseteq \mathcal{M}_{\triangle}$, the locus of triangles with one side of length 7.

Same baby question

What is $|X_{iso} \cap X_5 \cap X_7|$?

Intuitive intersection theory (a.k.a. cohomology)

- An $(N - d)$-dimensional subset of an N-dimensional space is said to have codimension d.
- A "generic" intersection between subsets with codimension d_1 and d_2 has codimension $d_1 + d_2$.
- A "generic" intersection between m subsets of an N-dimensional space with codimensions $d_1 + d_2 + \cdots + d_m = N$ is a set of points. The number of these points is called an intersection number.
- We will use the following notation for intersection numbers.

$$X_1 \cdot X_2 \cdots X_m = |X_1 \cap X_2 \cap \ldots \cap X_m|$$

What can you do to a surface?

It might depend on what you're interested in...

ALGEBRAIC GEOMETRY	**HYPERBOLIC GEOMETRY**
COMPLEX ANALYSIS	**DIFFERENTIAL GEOMETRY**

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ALGEBRAIC GEOMETRY algebraic structure	**HYPERBOLIC GEOMETRY** hyperbolic metric
COMPLEX ANALYSIS complex structure	**DIFFERENTIAL GEOMETRY** Riemannian metric

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ALGEBRAIC GEOMETRY algebraic structure up to isomorphism	**HYPERBOLIC GEOMETRY** hyperbolic metric up to hyperbolic isometry
COMPLEX ANALYSIS complex structure up to biholomorphic equivalence	**DIFFERENTIAL GEOMETRY** Riemannian metric up to conformal equivalence

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ALGEBRAIC GEOMETRY algebraic structure up to isomorphism = algebraic curve	**HYPERBOLIC GEOMETRY** hyperbolic metric up to hyperbolic isometry = hyperbolic surface
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...but actually it doesn't! **THEY ARE ALL THE SAME!**

An introduction to moduli spaces of curves

$\mathcal{M}_{g,n}$	$=$	moduli space of genus g	algebraic curves	with n marked points
	$=$	moduli space of genus g	hyperbolic surfaces	with n cusps
	$=$	moduli space of genus g	Riemann surfaces	with n marked points

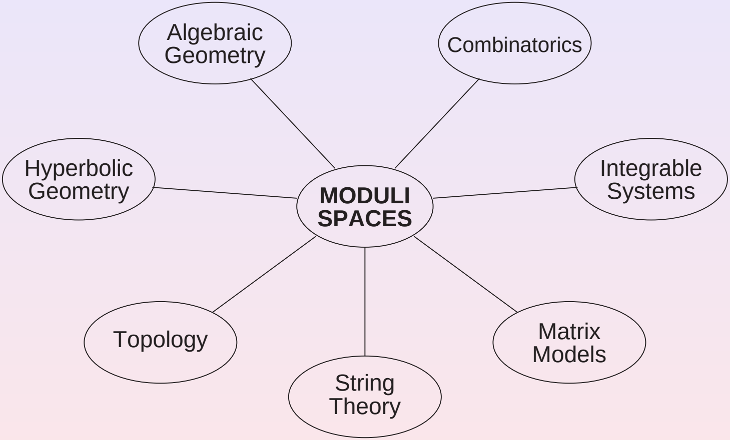
Two technical problems

- Problem: $\mathcal{M}_{g,n}$ is not compact.
  Solution: Use the (Deligne–Mumford) compactification $\overline{\mathcal{M}}_{g,n}$.
  Points in $\overline{\mathcal{M}}_{g,n}$ correspond to stable curves.
- Problem: $\overline{\mathcal{M}}_{g,n}$ is not a manifold.
  Solution: Treat $\overline{\mathcal{M}}_{g,n}$ like an orbifold.
  Allow intersection numbers to be rational.

Some facts

- The dimension of $\overline{\mathcal{M}}_{g,n}$ is $6g - 6 + 2n$.
- The structure of $\overline{\mathcal{M}}_{g,n}$ is generally very complicated!

Why do we care about moduli spaces of curves?



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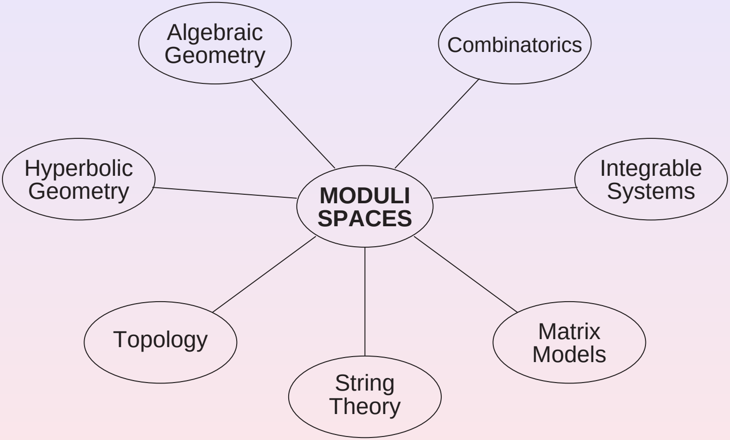


BECAUSE THEY ARE INTERESTING AND FUN!

Intersection numbers of psi-classes

- There are very important cohomology classes called psi-classes:

$$\psi_1, \psi_2, \ldots, \psi_n \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

- Translation: $\psi_1, \psi_2, \ldots, \psi_n$ are codimension 2 subsets of $\overline{\mathcal{M}}_{g,n}$.
- Choose $a_1 + a_2 + \cdots + a_n = 3g - 3 + n = \frac{1}{2} \dim \overline{\mathcal{M}}_{g,n}$.
- We are interested in intersection numbers of psi-classes:

$$\psi_1^{a_1} \cdot \psi_2^{a_2} \cdots \psi_n^{a_n} \in \mathbb{Q}.$$

Examples of intersection numbers of psi-classes

- On $\overline{\mathcal{M}}_{0,5}$, the intersection number $\psi_1 \cdot \psi_2$ is 2.
- On $\overline{\mathcal{M}}_{1,1}$, the intersection number ψ_1 is $\frac{1}{24}$.

A very brief history of Witten's conjecture

Witten (1991): I have a conjectured recursive formula which generates all intersection numbers of psi-classes.

Kontsevich (1992): Witten is right! I have a combinatorial formula which relates intersection numbers with ribbon graphs.

Okounkov and Pandharipande (2001): Witten is right! We have a formula which relates intersection numbers with Hurwitz numbers.

Mirzakhani (2004): Witten is right! I have a formula which relates intersection numbers with volumes of moduli spaces.

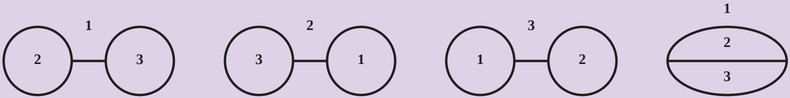
Do (2007): Witten is right! I can prove Kontsevich's combinatorial formula using Mirzakhani's volumes of moduli spaces.

What is a ribbon graph?

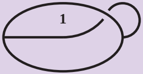
A ribbon graph of type (g, n) is

- a graph with a cyclic ordering of the edges at every vertex
- which can be thickened to give a surface of genus g and
- n boundary components labelled from 1 up to n.

Trivalent ribbon graphs of type $(0, 3)$



Trivalent ribbon graph of type $(1, 1)$



Kontsevich's combinatorial formula explained

Kontsevich's combinatorial formula

$$\sum_{|a|=3g-3+n} \psi_1^{a_1} \cdot \psi_2^{a_2} \cdots \psi_n^{a_n} \prod_{k=1}^{n} \frac{(2a_k - 1)!!}{s_k^{2a_k+1}} = \sum_{\Gamma \in TRG_{g,n}} \frac{2^{2g-2+n}}{|\mathrm{Aut}(\Gamma)|} \prod_{e \in \Gamma} \frac{1}{s_{\ell(e)} + s_{r(e)}}$$

- LHS: polynomial in $\frac{1}{s_1}, \frac{1}{s_2}, \ldots, \frac{1}{s_n}$
  - coefficients store all intersection numbers of psi-classes on $\overline{\mathcal{M}}_{g,n}$
- RHS: rational function in $s_1, s_2, \ldots, s_n$
  - strange enumeration over trivalent ribbon graphs of type (g, n)
- Kontsevich's combinatorial formula is incredible!

Kontsevich's combinatorial formula at work

Kontsevich's combinatorial formula for $g = 0$ and $n = 3$.

- The LHS is easy.

$$LHS = \psi_1^0 \cdot \psi_2^0 \cdot \psi_3^0 \, \frac{1}{s_1 s_2 s_3}$$

- The RHS has one term for each trivalent ribbon graph of type $(0, 3)$.

$$\begin{aligned} RHS &= \frac{2}{2s_1(s_1+s_2)(s_1+s_3)} + \frac{2}{2s_2(s_2+s_3)(s_2+s_1)} \\ &\quad + \frac{2}{2s_3(s_3+s_1)(s_3+s_2)} + \frac{2}{(s_1+s_2)(s_2+s_3)(s_3+s_1)} \\ &= \frac{s_2s_3(s_2+s_3) + s_3s_1(s_3+s_1) + s_1s_2(s_1+s_2) + 2s_1s_2s_3}{s_1s_2s_3(s_1+s_2)(s_2+s_3)(s_3+s_1)} \\ &= \frac{(s_1+s_2)(s_2+s_3)(s_3+s_1)}{s_1s_2s_3(s_1+s_2)(s_2+s_3)(s_3+s_1)} \\ &= \frac{1}{s_1s_2s_3} \end{aligned}$$

- Conclusion: $\psi_1^0 \cdot \psi_2^0 \cdot \psi_3^0 = 1$.

A new proof of Kontsevich's combinatorial formula

Kontsevich's combinatorial formula simplified

INTERSECTION NUMBERS OF PSI-CLASSES $\Longrightarrow$ RIBBON GRAPHS

Sketch proof: Step 1 of 3

INTERSECTION NUMBERS OF PSI-CLASSES $\Longrightarrow$ ASYMPTOTICS OF VOLUMES OF MODULI SPACES

Mirzakhani's Theorem: Let $\mathcal{M}_{g,n}(\ell_1, \ell_2, \ldots, \ell_n)$ be the moduli space of hyperbolic surfaces of genus g with n boundaries of lengths $\ell_1, \ell_2, \ldots, \ell_n$. Then its (Weil–Petersson) volume $V_{g,n}(\ell_1, \ell_2, \ldots, \ell_n)$ is a polynomial whose top degree coefficients store all intersection numbers of psi-classes on $\overline{\mathcal{M}}_{g,n}$.

Question: How do you access the top degree coefficients of a polynomial?
Answer: Asymptotics — more precisely, we consider the behaviour of $V_{g,n}(N\ell_1, N\ell_2, \ldots, N\ell_n)$ as N approaches infinity.

A new proof of Kontsevich's combinatorial formula

Sketch proof: Step 2 of 3

ASYMPTOTICS OF VOLUMES OF MODULI SPACES $\Longrightarrow$ HYPERBOLIC SURFACES WITH VERY LONG BOUNDARIES

To understand $V_{g,n}(N\ell_1, N\ell_2, \ldots, N\ell_n)$ for very large values of N, we must understand hyperbolic surfaces with boundaries of lengths $N\ell_1, N\ell_2, \ldots, N\ell_n$ for very large values of N.

Sketch proof: Step 3 of 3

HYPERBOLIC SURFACES WITH VERY LONG BOUNDARIES $\Longrightarrow$ RIBBON GRAPHS

Fact: All hyperbolic surfaces of genus g with n boundary components have the same surface area.

Crucial geometric reasoning: If you take a hyperbolic surface and send its boundary lengths to infinity, then you will obtain a ribbon graph (after rescaling).

Two interesting problems

Calculating volumes

Given m linear equations in n variables with positive coefficients, the set of non-negative solutions is an $(n-m)$-dimensional polytope. How can you calculate its volume?

Remark: Use the Laplace transform!

Graphs and determinants

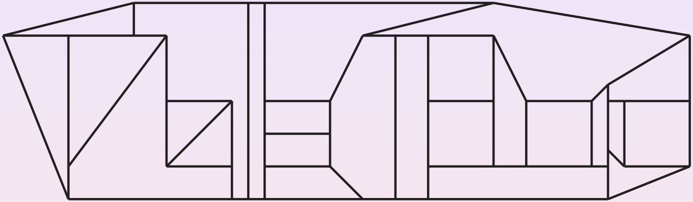
Consider a trivalent ribbon graph of type (g, n). Colour n of the edges blue and the remaining edges red. Let A be the matrix formed from the adjacency between the blue edges and the boundaries. Let B be the matrix formed from the oriented adjacency between the red edges. Then

$$\det B = 2^{2g-2}(\det A)^2.$$

Remark: There is no known combinatorial proof!

A problem about graphs and determinants

The following is a ribbon graph with $g = 0$ and $n = 26$.



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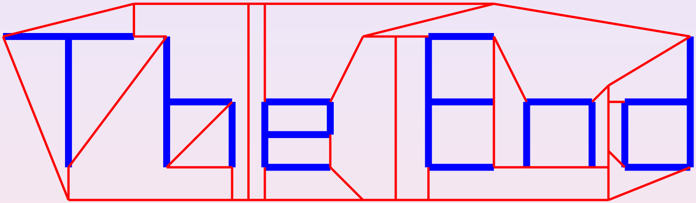

$$\begin{array}{cccccc} \det B & = & 2^{2g-2} & \times & (\det A)^2 \\ 256 & = & \frac{1}{4} & \times & 32^2 \end{array}$$