Intersection theory on moduli spaces of curves via hyperbolic geometry

Norman Do

The University of Melbourne

In the past few decades, moduli spaces of curves have become the centre of a rich confluence of rather disparate areas such as geometry, combinatorics, integrable systems and theoretical physics. Starting from baby principles, I will describe exactly what a moduli space is and motivate the study of its intersection theory. The talk will include a discussion of recent results from my PhD thesis, including a new proof of a formula of Kontsevich.

8 April 2008

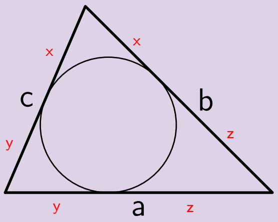
What is a moduli space?

- A moduli space parametrises a family of geometric objects.
- Different points in a moduli space represent different geometric objects and nearby points represent objects with similar structure.

Toy example: The moduli space of triangles

Consider a triangle with side lengths a, b and c.

$$\mathcal{M}_\triangle = \{(a,b,c) \in \mathbb{R}^3_+ \mid a+b>c, b+c>a, \text{ and } c+a>b\}$$



$$\begin{aligned} a &= y+z \\ b &= z+x \\ c &= x+y \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= \tfrac{b+c-a}{2} \\ y &= \tfrac{c+a-b}{2} \\ z &= \tfrac{a+b-c}{2} \end{aligned}$$

$$\mathcal{M}_\triangle = \{(x,y,z) \in \mathbb{R}^3_+\}$$

What is a moduli space good for?

Baby enumerative geometry question

How many triangles with vertices labelled A, B and C

- are isosceles;
- have at least one side of length 5; and
- have at least one side of length 7?

Define $X_{\mathrm{iso}} \subseteq \mathcal{M}_\triangle$, the locus of isosceles triangles.
Define $X_5 \subseteq \mathcal{M}_\triangle$, the locus of triangles with a side of length 5.
Define $X_7 \subseteq \mathcal{M}_\triangle$, the locus of triangles with a side of length 7.

Same question

What is $|X_{\mathrm{iso}} \cap X_5 \cap X_7|$?

Intuitive intersection theory (a.k.a. cohomology)

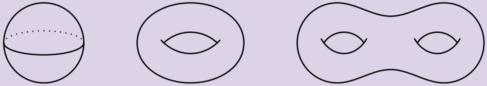
- An $(N-d)$-dimensional subset of an N-dimensional space is said to have codimension d.
- A "generic" intersection between subsets with codimension d_1 and codimension d_2 has codimension $d_1 + d_2$.
- A "generic" intersection between m subsets of an N-dimensional space with codimensions $d_1 + d_2 + \cdots + d_m = N$ is a set of points. The number of these points is called an intersection number.
- In order to obtain a well-defined intersection number, it is necessary to "jiggle the picture", "live in a compact space" and "count with signs".
- We will use the following (non-standard) notation for intersection numbers.

$$X_1 \cdot X_2 \cdots X_m = \begin{cases} |X_1 \cap X_2 \cap \ldots \cap X_m| & \text{if finite} \\ 0 & \text{otherwise} \end{cases}$$

The geometry of surfaces

TOPOLOGY

Topologists classified (compact, connected, orientable) surfaces by genus.



genus = 0 genus = 1 genus = 2

GEOMETRY

What does a geometer do with a surface? It depends...

algebraic geometry	**complex analysis**	**hyperbolic geometry**
algebraic structure up to isomorphism	complex structure up to biholomorphism	hyperbolic metric up to isometry
$\Downarrow$	$\Downarrow$	$\Downarrow$
algebraic curve	Riemann surface	hyperbolic surface

...but actually it doesn't, since these objects are all the same!

Moduli spaces of curves

$\mathcal{M}_{g,n}$ can be defined as the moduli space of

- genus g smooth algebraic curves with n marked points;
- genus g Riemann surfaces with n punctures; or
- genus g hyperbolic surfaces with n cusps.

The marked points or punctures or cusps are labelled from 1 up to n.

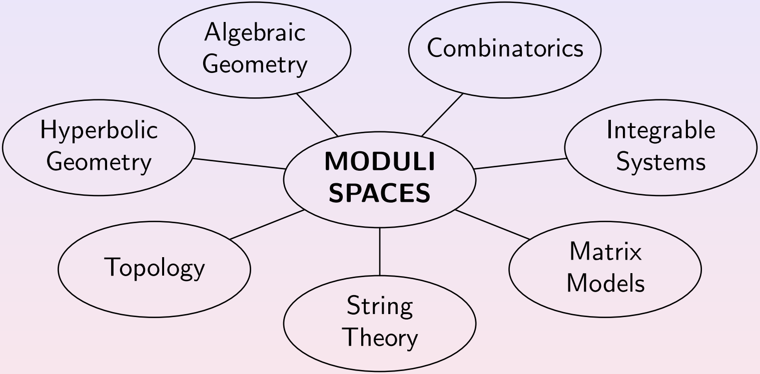
Three technical problems

Problem: $\mathcal{M}_{g,n}$ does not always exist
Solution: Do not allow $(g, n) = (0, 0)$, $(0, 1)$, $(0, 2)$ or $(1, 0)$

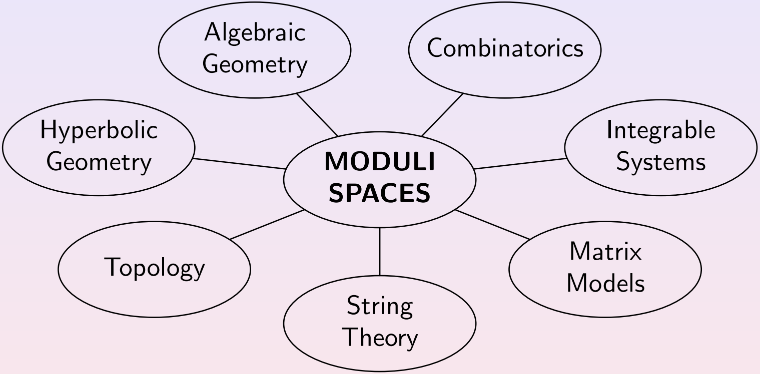
Problem: $\mathcal{M}_{g,n}$ is not compact
Solution: Use the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$
Points in $\overline{\mathcal{M}}_{g,n}$ correspond to stable algebraic curves

Problem: $\overline{\mathcal{M}}_{g,n}$ is not a manifold
Solution: Treat $\overline{\mathcal{M}}_{g,n}$ like an orbifold
Intersection numbers may be rational

Why do we care about moduli spaces of curves?



Why do we care about moduli spaces of curves?



BECAUSE THEY ARE INTERESTING AND FUN!

Psi-classes

Important fact

The dimension of $\overline{\mathcal{M}}_{g,n}$ is $6g - 6 + 2n$.

- The psi-classes $\psi_1, \psi_2, \ldots, \psi_n$ are codimension 2 subsets of $\overline{\mathcal{M}}_{g,n}$. In the more technical language of cohomology,

$$\psi_1, \psi_2, \ldots, \psi_n \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

- Choose non-negative integers $a_1 + a_2 + \cdots + a_n = 3g - 3 + n$ and consider the intersection number of psi-classes

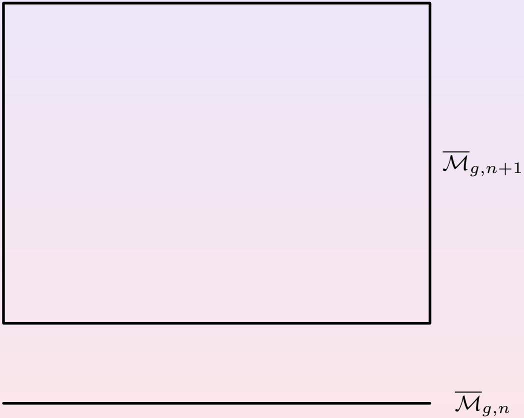
$$\psi_1^{a_1} \cdot \psi_2^{a_2} \cdots \psi_n^{a_n} \in \mathbb{Q}.$$

- There are various other important subsets of $\overline{\mathcal{M}}_{g,n}$ such as κ_1.

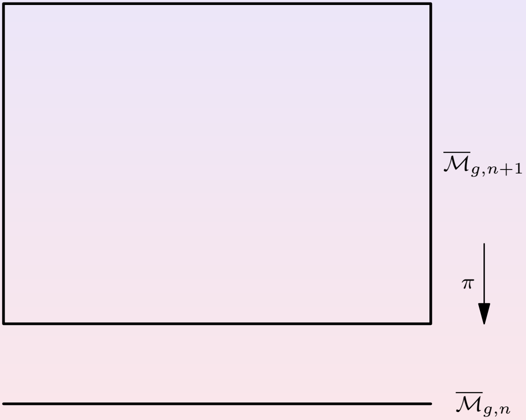
Examples of psi-class intersection numbers

On $\overline{\mathcal{M}}_{0,5}$, the intersection number $\psi_1 \cdot \psi_2$ is 2.
On $\overline{\mathcal{M}}_{1,1}$, the intersection number ψ_1 is $\frac{1}{24}$.

Constructing the psi-classes

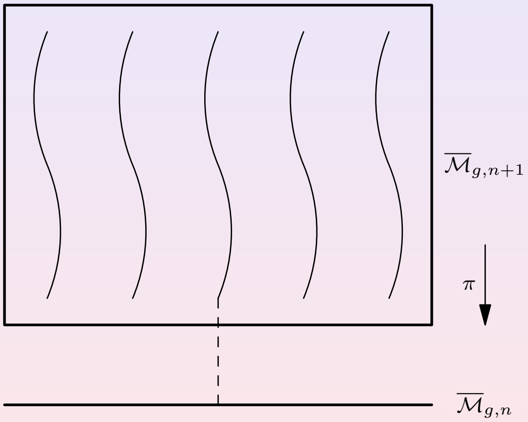


Constructing the psi-classes



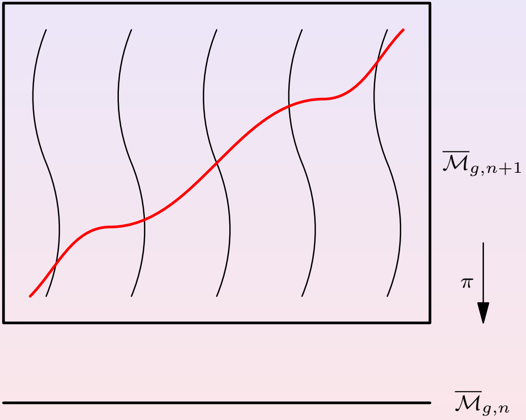
$\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is known as the forgetful map

Constructing the psi-classes



$\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is known as the forgetful map

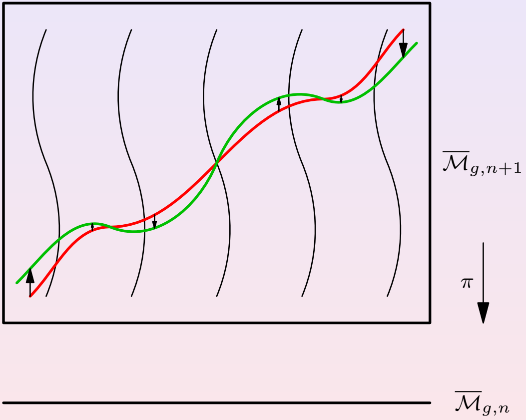
Constructing the psi-classes



$\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is known as the forgetful map

$\dim D_k = 6g - 6 + 2n$
$\text{codim } D_k = 2$ in $\overline{\mathcal{M}}_{g,n+1}$

Constructing the psi-classes

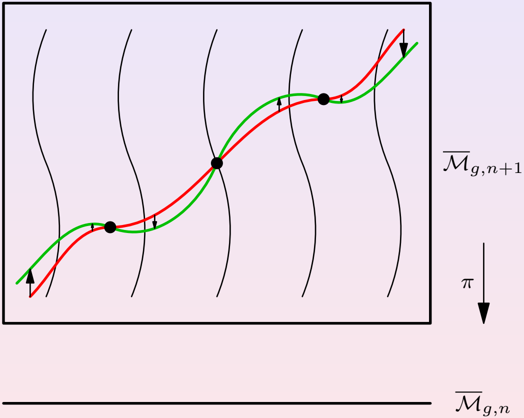


$\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is known as the forgetful map

$\dim D_k = 6g - 6 + 2n$
$\text{codim } D_k = 2$ in $\overline{\mathcal{M}}_{g,n+1}$

$\dim E_k = 6g - 6 + 2n$
$\text{codim } E_k = 2$ in $\overline{\mathcal{M}}_{g,n+1}$

Constructing the psi-classes



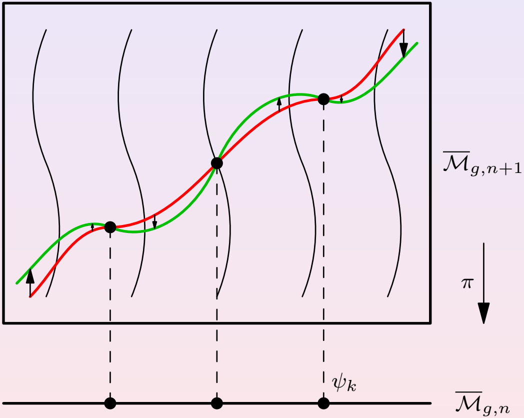
$\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is known as the forgetful map

$\dim D_k = 6g - 6 + 2n$
$\text{codim } D_k = 2$ in $\overline{\mathcal{M}}_{g,n+1}$

$\dim E_k = 6g - 6 + 2n$
$\text{codim } E_k = 2$ in $\overline{\mathcal{M}}_{g,n+1}$

$\text{codim } D_k \cap E_k = 4$ in $\overline{\mathcal{M}}_{g,n+1}$
$\text{codim } D_k \cap E_k = 2$ in D_k

Constructing the psi-classes



$\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is known as the forgetful map

$\dim D_k = 6g - 6 + 2n$
$\text{codim } D_k = 2$ in $\overline{\mathcal{M}}_{g,n+1}$

$\dim E_k = 6g - 6 + 2n$
$\text{codim } E_k = 2$ in $\overline{\mathcal{M}}_{g,n+1}$

$\text{codim } D_k \cap E_k = 4$ in $\overline{\mathcal{M}}_{g,n+1}$
$\text{codim } D_k \cap E_k = 2$ in D_k
$\text{codim } \psi_k = 2$ in $\overline{\mathcal{M}}_{g,n}$

A brief history of Witten's conjecture

Big question

How do you calculate intersection numbers of psi-classes?

- **Witten (1991):** I conjecture that if we put all of the intersection numbers of psi-classes into a generating function F, then F satisfies the infinite sequence of partial differential equations known as the KdV hierarchy.
- **Kontsevich (1992):** Witten is right! I have a formula which relates intersection numbers with ribbon graphs.
- **Okounkov and Pandharipande (2001):** Witten is right! We have a formula which relates intersection numbers with Hurwitz numbers.
- **Mirzakhani (2004):** Witten is right! I have a formula which relates intersection numbers with volumes of moduli spaces.

Volumes of moduli spaces

- Let $\mathcal{M}_{g,n}(L_1, L_2, \ldots, L_n)$ be the moduli space of genus g hyperbolic surfaces with n boundary components of lengths $L_1, L_2, \ldots, L_n$.
- The moduli space $\mathcal{M}_{g,n}(L_1, L_2, \ldots, L_n)$ has a natural symplectic structure — so one can measure its volume.
- Let $V_{g,n}(L_1, L_2, \ldots, L_n)$ be the volume of $\mathcal{M}_{g,n}(L_1, L_2, \ldots, L_n)$.

Mirzakhani's recursion

The volumes $V_{g,n}(L_1, L_2, \ldots, L_n)$ satisfy the following recursive formula.

$$2\frac{\partial}{\partial L_1} L_1 V_{g,n}(L_1, \ldots, L_n) = \int_0^\infty \int_0^\infty xy\ H(x+y, L_1)\ V_{g-1,n+1}(x, y, L_2, \ldots, L_n)\ dx\ dy$$

$$+ \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = [2,n]}} \int_0^\infty \int_0^\infty xy\ H(x+y, L_1)\ V_{g_1,|I_1|+1}(x, L_{I_1})\ V_{g_2,|I_2|+1}(y, L_{I_2})\ dx\ dy$$

$$+ \sum_{k=2}^n \int_0^\infty x[H(x, L_1+L_k) + H(x, L_1-L_k)]\ V_{g,n-1}(x, L_2, \ldots, \widehat{L}_k, \ldots, L_n)\ dx$$

One corollary of this formula is that $V_{g,n}(L_1, L_2, \ldots, L_n)$ is a polynomial.

From volumes to intersection numbers

Mirzakhani's theorem

The volume $V_{g,n}(L_1, L_2, \ldots, L_n)$ is given by the following formula.

$$\sum_{|a|+m=3g-3+n} \frac{(2\pi^2)^m \; \psi_1^{a_1} \cdot \psi_2^{a_2} \cdots \psi_n^{a_n} \cdot \kappa_1^m}{2^{|a|} a_1! a_2! \ldots a_n! m!} \; L_1^{2a_1} L_2^{2a_2} \ldots L_n^{2a_n}.$$

- Mirzakhani's recursion lets you calculate $V_{g,n}(L_1, L_2, \ldots, L_n)$.
- Mirzakhani's theorem says that $V_{g,n}(L_1, L_2, \ldots, L_n)$ is a polynomial whose coefficients store intersection numbers on $\overline{\mathcal{M}}_{g,n}$. The psi-class intersection numbers are stored in the top degree.
- Mirzakhani's recursion $+$ Mirzakhani's theorem $=$ Witten's conjecture
- **Philosophy: Any meaningful statement about $V_{g,n}$ gives a meaningful statement about intersection numbers on $\overline{\mathcal{M}}_{g,n}$.**

Examples of volume polynomials

$$V_{0,3} = 1$$

$$V_{0,4} = \frac{1}{2}(L_1^2 + L_2^2 + L_3^2 + L_4^2 + 4\pi^2)$$

$$V_{0,5} = \frac{1}{8}\sum L_i^4 + \frac{1}{2}\sum L_i^2 L_j^2 + 3\pi^2 \sum L_i^2 + 10\pi^4$$

$$V_{1,1} = \frac{1}{48}(L_1^2 + 4\pi^2)$$

$$V_{1,2} = \frac{1}{192}L_1^4 + \frac{\pi^2}{12}L_1^2 + \frac{\pi^4}{4} + \frac{\pi^2}{12}L_2^2 + \frac{1}{192}L_2^4 + \frac{1}{96}L_1^2 L_2^2$$

$$V_{2,1} = \frac{139\pi^4}{23040}L_1^4 + \frac{169\pi^6}{2880}L_1^2 + \frac{29\pi^8}{192} + \frac{29\pi^2}{138240}L_1^6 + \frac{1}{442368}L_1^8$$

New volume polynomial relations

Generalised string equation, generalised dilaton equation and more

The volume polynomials $V_{g,n+1}$ and $V_{g,n}$ satisfy the following relations.

$$V_{g,n+1}(L_1, \ldots, L_n, 2\pi i) = \sum_{k=1}^{n} \int L_k V_{g,n} \, dL_k \qquad \text{(GSE)}$$

$$\frac{\partial V_{g,n+1}}{\partial L_{n+1}}(L_1, \ldots, L_n, 2\pi i) = 2\pi i(2g-2+n) V_{g,n} \qquad \text{(GDE)}$$

$$\frac{\partial^2 V_{g,n+1}}{\partial L_{n+1}^2}(L_1, \ldots, L_n, 2\pi i) = \sum_{k=1}^{n} L_k \frac{\partial V_{g,n}}{\partial L_k} - (4g-4+n) V_{g,n}$$

$$\vdots$$

$$\frac{\partial^k V_{g,n+1}}{\partial L_{n+1}^k}(L_1, \ldots, L_n, 2\pi i) = [\ ???\ ]\, V_{g,n}$$

Three proofs

- **Algebraic geometry**
  Mirzakhani's theorem translates these results into relations between intersection numbers on $\overline{\mathcal{M}}_{g,n+1}$ and $\overline{\mathcal{M}}_{g,n}$. Such relations emerge from analysing the forgetful map $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$.

- **Mirzakhani's recursion**
  Mirzakhani's recursion determines all volumes $V_{g,n}(L_1, L_2, \ldots, L_n)$, so it should encode these relations in some sense. Interestingly, these proofs use identities among the Bernoulli numbers.

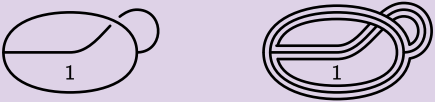
- **Hyperbolic geometry?**
  A purely imaginary length usually corresponds to an angle. So these results suggest a connection between intersection numbers on $\overline{\mathcal{M}}_{g,n}$ and the geometry of hyperbolic surfaces with cone points.

What is a ribbon graph?

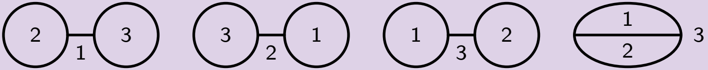
A ribbon graph of type (g, n) is

- a graph with a cyclic ordering of the edges at every vertex
- which can be thickened to give a surface of genus g and
- n boundary components labelled from 1 up to n.

Trivalent ribbon graph of type $(1, 1)$



Trivalent ribbon graphs of type $(0, 3)$



Kontsevich's combinatorial formula explained

Kontsevich's combinatorial formula

$$\sum_{|a|=3g-3+n} \psi_1^{a_1} \cdot \psi_2^{a_2} \cdots \psi_n^{a_n} \prod_{k=1}^{n} \frac{(2a_k-1)!!}{s_k^{2a_k+1}} = \sum_{\Gamma \in TRG_{g,n}} \frac{2^{2g-2+n}}{|\mathrm{Aut}(\Gamma)|} \prod_{e \in \Gamma} \frac{1}{s_{\ell(e)} + s_{r(e)}}$$

- LHS: polynomial in $\frac{1}{s_1}, \frac{1}{s_2}, \ldots, \frac{1}{s_n}$
  - coefficients store all intersection numbers of psi-classes on $\overline{\mathcal{M}}_{g,n}$
- RHS: rational function in $s_1, s_2, \ldots, s_n$
  - strange enumeration over trivalent ribbon graphs of type (g, n)
- Kontsevich's combinatorial formula is incredible!

Kontsevich's combinatorial formula for $g = 0$ and $n = 3$

- The LHS is easy.

$$LHS = \psi_1^0 \cdot \psi_2^0 \cdot \psi_3^0 \; \frac{1}{s_1 s_2 s_3}$$

- The RHS has one term for each trivalent ribbon graph of type $(0, 3)$.

$$\begin{aligned} RHS &= \frac{2}{2s_1(s_1+s_2)(s_1+s_3)} + \frac{2}{2s_2(s_2+s_3)(s_2+s_1)} \\ &\quad + \frac{2}{2s_3(s_3+s_1)(s_3+s_2)} + \frac{2}{(s_1+s_2)(s_2+s_3)(s_3+s_1)} \\ &= \frac{s_2 s_3(s_2+s_3) + s_3 s_1(s_3+s_1) + s_1 s_2(s_1+s_2) + 2 s_1 s_2 s_3}{s_1 s_2 s_3 (s_1+s_2)(s_2+s_3)(s_3+s_1)} \\ &= \frac{(s_1+s_2)(s_2+s_3)(s_3+s_1)}{s_1 s_2 s_3 (s_1+s_2)(s_2+s_3)(s_3+s_1)} \\ &= \frac{1}{s_1 s_2 s_3} \end{aligned}$$

- Conclusion: $\psi_1^0 \cdot \psi_2^0 \cdot \psi_3^0 = 1$.

A new approach to Kontsevich's combinatorial formula

Kontsevich's combinatorial formula simplified

INTERSECTION NUMBERS OF PSI-CLASSES	$\Longrightarrow$	RIBBON GRAPHS

Sketch proof: Step 1 of 3

INTERSECTION NUMBERS OF PSI-CLASSES	$\Longrightarrow$	ASYMPTOTICS OF VOLUMES OF MODULI SPACES

Recall that the intersection numbers of psi-classes are stored in the top degree coefficients of $V_{g,n}(L_1, L_2, \ldots, L_n)$. You can access the top degree coefficients of a polynomial using asymptotics.

Sketch proof: Step 2 of 3

ASYMPTOTICS OF VOLUMES OF MODULI SPACES	$\Longrightarrow$	HYPERBOLIC SURFACES WITH VERY LONG BOUNDARIES

To understand $V_{g,n}(N\ell_1, N\ell_2, \ldots, N\ell_n)$ for large N, one must understand hyperbolic surfaces with boundaries of lengths $N\ell_1, N\ell_2, \ldots, N\ell_n$ for large N.

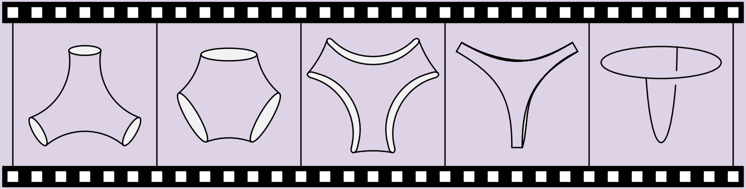
A new approach to Kontsevich's combinatorial formula

Sketch proof: Step 3 of 3

HYPERBOLIC SURFACES WITH VERY LONG BOUNDARIES $\Longrightarrow$ RIBBON GRAPHS

Fact: The Gauss–Bonnet theorem implies that all hyperbolic surfaces of genus g with n boundary components have the same surface area.

Crucial geometric reasoning: If you take a hyperbolic surface and stretch its boundary lengths to infinity, then you will obtain a ribbon graph after rescaling.



A problem about ribbon graphs and determinants

Ribbon graphs and determinants

Consider a trivalent ribbon graph of type (g, n). Colour n of the edges blue and the remaining edges red. Let A be the matrix formed from the adjacency between the blue edges and the boundaries. Let B be the matrix formed from the oriented adjacency between the red edges. Then

$$\det B = 2^{2g-2}(\det A)^2.$$

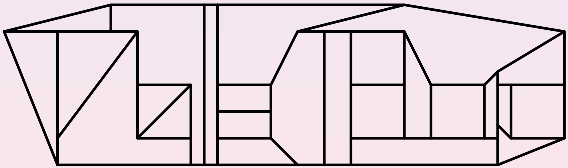
A problem about ribbon graphs and determinants

Ribbon graphs and determinants

Consider a trivalent ribbon graph of type (g, n). Colour n of the edges blue and the remaining edges red. Let A be the matrix formed from the adjacency between the blue edges and the boundaries. Let B be the matrix formed from the oriented adjacency between the red edges. Then

$$\det B = 2^{2g-2}(\det A)^2.$$

The following is a ribbon graph with $g = 0$ and $n = 26$.



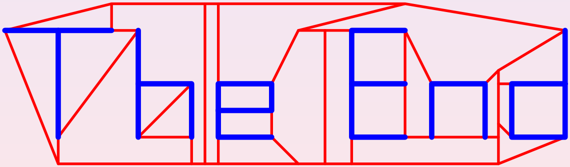
A problem about ribbon graphs and determinants

Ribbon graphs and determinants

Consider a trivalent ribbon graph of type (g, n). Colour n of the edges blue and the remaining edges red. Let A be the matrix formed from the adjacency between the blue edges and the boundaries. Let B be the matrix formed from the oriented adjacency between the red edges. Then

$$\det B = 2^{2g-2}(\det A)^2.$$

The following is a ribbon graph with $g = 0$ and $n = 26$.



$$\begin{array}{cccc} \det B & = & 2^{2g-2} & \times & (\det A)^2 \\ 256 & = & \frac{1}{4} & \times & 32^2 \end{array}$$

End matter

- **Slides**
  `http://www.ms.unimelb.edu.au/~norm`
- **Article**
  A tourist's guide to intersection theory on moduli spaces of curves
  To appear in the Australian Mathematical Society Gazette
  Volume 35, No. 2 (May 2008) or No. 3 (July 2008)
- **Seminar**
  Geometry and Topology seminar on 6 May 2008
- **Thesis**
  Coming soon!

Thanks to my supervisors:
Paul Norbury, Craig Hodgson and Iain Aitchison.