COUNTING LATTICE POINTS IN COMPACTIFIED MODULI SPACES OF CURVES

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How many ways are there to obtain a genus g surface by gluing together the edges of n given polygons? For geometric reasons, it is natural to generalise this problem to the case of stable surfaces. We show that such an augmented enumeration yields *lattice point polynomials* which can be recursively computed. Their top degree coefficients are intersection numbers on compactified moduli spaces of curves while their constant terms are Euler characteristics of compactified moduli spaces of curves. On the other hand, the geometric meaning of the intermediate coefficients remains a complete mystery. In this talk, we will define the lattice point polynomials, present some of their properties, and indicate possible connections to topics such as Gromov-Witten theory and topological recursion.

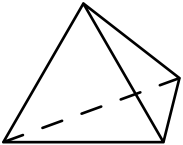
BUILDING A SURFACE FROM POLYGONS

How many ways are there to obtain a genus g surface by gluing together the edges of n given polygons?

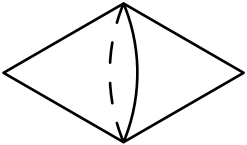
- Let the polygons be numbered $1, 2, \ldots, n$ and have $b_1, b_2, \ldots, b_n$ edges.
- The edges form a combinatorial object called a ribbon graph.
- We won't allow vertices of degree one.
- Denote this weighted enumeration of ribbon graphs by $N_{g,n}(b_1, b_2, \ldots, b_n)$.

A baby example

You should be able to check that $N_{0,4}(3, 3, 3, 3) = 8$.



2 labellings



6 labellings

TOPOLOGICAL RECURSION

Theorem (Norbury, 2008)

There is a topological recursion by which you can compute $N_{g,n}$ from $N_{g,n-1}$, $N_{g-1,n+1}$, and $N_{g_1,n_1} \times N_{g_2,n_2}$ for $g_1 + g_2 = g$ and $n_1 + n_2 = n + 1$.

Where does the recursion come from?

When you remove an edge from a ribbon graph,

- two faces could become one;
- one face could become two with a loss of genus; or
- the ribbon graph could become disconnected.

Corollary

If we restrict to even $b_1, b_2, \ldots, b_n$, then $N_{g,n}(b_1, b_2, \ldots, b_n)$ is an even polynomial of degree $6g - 6 + 2n$.

Bonus fact

Eynard–Orantin technology stores the recursion in the curve $xy - y^2 = 1$.

EXAMPLES OF LATTICE POINT POLYNOMIALS

g	n	$N_{g,n}(b_1, b_2, \ldots, b_n)$
0	3	1
1	1	$\frac{1}{48}(b_1^2 - 4)$
0	4	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 - 4)$
1	2	$\frac{1}{384}(b_1^4 + b_2^4 + 2b_1^2 b_2^2 - 12b_1 - 12b_2 + 32)$
2	1	$\frac{1}{8847360}(5b_1^8 - 312b_1^6 + 5712b_1^4 - 36608b_1^2 + 73728)$

What do the coefficients of the lattice point polynomials mean?

MODULI SPACES OF CURVES

$$\begin{aligned}\mathcal{M}_{g,n} &= \left\{\begin{array}{l}\text{genus } g \text{ smooth algebraic curves with distinct} \\ \text{points labelled from 1 up to } n\end{array}\right\} \\ &= \left\{\begin{array}{l}\text{genus } g \text{ Riemann surfaces with distinct} \\ \text{punctures labelled from 1 up to } n\end{array}\right\} \\ &= \left\{\begin{array}{l}\text{genus } g \text{ hyperbolic surfaces with distinct} \\ \text{cusps labelled from 1 up to } n\end{array}\right\}\end{aligned}$$

$$\dim \mathcal{M}_{g,n} = 6g - 6 + 2n$$

Strebel's theorem

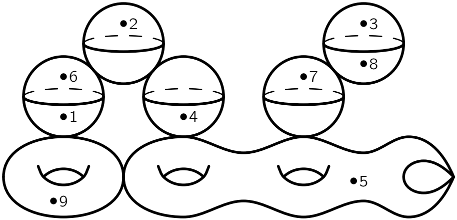
For fixed $r_1, r_2, \ldots, r_n > 0$, we can bijectively associate a point in $\mathcal{M}_{g,n}$ to a ribbon graph where

- every vertex has degree at least three;
- a positive length is assigned to every edge; and
- the perimeter of face k is r_k.

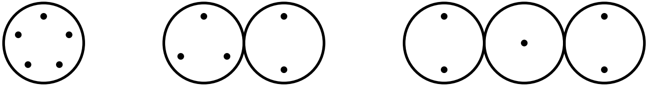
DELIGNE–MUMFORD COMPACTIFICATION

$$\overline{\mathcal{M}}_{g,n} = \left\{ \begin{array}{l} \text{genus } g \text{ stable algebraic curves with distinct} \\ \text{smooth points labelled from 1 up to } n \end{array} \right\}$$

- A stable curve may be nodal but its components satisfy $2g - 2 + n > 0$.



- There is a stratification of $\overline{\mathcal{M}}_{g,n}$ by smaller moduli spaces of curves.



$$\overline{\mathcal{M}}_{0,5} = \underset{\text{1 labelling}}{\mathcal{M}_{0,5}} \sqcup \underset{\text{10 labellings}}{\mathcal{M}_{0,4} \times \mathcal{M}_{0,3}} \sqcup \underset{\text{15 labellings}}{\mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \times \mathcal{M}_{0,3}}$$

LATTICE POINTS IN MODULI SPACES OF CURVES

Interpret ribbon graphs with integer edge lengths as lattice points in moduli spaces of curves.

Theorem (Norbury, 2008)

- The top degree part of $N_{g,n}$ stores psi-class intersection numbers

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \psi_2^{a_2} \cdots \psi_n^{a_n},$$

  where $\psi_1, \psi_2, \ldots, \psi_n \in H^2(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$.
- The constant term of $N_{g,n}$ is $\chi(\mathcal{M}_{g,n})$.

Corollary

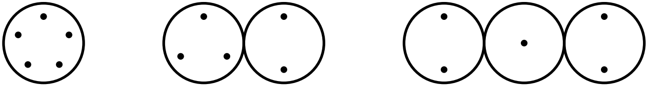
Combining this theorem with the topological recursion yields another proof of the Witten–Kontsevich theorem, which governs psi-class intersection numbers.

LATTICE POINTS IN COMPACTIFIED MODULI SPACES

Count lattice points in the compactified moduli space $\overline{\mathcal{M}}_{g,n}$ to obtain $\overline{N}_{g,n}$.

Example

Points in $\overline{\mathcal{M}}_{0,5}$ represent curves of the following types.



$$\overline{\mathcal{M}}_{0,5} = \underset{\text{1 labelling}}{\mathcal{M}_{0,5}} \sqcup \underset{\text{10 labellings}}{\mathcal{M}_{0,4} \times \mathcal{M}_{0,3}} \sqcup \underset{\text{15 labellings}}{\mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \times \mathcal{M}_{0,3}}$$

$$\begin{aligned}\overline{N}_{0,5}(b_1, b_2, b_3, b_4, b_5) = {} & N_{0,5}(b_1, b_2, b_3, b_4, b_5) \\ & + \sum_{\text{10 terms}} N_{0,4}(b_i, b_j, b_k, 0) \times N_{0,3}(b_\ell, b_m, 0) \\ & + \sum_{\text{15 terms}} N_{0,3}(b_i, b_j, 0) \times N_{0,3}(b_k, 0, 0) \times N_{0,3}(b_\ell, b_m, 0)\end{aligned}$$

COMPACTIFIED LATTICE POINT POLYNOMIALS

Some simple facts

- If we restrict to even $b_1, b_2, \ldots, b_n$, then $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$ is an even polynomial of degree $6g - 6 + 2n$.
- The top degree part of $\overline{N}_{g,n}$ stores psi-class intersection numbers on $\overline{\mathcal{M}}_{g,n}$.
- The constant term of $\overline{N}_{g,n}$ is $\chi(\overline{\mathcal{M}}_{g,n})$.

Theorem (Do–Norbury, 2011)

- The polynomial $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$ enumerates stable ribbon graphs.
- There is a topological recursion by which you can compute $\overline{N}_{g,n}$ from $\overline{N}_{g,n-1}$, $\overline{N}_{g-1,n+1}$, and $\overline{N}_{g_1,n_1} \times \overline{N}_{g_2,n_2}$ for $g_1 + g_2 = g$ and $n_1 + n_2 = n + 1$.
- If we write $\overline{\chi}_{g,n} = \chi(\overline{\mathcal{M}}_{g,n})$, then

$$\overline{\chi}_{g,n+1} = (2-2g-n)\,\overline{\chi}_{g,n} + \frac{1}{2}\,\overline{\chi}_{g-1,n+2} + \frac{1}{2}\sum_{h=0}^{g}\sum_{k=0}^{n}\binom{n}{k}\,\overline{\chi}_{h,k+1}\,\overline{\chi}_{g-h,n-k+1}.$$

EXAMPLES OF COMPACTIFIED LATTICE POINT POLYNOMIALS

g	n	$\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$
0	3	1
1	1	$\frac{1}{48}(b_1^2 + 20)$
0	4	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 + 8)$
1	2	$\frac{1}{384}(b_1^4 + b_2^4 + 2b_1^2 b_2^2 + 48b_1^2 + 48b_2^2 + 192)$
2	1	$\frac{1}{8847360}(5b_1^8 + 648b_1^6 + 19152b_1^4 + 278272b_1^2 + 1517568)$

It appears that the compactified lattice point polynomials may be the right objects to study. What do their coefficients mean?

Bonus question

Does Eynard–Orantin technology store the recursion in a curve?

ADDING DEGREE ONE VERTICES

Define $\overline{M}_{g,n}(b_1, b_2, \ldots, b_n)$ analogously to $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$ except we now

- allow degree one vertices; and
- divide by the combinatorial constant $\prod \binom{b_k - 1}{\frac{1}{2}b_k}$.

Then $\overline{M}_{g,n}(b_1, b_2, \ldots, b_n)$ is a polynomial of degree $3g - 3 + n$.

g	n	$\overline{M}_{g,n}(b_1, b_2, \ldots, b_n)$
0	3	1
1	1	$\frac{1}{24}(b_1 + 10)$
0	4	$\frac{1}{2}(b_1 + b_2 + b_3 + b_4 + 4)$
1	2	$\frac{1}{48}(b_1^2 + b_2^2 + b_1 b_2 + 8b_1 + 8b_2 + 24)$
2	1	$\frac{1}{23040}(5b_1^4 + 56b_1^3 + 228b_1^2 + 1184b_1 + 3952)$

RIBBON GRAPHS AND BRANCHED COVERS OF $\mathbb{CP}^1$

Old fact

Let $Z_{g,n}(b_1, b_2, \ldots, b_n)$ be the set of smooth maps from a genus g curve to $\mathbb{CP}^1$ whose only branching is over $\{0, 1, \infty\}$ such that

- the branching profile over ∞ is $(b_1, b_2, \ldots, b_n)$;
- the branching profile over 1 is $(2, 2, \ldots, 2)$; and
- there are no points with branching of order 1 over 0.

Then $N_{g,n}(b_1, b_2, \ldots, b_n) = \#Z_{g,n}(b_1, b_2, \ldots, b_n)$.

New fact

Let $\overline{Z}_{g,n}(b_1, b_2, \ldots, b_n)$ be the set of stable maps from a genus g curve to $\mathbb{CP}^1$ whose only branching is over $\{0, 1, \infty\}$ such that

- the branching profile over ∞ is $(b_1, b_2, \ldots, b_n)$;
- the branching profile over 1 is $(2, 2, \ldots, 2)$; and
- every point with branching of order 1 over 0 is a node.

Then $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n) = \chi\left[\overline{Z}_{g,n}(b_1, b_2, \ldots, b_n)\right]$.

THE GROMOV–WITTEN/HURWITZ CORRESPONDENCE

There are two competing theories for counting genus g branched covers of $\mathbb{CP}^1$ with specified branching conditions.

- Hurwitz theory: Combinatorially count by multiplying cycles in the symmetric group algebra.
- Gromov–Witten theory: Geometrically count by doing intersection theory on a suitable space of maps.

GW/H correspondence (Okounkov–Pandharipande, 2006)

The GW count is a compactified version of the H count. It can be obtained by multiplying completed cycles in the symmetric group algebra.

The count $N_{g,n}(b_1, b_2, \ldots, b_n)$ is naturally a Hurwitz number. Is its compactified version $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$ a Gromov–Witten number?

THANKS

If you would like more information, you can

- find the slides from the talk at `http://www.ms.unimelb.edu.au/~nndo`
- download the preprint at `http://arxiv.org/abs/1012.5923`
- wait for the paper to appear in `Geometry and Topology`
- email me at `normdo@gmail.com`
- speak to me at `the front of the lecture theatre`