Intersection theory on moduli spaces of curves via hyperbolic geometry

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Submitted in total fulfilment of the requirements of the degree of Doctor of Philosophy

December 2008

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Produced on archival quality paper

Abstract

This thesis explores the intersection theory on $\overline{\mathcal{M}}_{g,n}$, the moduli space of genus g stable curves with n marked points. Our approach will be via hyperbolic geometry and our starting point will be the recent work of Mirzakhani.

One of the landmark results concerning the intersection theory on $\overline{\mathcal{M}}_{g,n}$ is Witten's conjecture. Kontsevich was the first to provide a proof, the crux of which is a formula involving combinatorial objects known as ribbon graphs. A subsequent proof, due to Mirzakhani, arises from considering $\mathcal{M}_{g,n}(\mathbf{L})$, the moduli space of genus g hyperbolic surfaces with n marked geodesic boundaries whose lengths are prescribed by $\mathbf{L} = (L_1, L_2, \ldots, L_n)$. Through the Weil–Petersson symplectic structure on this space, one can associate to it a volume $V_{g,n}(\mathbf{L})$. Mirzakhani produced a recursion which can be used to effectively calculate these volumes. Furthermore, she proved that $V_{g,n}(\mathbf{L})$ is a polynomial whose coefficients store intersection numbers on $\overline{\mathcal{M}}_{g,n}$. Her work allows us to adopt the philosophy that any meaningful statement about the volume $V_{g,n}(\mathbf{L})$ gives a meaningful statement about the intersection theory on $\overline{\mathcal{M}}_{g,n}$, and vice versa.

Two new results, known as the generalised string and dilaton equations, are introduced in this thesis. These take the form of relations between the Weil–Petersson volumes $V_{g,n}(\mathbf{L})$ and $V_{g,n+1}(\mathbf{L}, L_{n+1})$. Two distinct proofs are supplied — one arising from algebraic geometry and the other from Mirzakhani's recursion. However, the particular form of the generalised string and dilaton equations is highly suggestive of a third proof, using the geometry of hyperbolic cone surfaces. We briefly discuss ideas related to such an approach, although this largely remains work in progress. Applications of these relations include fast, effective algorithms to calculate the Weil–Petersson volumes in genus 0 and 1. We also deduce a formula for the volume $V_{g,0}$, a case not dealt with by Mirzakhani.

In this thesis, we also give a new proof of Kontsevich's combinatorial formula, relating the intersection theory on $\overline{\mathcal{M}}_{g,n}$ to the combinatorics of ribbon graphs. Mirzakhani's theorem suggests that the asymptotics of $V_{g,n}(\mathbf{L})$ store valuable information. We demonstrate that this information is precisely Kontsevich's combinatorial formula. Our proof involves using hyperbolic geometry to develop a combinatorial model for $\mathcal{M}_{g,n}(\mathbf{L})$ and to analyse the asymptotic behaviour of the Weil–Petersson symplectic form. The key geometric intuition involved is the fact that, as the boundary lengths of a hyperbolic surface approach infinity, the surface resembles a ribbon graph after appropriate rescaling of the metric. This work draws together Kontsevich's combinatorial approach and Mirzakhani's hyperbolic approach to Witten's conjecture into a coherent narrative.

Declaration

This is to certify that

(i) the thesis comprises only my original work towards the PhD except where indicated in the preface;

(ii) due acknowledgement has been made in the text to all other material used; and

(iii) the thesis is less than 100,000 words in length, exclusive of tables, maps, bibliographies and appendices.

Norman Do

Preface

Chapter 1 is a review of known results concerning moduli spaces of curves, with the appropriate references to the literature contained therein.

Chapter 2 includes results obtained in collaboration with Paul Norbury. More specifically, Section 2.1, Section 2.2 and parts of Section 2.4 contain the basic content of the joint paper [9], though substantially rewritten and elaborated on, from my own perspective. Section 2.3 and the remaining parts of Section 2.4 constitute wholly original work.

Chapter 3 is the product of my own research, although some of the ideas have been drawn from various sources in the literature. In particular, the proof of Theorem 3.9 parallels the work of Bowditch and Epstein [5]. Furthermore, the concluding arguments in this chapter follow the structure of Kontsevich's proof of his combinatorial formula [26]. We note that Section 3.4 essentially reproduces the results appearing in Appendix C of Kontsevich's paper, though the exposition is greatly expanded to include more thorough and more elementary proofs.

Acknowledgements

First and foremost, I would like to thank my principal supervisor, Paul Norbury. Over the years, he has been extremely generous with his time and remarkably insightful with his advice. Under his guidance, I have developed an increasingly accurate picture of what mathematics is about and what it is to be a mathematician.

I must have been quite a handful to supervise, since it was deemed necessary to assign two additional supervisors to me — namely, Craig Hodgson and Iain Aitchison. I would like to express my sincere gratitude to Craig for his patience, to Iain for his integrity, and to both for their generosity. Without them, my progress might have been even slower and my perspective more narrow.

The life of a graduate student in mathematics is often isolating, so I am indebted to those whom I could bounce ideas off. Particularly deserving of mention are Dan Mathews, Nick Sheridan and Tom Handscomb, even if they were not always able to fully grasp my unintelligible remarks. I would also like to thank Brad Safnuk for sharing his mathematical expertise with me, which greatly facilitated the writing of this thesis.

I have always been fortunate to have the overwhelming support of my family. In particular, I would like to dedicate this thesis to my parents. Without them, I would not be where I am today, in so many more ways than one.

Last, but certainly not least, I would like to thank my *raison d'être*, Denise Lin. Her steadfast support, unbounded love and beautiful smile inspire me each and every day.

Contents

Chapter 0

Overview

In this chapter, we provide an overview of the thesis, including a discussion of the most important results obtained, though without all of the gory details. The exposition is far from self-contained, so those without the requisite background may wish to skim through the chapter for a taste of what lies ahead. On the other hand, those acquainted with the language and methodology involved in the study of moduli spaces of curves should be able to ascertain the scope of this thesis.

A gentle introduction to moduli spaces of curves

In this thesis, we explore the fascinating world of intersection theory on moduli spaces of curves. The focus will be on the moduli space of genus g stable curves with n marked points, denoted by $\overline{\mathcal{M}}_{g,n}$. These geometric objects possess a rich structure and arise naturally in the study of algebraic curves and how they vary in families. One of the earliest results in the area dates back to Riemann, who essentially calculated that the real dimension of $\overline{\mathcal{M}}_{g,n}$ is $6g - 6 + 2n$. Subsequently, moduli spaces of curves have been studied by analysts, topologists and algebraic geometers, with each group contributing their own set of tools and techniques. There has been a recent surge of interest in moduli spaces of curves, catalysed by the discovery of their connection with string theory. As a result, they have become rather important objects of study in mathematics over the past couple of decades. In fact, moduli spaces of curves now lie at the centre of a rich confluence of somewhat disparate areas such as geometry, topology, combinatorics, integrable systems, matrix models and theoretical physics.

A natural approach to understanding the structure of geometric objects is through algebraic invariants, such as homology and cohomology. To this end, a great deal of research has been dedicated to the tautological ring of $\overline{\mathcal{M}}_{g,n}$. This is a subset of the cohomology ring $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$

which is far more tractable, yet retains much of the geometrically valuable information. Of central importance in the tautological ring are the psi-classes $\psi_1, \psi_2, \ldots, \psi_n \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$, which are defined as the Chern classes of certain natural complex line bundles over $\overline{\mathcal{M}}_{g,n}$. Taking cup products of the psi-classes and evaluating against the fundamental class, one obtains intersection numbers of the form

$$\langle \tau_{\alpha_1} \tau_{\alpha_2} \ldots \tau_{\alpha_n} \rangle = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \psi_2^{\alpha_2} \ldots \psi_n^{\alpha_n} \in \mathbb{Q}.$$

There is reason to believe that these psi-class intersection numbers, in some precise sense, store all of the information of the tautological ring. Therefore, their calculation is of tremendous significance to understanding the moduli space $\overline{\mathcal{M}}_{g,n}$.

One of the landmark results concerning the intersection theory on moduli spaces of curves is Witten's conjecture, now Kontsevich's theorem. In his foundational paper [57], Witten posited that a particular generating function for the psi-class intersection numbers is governed by the KdV hierarchy. There are two rather striking aspects of Witten's conjecture. The first is the fact that it arose from the analysis of a model of two-dimensional quantum gravity. This highlights the amazing interplay between pure mathematics and theoretical physics that emerges from the study of moduli spaces of curves. The second is the appearance of the KdV hierarchy, an infinite sequence of non-linear partial differential equations which begins with the KdV equation, the prototypical example of an exactly solvable model. This hints at an extraordinary amount of structure underlying moduli spaces of curves.

The year after Witten stated his conjecture, Kontsevich produced a proof as part of his doctoral thesis [26]. One of the main tools used was a cell decomposition of the decorated uncompactified moduli space $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$, where the cells are indexed by combinatorial objects known as ribbon graphs.[1] This allowed him to deduce a formula which combinatorialises the psi-class intersection numbers into an unconventional enumeration of trivalent ribbon graphs. From this point, Kontsevich was able to use Feynman diagram techniques and a particular matrix model to show that Witten's conjecture follows as a corollary.

Subsequently, several distinct proofs of Witten's conjecture have emerged. However, of particular relevance to this thesis is Mirzakhani's proof [33, 34], which adopts the approach of hyperbolic geometry. For an n-tuple of positive real numbers $\mathbf{L} = (L_1, L_2, \ldots, L_n)$, let $\mathcal{M}_{g,n}(\mathbf{L})$ denote the moduli space of genus g hyperbolic surfaces with n marked geodesic boundaries of lengths $L_1, L_2, \ldots, L_n$. This space possesses a symplectic structure via the Weil–Petersson symplectic form ω. Therefore, one can endow $\mathcal{M}_{g,n}(\mathbf{L})$ with a well-defined volume, which we denote by $V_{g,n}(\mathbf{L})$. It is natural to explore the behaviour of the symplectic structure and volume of $\mathcal{M}_{g,n}(\mathbf{L})$ as one varies the boundary lengths.

[1]The terms fatgraph and ribbon graph are used interchangeably in the literature. The notion was introduced by Penner [48] who coined the former, while Kontsevich adopted the latter. Which term to use is largely a matter of taste or, in the case of this author, a matter of habit.

The calculation of $V_{g,n}(\mathbf{L})$ in all generality was first performed by Mirzakhani, who accomplished the following.

- She produced a scheme for integrating a special class of functions over the moduli space $\mathcal{M}_{g,n}(\mathbf{L})$ and generalised McShane's identity, a remarkable formula concerning lengths of geodesics on a hyperbolic surface. Using these in conjunction, Mirzakhani managed to deduce a recursive formula from which the Weil–Petersson volumes could be effectively calculated. One corollary is the fact that $V_{g,n}(\mathbf{L})$ is an even symmetric polynomial in the variables $L_1, L_2, \ldots, L_n$ of degree $6g - 6 + 2n$.
- Mirzakhani used certain results concerning symplectic reduction in order to prove that

$$V_{g,n}(\mathbf{L}) = \sum_{|\boldsymbol{\alpha}|+m=3g-3+n} \frac{(2\pi^2)^m \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \psi_2^{\alpha_2} \ldots \psi_n^{\alpha_n} \kappa_1^m}{2^{|\boldsymbol{\alpha}|} \boldsymbol{\alpha}! m!} L_1^{2\alpha_1} L_2^{2\alpha_2} \ldots L_n^{2\alpha_n}.$$

  Here, and subsequently in this thesis, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ represents an n-tuple of non-negative integers. In addition, we will adopt the shorthand $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and $\boldsymbol{\alpha}! = \alpha_1! \alpha_2! \ldots \alpha_n!$. Note the appearance of $\kappa_1 \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$, which denotes the first Mumford–Morita–Miller class. The upshot of Mirzakhani's theorem is the fact that the volume $V_{g,n}(\mathbf{L})$ is a polynomial whose coefficients store intersection numbers on $\overline{\mathcal{M}}_{g,n}$. In particular, note that the intersection numbers of psi-classes alone are stored in the top degree part.

Of course, combining these two results yields a recursive procedure for calculating all psi-class intersection numbers. Therefore, it should come as little surprise that Mirzakhani was able to prove Witten's conjecture. What is surprising is that she accomplished this by directly verifying that Witten's generating function for the psi-class intersection numbers satisfies certain equations known as Virasoro constraints. Mirzakhani's proof was also the first to appear which did not require the use of a matrix model. But perhaps the most notable aspect of Mirzakhani's work is the fact that it is deeply rooted in hyperbolic geometry.

Weil–Petersson volume relations and hyperbolic cone surfaces

The main goal of this thesis is to explore intersection theory on moduli spaces of curves, using the work of Mirzakhani as a starting point. Her results allow us to adopt the philosophy that any meaningful statement about the volume $V_{g,n}(\mathbf{L})$ gives a meaningful statement about the intersection theory on $\overline{\mathcal{M}}_{g,n}$, and vice versa. The guiding viewpoint is that the approach of hyperbolic geometry has something to contribute to this theory. A particular consequence is that one may be able to find new relations among the intersection numbers on moduli spaces of curves. In this thesis, we introduce two such results, which relate the Weil–Petersson volumes $V_{g,n}(\mathbf{L})$ and $V_{g,n+1}(\mathbf{L}, L_{n+1})$. For reasons which will hopefully become clear, we refer to these as the generalised string and dilaton equations.[2]

Theorem 2.1 (Generalised string equation). *For $2g-2+n>0$, the Weil–Petersson volumes satisfy the following relation.*

$$V_{g,n+1}(\mathbf{L}, 2\pi i) = \sum_{k=1}^{n} \int L_k V_{g,n}(\mathbf{L})\, dL_k$$

Theorem 2.2 (Generalised dilaton equation). *For $2g-2+n>0$, the Weil–Petersson volumes satisfy the following relation.*

$$\frac{\partial V_{g,n+1}}{\partial L_{n+1}}(\mathbf{L}, 2\pi i) = 2\pi i(2g-2+n)V_{g,n}(\mathbf{L})$$

The particular form of these relations suggests various things. For example, their succinct nature is evidence that the volume polynomial $V_{g,n}(\mathbf{L})$ is a valuable way to package intersection numbers on $\overline{\mathcal{M}}_{g,n}$. Furthermore, it appears that these may be the first two in a sequence of equations which describe the derivatives of the Weil–Petersson volumes, with one argument evaluated at $2\pi i$. The appearance of the number $2\pi i$ itself indicates that there should be some interesting geometry underlying these results. In fact, we claim that a hyperbolic geometric approach is one of three natural strategies with which to prove the generalised string and dilaton equations.

- *Algebraic geometry.* By Mirzakhani's theorem, the coefficients of the volume polynomials $V_{g,n}(\mathbf{L})$ and $V_{g,n+1}(\mathbf{L}, L_{n+1})$ store intersection numbers on moduli spaces of curves. Thus, we may unravel the string and dilaton equations to obtain equivalent statements involving the intersection theory on $\overline{\mathcal{M}}_{g,n}$ and $\overline{\mathcal{M}}_{g,n+1}$. The advantage of the algebro-geometric approach is that there is a natural way to relate intersection numbers on these two spaces. Such relations arise from the analysis of cohomology classes under pull-back and push-forward by the forgetful morphism $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$, which forgets the last marked point. We prove the generalised string and dilaton equations in this manner.

- *Mirzakhani's recursion.* One way to pass from a hyperbolic surface with $n+1$ boundary components to a hyperbolic surface with n boundary components is to remove a pair of pants. This is essentially the mechanism by which Mirzakhani's recursive formula inductively reduces the calculation of $V_{g,n}(\mathbf{L})$. Since it governs all of the Weil–Petersson volumes, the generalised string and dilaton equations should be encapsulated, in some

[2] It is quite reasonable to wonder why the first theorem in Chapter 0 has been labelled Theorem 2.1. This is due to the fact that it also appears as the first theorem in Chapter 2. Indeed, throughout this chapter, all results have been numbered according to their subsequent appearance in the thesis.

sense, in her recursion. We show that this is indeed true and, furthermore, that this approach requires certain interesting relations concerning the Bernoulli numbers.

- *Hyperbolic cone surfaces.* Another way to pass from a hyperbolic surface with $n+1$ boundary components to a hyperbolic surface with n boundary components is to degenerate one of them to a cone point with cone angle 2π. The evaluation $L_{n+1} = 2\pi i$ in the generalised string and dilaton equations is highly suggestive that these relations may be proven using the geometry of hyperbolic cone surfaces. We briefly discuss these ideas, which should lead to new proofs and insights, although such an approach largely remains work in progress.

There are various simple applications of the generalised string and dilaton equations. For example, we prove that they uniquely determine $V_{0,n+1}(\mathbf{L}, L_{n+1})$ from $V_{0,n}(\mathbf{L})$ and $V_{1,n+1}(\mathbf{L}, L_{n+1})$ from $V_{1,n}(\mathbf{L})$. Together with the base cases $V_{0,3}(L_1, L_2, L_3)$ and $V_{1,1}(L_1) = \frac{1}{48}(L_1^2 + 4\pi^2)$, this results in an effective algorithm to compute the Weil–Petersson volumes in genus 0 and genus 1 much faster than implementing Mirzakhani's recursion. We also prove the following theorem, which includes a formula for $V_{g,0}$, a case not dealt with by Mirzakhani.

Theorem 2.12.

(i) When $n = 1$, the volume factorises as $V_{g,1}(L) = (L^2 + 4\pi^2)P_g(L)$ for some polynomial P_g.

(ii) For $g \geq 2$, we have the following formula.

$$V_{g,0} = \frac{1}{4\pi i(g-1)} \frac{\partial V_{g,1}}{\partial L}(2\pi i) = \frac{P_g(2\pi i)}{g-1}$$

As mentioned earlier, the generalised string and dilaton equations appear to be part of a potentially infinite sequence of relations. The search for these has uncovered the following equation involving the second derivative of the Weil–Petersson volumes. Of course, the hope is that there will be more to follow.

Proposition 2.14. *For $2g - 2 + n > 0$, the Weil–Petersson volumes satisfy the following relation.*

$$\frac{\partial^2 V_{g,n+1}}{\partial L_{n+1}^2}(\mathbf{L}, 2\pi i) = \left[\sum_{k=1}^{n} L_k \frac{\partial}{\partial L_k} - (4g - 4 + n) \right] V_{g,n}(\mathbf{L})$$

A new approach to Kontsevich's combinatorial formula

As mentioned earlier, the crux of Kontsevich's proof of Witten's conjecture is a formula which relates psi-class intersection numbers to combinatorial objects known as ribbon graphs. These are defined to be graphs with all vertices of degree at least three such that there is a cyclic ordering of the half-edges meeting at each vertex. This cyclic ordering allows one to thicken the graph in a well-defined manner to obtain a surface with boundary. We say that the ribbon graph is of type (g,n) if this results in a connected surface with genus g and n boundary components labelled from 1 up to n. For a ribbon graph Γ, there is the notion of its group of automorphisms, which is denoted by Aut Γ. Let the set of ribbon graphs of type (g,n) be $RG_{g,n}$ and the subset consisting of trivalent graphs be $TRG_{g,n}$. With this notation in place, we may state Kontsevich's combinatorial formula as follows.

Theorem 3.4 (Kontsevich's combinatorial formula). *The psi-class intersection numbers on $\overline{\mathcal{M}}_{g,n}$ satisfy the following formula.*

$$\sum_{|\boldsymbol{\alpha}|=3g-3+n} \langle \tau_{\alpha_1}\tau_{\alpha_2}\ldots\tau_{\alpha_n}\rangle \prod_{k=1}^{n} \frac{(2\alpha_k-1)!!}{s_k^{2\alpha_k+1}} = \sum_{\Gamma\in TRG_{g,n}} \frac{2^{2g-2+n}}{|Aut\ \Gamma|} \prod_{e\in E(\Gamma)} \frac{1}{s_{\ell(e)}+s_{r(e)}}$$

Here, $E(\Gamma)$ denotes the set of edges of Γ and the expression $(2\alpha-1)!!$ is a shorthand for $\frac{(2\alpha)!}{2^\alpha \alpha!}$. For an edge e, the terms $\ell(e)$ and $r(e)$ are the labels of the boundaries on its left and right.

Observe that the left hand side of Kontsevich's combinatorial formula is a polynomial in the variables $\frac{1}{s_1}, \frac{1}{s_2}, \ldots, \frac{1}{s_n}$ whose coefficients store all psi-class intersection numbers on $\overline{\mathcal{M}}_{g,n}$. The right hand side can be considered a particular enumeration of trivalent ribbon graphs of type (g,n) which, a priori, appears only to be a rational function of $s_1, s_2, \ldots, s_n$. That the two sides concur is quite a remarkable phenomenon.

In this thesis, we give a new proof of Kontsevich's combinatorial formula via hyperbolic geometry. Our starting point is Mirzakhani's theorem, which relates intersection numbers on moduli spaces of curves to volumes of moduli spaces of hyperbolic surfaces. In particular, we observed earlier that the psi-class intersection numbers on $\overline{\mathcal{M}}_{g,n}$ are stored in the top degree part of $V_{g,n}(\mathbf{L})$. This suggests that it may be useful to consider the asymptotics of the Weil–Petersson volumes. In fact, a directly corollary of Mirzakhani's theorem is the fact that the Laplace transform of the asymptotics

$$\mathcal{L}\left\{\lim_{N\to\infty} \frac{V_{g,n}(Nx_1, Nx_2, \ldots, Nx_n)}{N^{6g-6+2n}}\right\}$$

is precisely the left hand side of Kontsevich's combinatorial formula. It practically goes without saying that we then wish to prove that this expression is also equal to the right hand side. The proof can be conceptually divided into three main parts.

- *Part 1: Why do we obtain a sum over trivalent ribbon graphs?*

  Embedded on a hyperbolic surface with geodesic boundary is a ribbon graph, formed from the set of points with at least two shortest paths to the boundary. In fact, there is a way to use the hyperbolic structure to assign a positive real number to every edge so that one obtains what is referred to as a metric ribbon graph. We refer to this number as the length of the edge and the sum of the numbers around a boundary as the length of the boundary. The space of metric ribbon graphs of type (g, n) with boundary lengths prescribed by $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ forms a topological space — in fact, an orbifold — which we denote by $\mathcal{MRG}_{g,n}(\mathbf{x})$. Using an argument analogous to that of Bowditch and Epstein which appears in [5], we prove the following.

Theorem 3.9. *The spaces $\mathcal{M}_{g,n}(\mathbf{x})$ and $\mathcal{MRG}_{g,n}(\mathbf{x})$ are homeomorphic as orbifolds.*

Therefore, one may equivalently consider $\mathcal{MRG}_{g,n}(\mathbf{x})$ rather than $\mathcal{M}_{g,n}(\mathbf{x})$. One advantage is that the space of metric ribbon graphs possesses the natural orbifold cell decomposition

$$\mathcal{MRG}_{g,n}(\mathbf{x}) = \bigcup_{\Gamma \in RG_{g,n}} \mathcal{MRG}_{\Gamma}(\mathbf{x}),$$

where a metric ribbon graph lies in the set $\mathcal{MRG}_{\Gamma}(\mathbf{x})$ if its underlying ribbon graph coincides with Γ. Furthermore, $\mathcal{MRG}_{\Gamma}(\mathbf{x})$ is top-dimensional if and only if the ribbon graph Γ is trivalent. Since the volume does not care about cells of positive codimension, it can be expressed as a sum over trivalent ribbon graphs as follows, where $V_{\Gamma}(\mathbf{x})$ denotes the Weil–Petersson volume of $\mathcal{MRG}_{\Gamma}(\mathbf{x})$.

$$V_{g,n}(\mathbf{x}) = \sum_{\Gamma \in TRG_{g,n}} V_{\Gamma}(\mathbf{x})$$

- *Part 2: Why do we obtain a product over the edges of the trivalent ribbon graph?*

Fix a ribbon graph $\Gamma \in TRG_{g,n}$ and an n-tuple of positive real numbers $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Then for every positive real number N, we have the map

$$f : \mathcal{MRG}_{\Gamma}(\mathbf{x}) \to \mathcal{MRG}_{\Gamma}(N\mathbf{x}) \to \mathcal{M}_{g,n}(N\mathbf{x}),$$

which is a homeomorphism onto its image. This is the composition of two maps — the first scales the ribbon graph metric by a factor of N while the second uses the Bowditch–Epstein construction. Consider the normalised Weil–Petersson symplectic form $\frac{\omega}{N^2}$ on $\mathcal{M}_{g,n}(N\mathbf{x})$ and note that it pulls back via f to a symplectic form on $\mathcal{MRG}_{\Gamma}(\mathbf{x})$. The behaviour of this 2-form as N approaches infinity is described by the theorem below. The proof of this statement, which relies on a mixture of hyperbolic geometry and combinatorics, is one of the main technical contributions in this part of the thesis. The key geometric intuition involved is the fact that, as the boundary lengths of a hyperbolic surface approach infinity, the surface resembles a ribbon graph after appropriate rescaling of the metric. It has been brought to our attention that this result, with an alternative proof, has also appeared recently in the work of Mondello [36].

Theorem 3.20. *In the $N \to \infty$ limit, $\frac{f^*\omega}{N^2}$ converges pointwise on $\mathcal{MRG}_{\Gamma}(\mathbf{x})$ to a 2-form Ω.*

Note that the space of metric ribbon graphs $\mathcal{MRG}_\Gamma(\mathbf{x})$ possesses a tractable system of coordinates, provided by the edge lengths. In particular, with respect to these coordinates, Ω is a constant 2-form. Furthermore, $\mathcal{MRG}_\Gamma(\mathbf{x})$ has a simple description — it is the quotient of a polytope by the action of the finite group Aut Γ. Therefore, modulo some analytical details, we have the equality

$$\lim_{N\to\infty} \frac{V_\Gamma(N\mathbf{x})}{N^{6g-6+2n}} = \int_{\mathcal{MRG}_\Gamma(\mathbf{x})} \frac{\Omega^{3g-3+n}}{(3g-3+n)!}.$$

From the previous discussion, this is essentially the integral of a constant volume form over a polytope. The integration is most easily evaluated after taking the Laplace transform, which results in the desired product over edges of Γ. Furthermore, the action of the finite group Aut Γ on this polytope naturally introduces the factor of $\frac{1}{|\text{Aut}\,\Gamma|}$ which appears on the right hand side of Kontsevich's combinatorial formula.

- *Part 3: Where does the combinatorial constant come from?*

Interestingly, the remaining factor of 2^{2g-2+n} on the right hand side of Kontsevich's combinatorial formula is no simple matter to explain. In fact, its appearance boils down to the following statement.

Theorem 3.22. *Let Γ be a trivalent ribbon graph of type (g, n) with n edges coloured white and the remaining $6g-6+2n$ edges coloured black. Let A be the $n \times n$ adjacency matrix formed between the faces and the white edges. Let B be the $(6g-6+2n) \times (6g-6+2n)$ oriented adjacency matrix formed between the black edges. Then*

$$\det B = 2^{2g-2}(\det A)^2.$$

Perhaps surprisingly, there does not exist a purely combinatorial proof of this statement in the literature. Instead, we essentially follow the argument from Appendix C of Kontsevich's paper [26], which uses the torsion of an acyclic chain complex associated to a trivalent ribbon graph. However, our exposition is greatly expanded to make the proof both more thorough and more elementary.

Of course, Kontsevich's combinatorial formula in itself is not a new result. What is novel, in this part of the thesis, is the hyperbolic geometric approach and the explicit connection between the work of Kontsevich and Mirzakhani. We believe that this proof of Kontsevich's combinatorial formula is rather intuitive in nature, avoids the technical difficulties which are inherent in the original proof, and may lead to further insights. Indeed, as a part of joint work with Safnuk [10], we have extended these ideas to produce a recursive formula à la Mirzakhani which computes the asymptotics of $V_{g,n}(\mathbf{L})$. The differential version of this formula is the Virasoro constraint condition, thereby providing a new path to Witten's conjecture. We also believe that it will not be difficult to extend these ideas to integration over the combinatorially defined Witten cycles.

Chapter 1

A gentle introduction to moduli spaces of curves

In this chapter, we introduce the main characters of our story, moduli spaces of curves. The aim is to provide a concise exposition of the important results and ideas which form the background to this thesis. Newcomers to the area will hopefully find this chapter a suitable point of entry to the now vast body of knowledge concerning moduli spaces of curves. However, the selection of material presented here is necessarily only a small subset, chosen to suit our specific needs and reflect our particular point of view. For example, a great deal of attention has been paid to intersection theory on moduli spaces of curves, to the interaction between algebraic and hyperbolic geometry, and to the recent results of Mirzakhani. Throughout the chapter, details and proofs have often been omitted for the sake of clarity and space. For those interested in further information, there are numerous references to the relevant sources in the literature.[1]

1.1 Moduli spaces of curves

First principles

Informally, the points of a moduli space classify objects of a certain type, while its geometry reflects the way in which these objects can vary in families. For example, consider the moduli space

$$\mathcal{M}_g = \{C \mid C \text{ is a smooth algebraic curve of genus } g\} / \sim$$

[1]In particular, we start by mentioning the articles [43, 55, 56] which have influenced our exposition and which are suitable for those wishing to discover this remarkable area of mathematics for the first time.

where $C \sim D$ if and only if there exists an isomorphism from C to D.[2] Now suppose that $\phi : X \to B$ is a family of smooth curves of genus g. (More precisely, ϕ should be a proper flat morphism of schemes whose fibres are smooth curves of genus g.) Then we can form the map $f : B \to \mathcal{M}_g$ which sends b to the point in $\mathcal{M}_g$ that represents the equivalence class of the fibre over b. One would certainly expect this map to be continuous, and the set $\mathcal{M}_g$ can be endowed with a well-defined topology such that this is true. Furthermore, there should be a family $\pi : \mathcal{C}_g \to \mathcal{M}_g$ of smooth curves of genus g such that its pull-back via f produces the original family over B. Such a family is referred to as the universal family over $\mathcal{M}_g$.

$$\begin{array}{ccc} X & & \mathcal{C}_g \\ \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle \pi} \\ B & \xrightarrow{\ f\ } & \mathcal{M}_g \end{array} \qquad\qquad X = f^*\mathcal{C}_g$$

Conversely, every map from B to $\mathcal{M}_g$ gives rise to a family of smooth curves of genus g over B by pulling back the universal family. This provides us with an extremely useful dictionary correspondence which translates statements about families of curves to statements about the geometry of the corresponding moduli space.[3]

Unfortunately, if one adopts this somewhat naive point of view, then certain technical issues arise in the construction of moduli spaces of curves. The root of these evils is the fact that some algebraic curves possess non-trivial automorphisms. We will address three particular problems caused by such curves, and these will lead us to develop the more refined notion of the moduli stack $\mathcal{M}_{g,n}$.

- Since there is only one smooth genus 0 curve up to isomorphism, one might expect $\mathcal{M}_0$ to be a point. This, in turn, would imply that the pull-back of the universal family over any base would be trivial. However, it is clear that there exist locally trivial yet globally non-trivial families with fibre $\mathbb{CP}^1$. The discrepancy is due to the fact that $\mathbb{CP}^1$ has a large automorphism group which allows trivial pieces to be glued together in a non-trivial way. One solution to this problem is to consider curves with sufficiently many marked points to ensure that their automorphism groups are finite. Therefore, we define the moduli space

$$\mathcal{M}_{g,n} = \left\{ (C, p_1, p_2, \ldots, p_n) \;\middle|\; \begin{array}{l} C \text{ is a smooth algebraic curve of genus } g \\ \text{with } n \text{ distinct points } p_1, p_2, \ldots, p_n \end{array} \right\} \Big/ \sim$$

[2]We decree that all algebraic curves referred to in this thesis are to be complex, connected and complete.

[3]The informed reader will hopefully be reminded of the relationship between vector bundles and Grassmannians, whose theory may be considered a paradigm for the theory of moduli spaces of curves. When studying Grassmannians, one is normally motivated to consider their intersection theory, which has a rich structure relating to combinatorics and representation theory. In a similar vain, we consider the intersection theory on moduli spaces of curves, which has a similarly rich structure, but of a very different nature.

where $(C, p_1, p_2, \ldots, p_n) \sim (D, q_1, q_2, \ldots, q_n)$ if and only if there exists an isomorphism from C to D which sends p_k to q_k for all k. The resulting equivalence classes are referred to as pointed curves. Note that the automorphism group of a pointed curve is finite as long as there are at least three marked points in the case of genus 0 and at least one marked point in the case of genus 1. As a result, we will only consider the moduli spaces $\mathcal{M}_{g,n}$ which satisfy the Euler characteristic condition $2 - 2g - n < 0$. Pointed curves arise naturally in many geometric situations — for example, given a family of curves with a number of disjoint sections — so the added effort required to keep track of the marked points is outweighed by the added benefit.

- Later in this chapter, we discuss the construction of the moduli space $\mathcal{M}_{g,n}$ using an approach from Teichmüller theory. There are actually a few different constructions, although there is one common feature underlying them all. They each consider pointed curves endowed with some extra structure, so that the corresponding parameter space is a manifold. Taking the quotient of this space by the relation which identifies the additional structures then yields the moduli space as the quotient of a manifold by a group action. In all such constructions, there necessarily exist points where the action is not free, and these correspond precisely to those curves with non-trivial automorphism group. Therefore, $\mathcal{M}_{g,n}$ is naturally an orbifold, where the orbifold group at a point is canonically isomorphic to the automorphism group of the corresponding pointed curve. However, the situation is not so bad, since the following theorem — due to Boggi and Pikaart [4] and also to Looijenga [28] — allows one to make sense of calculations on the orbifold by lifting to a finite cover.

  Theorem 1.1. *There exists a finite cover $\widetilde{\mathcal{M}}_{g,n} \to \mathcal{M}_{g,n}$ such that $\widetilde{\mathcal{M}}_{g,n}$ is a smooth manifold.*

  The upshot is that many of the techniques and theorems which apply to manifolds can be carried over to the orbifold setting, with only the obvious modifications required. For our purposes, note that it is necessary to consider the cohomology of $\mathcal{M}_{g,n}$ with rational rather than integral coefficients.

- There is no universal family $\mathcal{C}_{g,n} \to \mathcal{M}_{g,n}$, at least not in the sense that we have described. In fact, the fibre over a point in $\mathcal{M}_{g,n}$ corresponding to a pointed curve C with automorphism group G is C/G. Since curves with non-trivial automorphisms occur naturally in families, this causes a real problem. The crux of the matter is that the universal family cannot be constructed in the category of schemes, where most algebraic geometers live. However, the issue disappears as long as one is willing to work in the category of Deligne–Mumford stacks, first introduced in [8]. This category is ideal in the sense that it is just large enough to allow for the construction of the universal family while still being restricted enough to retain geometric concepts such as smoothness, vector bundles, cohomology, and so on.

Admittedly, the definition of a Deligne–Mumford stack is rather technical. To present it here would take us too far afield from our goal and may even obscure the geometric nature of moduli spaces. However, one will not go too far wrong thinking of a smooth Deligne–Mumford stack as a complex orbifold. The stack structure takes care of the extra bookkeeping required to deal with curves with non-trivial automorphisms. Essentially, this will allow us to treat the moduli space as smooth and ensure that there is a universal family over it. The reader interested in discovering more on stacks is encouraged to consult [14] and the references contained therein.

Those troublesome curves with non-trivial automorphisms have led us to consider moduli stacks of pointed curves. However, there is still one outstanding issue which needs to be addressed — $\mathcal{M}_{g,n}$ is not compact. To see this, observe that when two marked points on a pointed curve approach each other, then the corresponding limit does not exist in $\mathcal{M}_{g,n}$. Of course, there are numerous advantages in working with a compact space. And although there are various ways to compactify the moduli space, it is becoming increasingly clear that the most profitable is to use what is known as the Deligne–Mumford compactification. One of its virtues is that it is modular — in other words, it is a moduli space for a well-behaved, easy to describe, class of curves. In fact, all we need to do is gently relax our smoothness condition. Thus, we define the Deligne–Mumford compactification of the moduli space

$$\overline{\mathcal{M}}_{g,n} = \left\{ (C, p_1, p_2, \ldots, p_n) \;\middle|\; \begin{array}{l} C \text{ is a stable algebraic curve of genus } g \text{ with} \\ n \text{ distinct smooth points } p_1, p_2, \ldots, p_n \end{array} \right\} \Big/ \sim$$

where $(C, p_1, p_2, \ldots, p_n) \sim (D, q_1, q_2, \ldots, q_n)$ if and only if there exists an isomorphism from C to D which sends p_k to q_k for all k. Here, an algebraic curve is called stable if it has at worst nodal singularities and a finite automorphism group. The practical interpretation of this latter condition is that every rational component of the curve must have at least three special points, where a special point refers to a node or a marked point. It should be clear that $\mathcal{M}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n}$, and we refer to $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ as the boundary divisor. Theorem 1.1 can be extended to the compactification as follows.

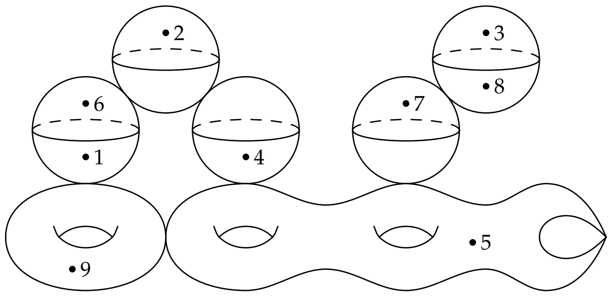
Theorem 1.2. *There exists a finite cover $\widetilde{\overline{\mathcal{M}}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ such that $\widetilde{\overline{\mathcal{M}}}_{g,n}$ is a smooth manifold. Furthermore, the boundary divisor lifts to a union of codimension two submanifolds intersecting transversally.*

The question still remains as to which curve arises in the limit when two marked points approach each other. To see what the correct answer should be, consider a family of genus g curves with n marked points $\phi : X \to B$, where B is smooth and of complex dimension 1. The marked points give rise to n sections which we denote by $\sigma_1, \sigma_2, \ldots, \sigma_n : B \to X$. Suppose that all fibres are stable pointed curves apart from over the point b where the sections σ_i and σ_j intersect. In order to obtain a family of stable curves, one simply needs to blow up the surface X at the point $\sigma_i(b) = \sigma_j(b)$ to obtain $\pi : \widetilde{X} \to X$. The new family of curves is given by

$\widetilde{\phi} : \widetilde{X} \to B$ where $\widetilde{\phi} = \phi \circ \pi$. Note that the fibres are unchanged away from b, whereas the new fibre over b consists of the old fibre plus the exceptional divisor obtained from blowing up. Furthermore, the points $\sigma_i(b)$ and $\sigma_j(b)$ now lie on the exceptional divisor and are distinct as long as the sections σ_i and σ_j intersected transversally. Therefore, in the limit when two marked points approach each other, a $\mathbb{CP}^1$ bubbles off, containing these two marked points. This is a particular instance of the more general process known as stable reduction.[4]

Theorem 1.3 (Stable reduction). *Let b be a point on a smooth curve B and suppose that X is a family of stable curves over $B \setminus \{b\}$. Then after a sequence of blow-ups and blow-downs and passing to a branched cover of B, one can obtain a new family where all fibres are stable curves. Furthermore, the fibre over b in this new family is uniquely determined.*

More complicated stable pointed curves arise from more complicated limits, such as the example shown in the following diagram.[5]



We note now the important foundational result that

$$\dim_{\mathbb{R}} \overline{\mathcal{M}}_{g,n} = 6g - 6 + 2n,$$

a calculation which dates back to Riemann. Later in this chapter, we will see how the uncompactified moduli space $\mathcal{M}_{g,n}$ actually arises as the quotient of an open ball of real dimension $6g - 6 + 2n$ by a properly discontinuous group action.

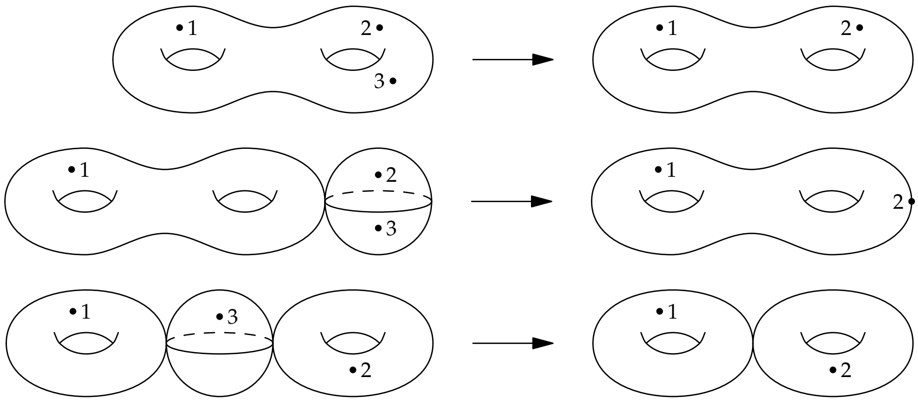
Natural morphisms

One interesting and useful aspect of moduli spaces of pointed curves is the interplay between them. As a simple example of this phenomenon, note that if $g \leq g'$ and $n \leq n'$, then $\overline{\mathcal{M}}_{g,n}$ can be considered a subvariety of $\overline{\mathcal{M}}_{g',n'}$. The following natural morphisms between moduli spaces of curves are of particular importance.

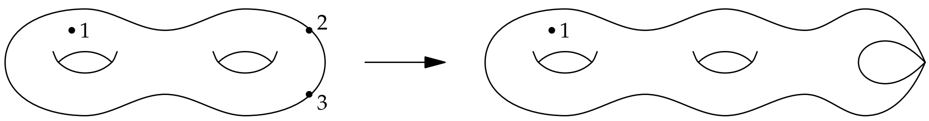
[4] For further details on stable reduction, one need look no further than [22]. In fact, the book is an excellent reference on the algebraic geometry of moduli spaces of curves in general.

[5] Recall that the singularities of a stable pointed curve must be nodal and, hence, locally look like $xy = 0$ at the origin. In order to represent such a singularity on a two-dimensional page, one usually resorts to drawing the two curves as pinched or tangent, both of which are misleading.

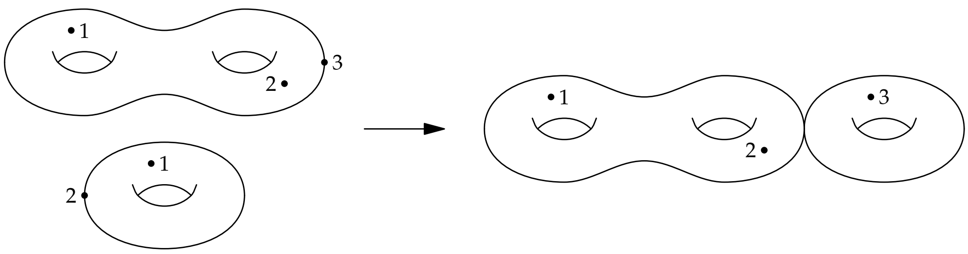
- *Forgetful morphism.* Given a stable genus g curve with $n+1$ marked points, one can forget the point labelled $n+1$ to obtain a genus g curve with n marked points. Unfortunately, the resulting curve may not be stable, but gives rise to a well-defined stable curve after contracting all rational components with only two special points. This gives a map $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$, which we refer to as a forgetful morphism.



- *Gluing morphism I.* Given a stable genus g curve with $n+2$ marked points, one can glue together the points labelled $n+1$ and $n+2$ to obtain a stable genus g curve with n marked points. This gives a map $gl_1 : \overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g+1,n}$ which we refer to as a gluing morphism.



- *Gluing morphism II.* Given a stable genus g_1 curve with n_1+1 marked points and a stable genus g_2 curve with n_2+1 marked points, one can glue together the last two points to obtain a stable genus g_1+g_2 curve with n_1+n_2 marked points. This gives a map $gl_2 : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$ which we also refer to as a gluing morphism.



- *Permutation morphism.* Given a stable genus g curve with n marked points and a permutation $\sigma \in S_n$, one can use σ to permute the labels on the marked points. This gives a map — in fact, an isomorphism — $P_\sigma : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ which we refer to as a permutation morphism.

Note that the forgetful morphism $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ can be interpreted as the universal family $\pi : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$. In other words, one can take a stable curve $C \in \overline{\mathcal{M}}_{g,n}$ along with a point p on C and associate a stable curve $\widetilde{C} \in \overline{\mathcal{M}}_{g,n+1}$ to the pair (C, p).

- If p is a smooth unmarked point of C, then let $\widetilde{C}$ be the curve C with the point p labelled $n+1$.
- If p is the point of C labelled k, then let $\widetilde{C}$ be the curve C with a $\mathbb{CP}^1$ bubbled off at the point p, containing points labelled k and $n+1$.
- If p is a nodal point of C, then let $\widetilde{C}$ be the curve C with a $\mathbb{CP}^1$ bubbled off at the point p, containing a point labelled $n+1$.

In this way, the point labelled k defines a section $\sigma_k : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$ for $k = 1, 2, \ldots, n$. Furthermore, the image of σ_k consists of all curves in $\overline{\mathcal{M}}_{g,n+1}$ with a $\mathbb{CP}^1$ bubbled off, containing points labelled k and $n+1$, and no other marked points.

The forgetful morphism can be used to pull back cohomology classes, but it will also be useful to push them forward. This is possible via the Gysin map $\pi_* : H^*(\overline{\mathcal{M}}_{g,n+1}) \to H^*(\overline{\mathcal{M}}_{g,n})$, which is a homomorphism of graded rings with grading -2. One can interpret the Gysin map as integrating along fibres but it can be alternatively defined by the following composition of maps, where $d = \dim(\overline{\mathcal{M}}_{g,n+1})$ and the outer maps denote Poincaré duality.

$$\pi_* : H^k(\overline{\mathcal{M}}_{g,n+1}) \xrightarrow{PD} H_{d-k}(\overline{\mathcal{M}}_{g,n+1}) \xrightarrow{\pi_*} H_{d-k}(\overline{\mathcal{M}}_{g,n}) \xrightarrow{PD} H^{k-2}(\overline{\mathcal{M}}_{g,n})$$

One of the nice properties enjoyed by the Gysin map is the push-pull formula, which states that $\pi_*(\alpha \pi^* \beta) = \pi_*(\alpha)\beta$ for $\alpha \in H^*(\overline{\mathcal{M}}_{g,n+1})$ and $\beta \in H^*(\overline{\mathcal{M}}_{g,n})$. Another property that we will make use of is the fact that

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \eta = \int_{\overline{\mathcal{M}}_{g,n}} \pi_* \eta.$$

Small examples

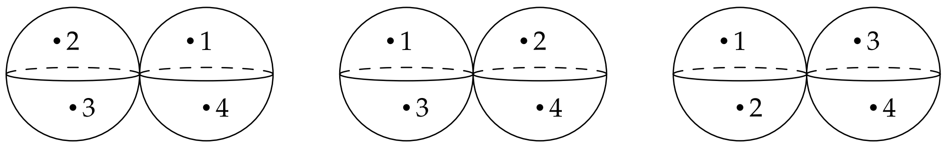
In general, moduli spaces of curves are not only of high dimension, but also possess a very complicated structure. However, there are a handful of cases for which we can provide a concrete description of their geometry. We conclude this section with some of these small examples, which are often useful to keep in mind.

Example 1.4 (The moduli space $\overline{\mathcal{M}}_{0,3}$)**.** The only smooth genus 0 algebraic curve up to isomorphism is $\mathbb{CP}^1$ and the action of its automorphism group is sharply transitive on triples of points. In other words, every smooth rational curve with three marked points (C, p_1, p_2, p_3) can be mapped isomorphically to $(\mathbb{CP}^1, 0, 1, \infty)$ in a unique manner. It follows that $\mathcal{M}_{0,3}$ is a point and, since there are no stable nodal curves of genus 0 with three marked points, it also follows that $\overline{\mathcal{M}}_{0,3}$ is a point.

Example 1.5 (The moduli space $\overline{\mathcal{M}}_{0,4}$)**.** Similarly, every smooth genus 0 curve with four marked points (C, p_1, p_2, p_3, p_4) can be mapped isomorphically to $(\mathbb{CP}^1, 0, 1, \infty, \lambda)$ in a unique manner for some $\lambda \notin \{0, 1, \infty\}$. In fact, λ can be interpreted as the cross-ratio of the quadruple (p_1, p_2, p_3, p_4) through the equation

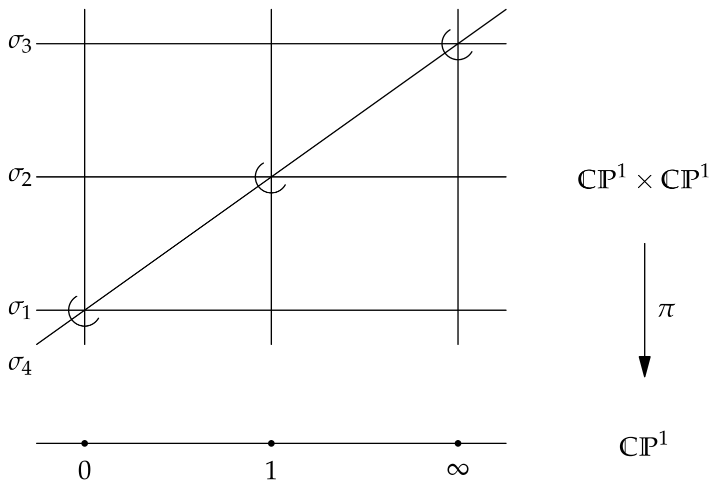
$$\lambda = \frac{(p_1 - p_4)(p_3 - p_2)}{(p_1 - p_2)(p_3 - p_4)}.$$

It follows that $\mathcal{M}_{0,4}$ is equal to $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$. The boundary divisor consists of three points which represent the following three nodal curves.



Respectively, these curves correspond to $\lambda = 0$, $\lambda = 1$ and $\lambda = \infty$, so $\overline{\mathcal{M}}_{0,4}$ is equal to $\mathbb{CP}^1$.

Example 1.6 (The moduli space $\overline{\mathcal{M}}_{0,5}$)**.** Let $\pi : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1$ be a family of curves, where π is defined by projection onto the first factor. Consider the four sections $\sigma_1(z) = (z, 0)$, $\sigma_2(z) = (z, 1)$, $\sigma_3(z) = (z, \infty)$ and $\sigma_4(z) = (z, z)$. Note that the fibre over a point $b \notin \{0, 1, \infty\}$ is a copy of $\mathbb{CP}^1$ with four distinct marked points which have cross-ratio equal to b. The section σ_4 meets σ_1, σ_2 and σ_3 transversally on the fibres over 0, 1 and ∞. To remove these intersections, blow up $\mathbb{CP}^1 \times \mathbb{CP}^1$ at the points $(0, 0)$, $(1, 1)$ and (∞, ∞), creating the exceptional divisors E_0, E_1 and E_∞, respectively. Now the three fibres over 0, 1 and ∞ are precisely the three nodal curves corresponding to the boundary divisor of $\overline{\mathcal{M}}_{0,4}$.



Therefore, the family we have described is precisely the universal family $\overline{\mathcal{C}}_{0,4} \to \overline{\mathcal{M}}_{0,4}$. However, as noted earlier, one may interpret this family as the forgetful map $\overline{\mathcal{M}}_{0,5} \to \overline{\mathcal{M}}_{0,4}$. So we may now conclude that $\overline{\mathcal{M}}_{0,5}$ is equal to $\mathbb{CP}^1 \times \mathbb{CP}^1$ blown up at the three points $(0,0)$, $(1,1)$ and (∞, ∞).

Example 1.7 (The moduli space $\overline{\mathcal{M}}_{1,1}$)**.** The uncompactified moduli space $\mathcal{M}_{1,1}$ is essentially the moduli space of complex elliptic curves. Every elliptic curve arises as a 2-fold branched covering of $\mathbb{CP}^1$ doubly ramified over ∞. Topological considerations force such a covering to have three additional ramification points which we can send to 0, 1 and $\lambda \notin \{0, 1, \infty\}$. The affine equation for such a curve is $y^2 = x(x-1)(x-\lambda)$, and we let the marked point correspond to the unique point on the curve over ∞. Note that there was some choice in normalising the ramification points, so we expect an S_3 symmetry to appear. Concretely, this manifests as one of the six transformations given by $\lambda \mapsto \lambda$, $\lambda \mapsto \frac{1}{\lambda}$, $\lambda \mapsto 1-\lambda$, $\lambda \mapsto \frac{1}{1-\lambda}$, $\lambda \mapsto \frac{\lambda-1}{\lambda}$ and $\lambda \mapsto \frac{\lambda}{\lambda-1}$. This symmetry can be dealt with by associating to each curve $y^2 = x(x-1)(x-\lambda)$ its j-invariant

$$j(\lambda) = 256\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2},$$

which satisfies $j(\lambda) = j(\lambda')$ if and only if $\lambda' \in \{\lambda, \frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, \frac{\lambda-1}{\lambda}, \frac{\lambda}{\lambda-1}\}$. Conversely, it is well-known that the j-invariant distinguishes between elliptic curves which are not isomorphic. Therefore, we deduce that $\mathcal{M}_{1,1} \cong \mathbb{C}$.

There is precisely one stable nodal curve of genus 1 with one marked point and this corresponds precisely to the $j \to \infty$ limit, so $\overline{\mathcal{M}}_{1,1} \cong \mathbb{CP}^1$. Our discussion up to this point has completely ignored the orbifold — or stack — structure of $\overline{\mathcal{M}}_{1,1}$. The case of $\overline{\mathcal{M}}_{1,1}$ is exceptional since every genus 1 curve with one marked point has at least one non-trivial automorphism — namely, the elliptic involution. Furthermore, the curve with j-invariant 0 actually has an automorphism group isomorphic to $\mathbb{Z}_6$ while the curve with j-invariant 1728 has an automorphism group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore, every point of $\overline{\mathcal{M}}_{1,1}$ is an orbifold point of order 2, apart from one point of order 4 and one point of order 6.

1.2 Intersection theory on moduli spaces

Characteristic classes

Given a space with interesting topological structure, one of the first compulsions of a geometer is often to calculate associated algebraic invariants. And so it is with moduli spaces of curves, except for the fact that their homology and cohomology are notoriously intractable in general. However, a great deal of progress can be made in this direction if one is willing to adopt the following two simplifications.

- We concentrate on a subset of the cohomology ring known as the tautological ring.
- We calculate the intersection theory, rather than the full ring structure, of the tautological ring with respect to certain characteristic classes.

Later we will see how these simplifications leave us with a much more manageable approach to understanding the structure of $\overline{\mathcal{M}}_{g,n}$ without forsaking too much of the interesting geometry.

The characteristic classes that we will consider live in the cohomology ring $H^*(\overline{\mathcal{M}}_{g,n})$ as well as its algebraic analogue, the Chow ring $A^*(\overline{\mathcal{M}}_{g,n})$. These are related by the natural map $A^k(\overline{\mathcal{M}}_{g,n}) \to H^{2k}(\overline{\mathcal{M}}_{g,n})$, where the doubling of the index is due to the fact that the grading of the Chow ring is by complex dimension. However, it is worth noting that the Chow ring is neither weaker nor stronger than the cohomology ring, since each carries information which the other does not. For example, although the Chow ring cannot detect odd-graded cohomology, it can distinguish between any two points on a smooth elliptic curve. For the remainder of this thesis, we will generally use the cohomological framework and language, which will be familiar to a wider audience. Practically all of the questions we consider are equivalent in either setting.

Many of the cohomology classes on $\overline{\mathcal{M}}_{g,n}$ of geometric interest arise from taking Chern classes of natural vector bundles. For example, consider the vertical cotangent bundle on $\overline{\mathcal{M}}_{g,n+1} = \overline{\mathcal{C}}_{g,n}$ with fibre at (C, p) equal to the cotangent line T_p^*C. Unfortunately, this definition is nonsensical when p is a singular point of C. Therefore, it is necessary to consider the relative dualising sheaf, the unique line bundle on $\overline{\mathcal{M}}_{g,n+1}$ which extends the vertical cotangent bundle. More precisely, it can be defined as

$$\mathcal{L} = \mathcal{K}_X \otimes \pi^*\mathcal{K}_B^{-1},$$

where $\mathcal{K}_X$ denotes the canonical line bundle on $\overline{\mathcal{M}}_{g,n+1}$ and $\mathcal{K}_B$ denotes the canonical line bundle on $\overline{\mathcal{M}}_{g,n}$. Sections of $\mathcal{L}$ along a non-singular fibre correspond precisely to holomorphic 1-forms on that fibre. However, sections of $\mathcal{L}$ along a singular fibre correspond to meromorphic 1-forms with at worst simple poles allowed at the nodes as well as an extra residue condition. This condition states that the two residues at the preimages of each node under normalisation must sum to zero.

One obtains natural line bundles on $\overline{\mathcal{M}}_{g,n}$ by pulling back $\mathcal{L}$ along the sections $\sigma_k : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$ for $k = 1, 2, \ldots, n$. Taking Chern classes of these line bundles, we obtain the psi-classes

$$\psi_k = c_1(\sigma_k^*\mathcal{L}) \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \text{ for } k = 1, 2, \ldots, n.$$

It is possible to define the Euler class $\widetilde{e} = c_1(\mathcal{L})$, but this completely ignores the marked points. Far more useful is the twisted Euler class given by $e = c_1(\mathcal{L}(D_1 + D_2 + \cdots + D_n))$, where $D_k \subseteq \overline{\mathcal{M}}_{g,n+1}$ denotes the image of the section $\sigma_k : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$. In a small and hopefully excusable abuse of notation, we will use D_k to represent the divisor, the corresponding homology class as well as its Poincaré dual cohomology class. Two important properties of the twisted Euler class are that $\langle e, \Sigma \rangle = -\chi(\Sigma - \cup D_k)$ on any fibre Σ and that it is actually identical to the class ψ_{n+1}. Taking the push-forward of its powers, one obtains the Mumford–Morita–Miller classes

$$\kappa_m = \pi_*(e^{m+1}) \in H^{2m}(\overline{\mathcal{M}}_{g,n}) \text{ for } m = 0, 1, 2, \ldots, 3g-3+n.$$

These were first introduced by Mumford [39] in the case $n = 0$ and are analogous to the characteristic classes of surface bundles dealt with by Miller [32] and Morita [37] in the topological setting. With due respect to these great mathematicians, we will subsequently refer to the Mumford–Morita–Miller classes simply as kappa-classes.

Another important construction on $\overline{\mathcal{M}}_{g,n}$ is the Hodge bundle Λ. Informally, it is the vector bundle whose fibre over a point $C \in \overline{\mathcal{M}}_{g,n}$ is the space of holomorphic 1-forms on the curve C. Once again, this definition is nonsensical over points corresponding to singular curves. A more precise definition is to express the Hodge bundle as $\Lambda = \pi_*(\mathcal{L})$, the direct image of the relative dualising sheaf under the forgetful morphism. The Chern classes of this rank g vector bundle are the Hodge classes

$$\lambda_k = c_k(\Lambda) \in H^{2k}(\overline{\mathcal{M}}_{g,n}) \text{ for } k = 0, 1, 2, \ldots, g.$$

The tautological ring

A great deal of attention has been paid to the subring of the cohomology ring $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ known as the tautological ring $R^*(\overline{\mathcal{M}}_{g,n})$. This is due to three main reasons.

- The tautological ring is much more tractable than the full cohomology ring.
- There is an extremely rich combinatorial structure underlying the tautological ring.
- The tautological ring, in some sense, captures all classes of geometric interest.

Vakil states in [56] that the tautological ring consists of "all the classes you can easily think of". Of course, this is a facetiously informal description but is supported by Vakil's heuristic argument that there is no class "that can be explicitly written down, that is provably not tautological, even though we expect that they exist". More precisely, consider the following two definitions for the system of tautological rings $R^*(\overline{\mathcal{M}}_{g,n})$ over all g and n.

Definition 1.8. The system of *tautological rings* $R^*(\overline{\mathcal{M}}_{g,n})$ is

- the smallest system of $\mathbb{Q}$-algebras closed under push-forwards by the natural morphisms;
- the smallest system of $\mathbb{Q}$-vector spaces closed under push-forwards by the natural morphisms, and which includes all monomials in the psi-classes.

The first definition is due to Faber and Pandharipande [13] and has the advantage of being more intrinsic. The second is due to Graber and Vakil [18], who prove that the two statements are equivalent. This latter definition serves to highlight the central part that the psi-classes play in the tautological ring and, hence, in the geometry of $\overline{\mathcal{M}}_{g,n}$.

Of course, one would certainly expect the kappa-classes to lie in the tautological ring. That this is the case follows from the following elegant and highly combinatorial formula relating the psi-classes and kappa-classes.

Proposition 1.9. *Consider the map $\pi^k_* : H^*(\overline{\mathcal{M}}_{g,n+k}) \to H^*(\overline{\mathcal{M}}_{g,n})$ obtained by iterating the forgetful morphism k times. If W is a product of the psi-classes $\psi_1, \psi_2, \ldots, \psi_n$, then*

$$\pi^k_* \left(W \cdot \psi_{n+1}^{\alpha_1+1} \psi_{n+2}^{\alpha_2+1} \cdots \psi_{n+k}^{\alpha_k+1} \right) = W \cdot \sum_{\sigma \in S_k} \kappa_\sigma(\alpha_1, \alpha_2, \ldots, \alpha_k).$$

Here, we write each permutation as a product of disjoint cycles $\sigma = s_1 s_2 \ldots s_m$, where all 1-cycles are included. For each cycle $s = (i_1 i_2 \ldots i_r)$, we let $|s| = \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_r}$ and define $\kappa_\sigma(\alpha_1, \alpha_2, \ldots, \alpha_k) = \kappa_{|s_1|} \kappa_{|s_2|} \cdots \kappa_{|s_m|}$.

An example may better illustrate the content of Proposition 1.9.

Example 1.10. Consider the map $\pi^3_* : H^*(\overline{\mathcal{M}}_{g,n+3}) \to H^*(\overline{\mathcal{M}}_{g,n})$. The six permutations in S_3, written as products of disjoint cycles, are: (1)(2)(3), (12)(3), (13)(2), (23)(1), (123) and (132). Therefore, we have the following.

$$\pi^3_* \left(W \cdot \psi_{n+1}^{\alpha_1+1} \psi_{n+2}^{\alpha_2+1} \psi_{n+3}^{\alpha_3+1} \right) = W \cdot (\kappa_{\alpha_1}\kappa_{\alpha_2}\kappa_{\alpha_3} + \kappa_{\alpha_1+\alpha_2}\kappa_{\alpha_3} + \kappa_{\alpha_1+\alpha_3}\kappa_{\alpha_2} + \kappa_{\alpha_2+\alpha_3}\kappa_{\alpha_1} + 2\kappa_{\alpha_1+\alpha_2+\alpha_3})$$

One can easily obtain the following important corollary of Proposition 1.9.

Corollary 1.11. *The intersection theory of psi-classes and kappa-classes on $\overline{\mathcal{M}}_{g,n}$ is equivalent to the intersection theory of psi-classes on all $\overline{\mathcal{M}}_{g,n+m}$ for non-negative integers m. In particular, the intersection theory of kappa-classes on $\overline{\mathcal{M}}_{g,0}$ is equivalent to the intersection theory of psi-classes on all $\overline{\mathcal{M}}_{g,n}$.*

Similarly, the following formula due to Faber [12] relates Hodge classes with kappa-classes and demonstrates that the Hodge classes lie in the tautological ring.

Proposition 1.12. *The Hodge classes and kappa-classes are related by the following formula, where $B_0, B_1, B_2, \ldots$ are the Bernoulli numbers.*

$$\sum_{k=0}^{\infty} \lambda_k t^k = \exp\left(\sum_{k=1}^{\infty} \frac{B_{2k}\, \kappa_{2k-1}}{2k(2k-1)} t^{2k-1} \right)$$

Apart from the actual content of Proposition 1.12, there are two important features of Faber's formula — namely, the use of generating functions and the appearance of the Bernoulli numbers. Generating functions are commonly used to encode information about the intersection theory on $\overline{\mathcal{M}}_{g,n}$. That this result can be expressed so succinctly using them highlights the combinatorial nature of the tautological ring. This fact is reinforced by the presence of the Bernoulli numbers, which will make another appearance in Chapter 2.

The following statement was first conjectured by Hain and Looijenga [20].

Conjecture 1.13. *The tautological ring $R^*(\overline{\mathcal{M}}_{g,n})$ is a Poincaré duality ring of dimension $3g-3+n$.*

To understand the content of this conjecture, we break it into three parts.

- *Vanishing conjecture.* For $k > 3g-3+n$, $R^k(\overline{\mathcal{M}}_{g,n}) \cong 0$.
- *Socle conjecture.* $R^{3g-3+n}(\overline{\mathcal{M}}_{g,n}) \cong \mathbb{Q}$.
- *Perfect pairing conjecture.* For $0 \leq k \leq 3g-3+n$, the following natural product is a perfect pairing.
$$R^k(\overline{\mathcal{M}}_{g,n}) \times R^{3g-3+n-k}(\overline{\mathcal{M}}_{g,n}) \to R^{3g-3+n}(\overline{\mathcal{M}}_{g,n}) \cong \mathbb{Q}$$

In order to appreciate the depth of these statements, one must consider the tautological rings as subsets of the corresponding Chow rings. For instance, if we were to consider the tautological ring as a subset of the cohomology ring, then the socle conjecture would be trivially true. On the other hand, the socle conjecture is neither obvious in the tautological Chow ring nor even true in the full Chow ring. At present, the conjecture remains unresolved, although there is a fair amount of low genus evidence. And if the conjecture is true, then one important consequence would be that it is possible to recover the structure of the entire tautological ring from the top intersections of tautological classes alone. Furthermore, from Definition 1.8, any top intersections in the tautological ring can be determined from the top intersections of psi-classes alone. Therefore, we are motivated to study intersection numbers of the form

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \psi_2^{\alpha_2} \ldots \psi_n^{\alpha_n} \in \mathbb{Q}$$

where $|\boldsymbol{\alpha}| = 3g-3+n$ or equivalently, $g = \frac{1}{3}(|\boldsymbol{\alpha}| - n + 3)$. It will be convenient to adopt Witten's notation for these psi-class intersection numbers, which suppresses the genus and encodes the symmetry between the psi-classes.

$$\langle \tau_{\alpha_1} \tau_{\alpha_2} \ldots \tau_{\alpha_n} \rangle = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \psi_2^{\alpha_2} \ldots \psi_n^{\alpha_n}$$

We treat the τ variables as commuting, so that we can write intersection numbers in the form $\langle \tau_0^{d_0} \tau_1^{d_1} \tau_2^{d_2} \ldots \rangle$ and we set $\langle \tau_{\alpha_1} \tau_{\alpha_2} \ldots \tau_{\alpha_n} \rangle = 0$ if $n = 0$ or if the genus $g = \frac{1}{3}(|\boldsymbol{\alpha}| - n + 3)$ is non-integral or negative. In this way, we have defined a linear functional

$$\langle \cdot \rangle : \mathbb{Q}[\tau_0, \tau_1, \tau_2, \ldots] \to \mathbb{Q}.$$

Example 1.14 (Psi-class intersection numbers on $\overline{\mathcal{M}}_{0,3}$)**.** The only non-zero intersection number on $\overline{\mathcal{M}}_{0,3}$ is $\langle \tau_0^3 \rangle$, which is equal to 1 by definition. This encodes the fact that there is a unique genus 0 curve with three marked points and that such a curve has trivial automorphism group.

Example 1.15 (Psi-class intersection numbers on $\overline{\mathcal{M}}_{0,4}$)**.** From Example 1.6, we know that $\overline{\mathcal{M}}_{0,5}$ is the blow up of $\mathbb{CP}^1 \times \mathbb{CP}^1$ at the three points $(0,0)$, $(1,1)$ and (∞, ∞). The forgetful map $\pi : \overline{\mathcal{M}}_{0,5} \to \overline{\mathcal{M}}_{0,4}$ is simply the projection onto the first $\mathbb{CP}^1$ factor. The singular fibres occur over 0, 1 and ∞ and correspond to the three nodal curves illustrated in Example 1.5. The four sections are given by $\sigma_1(z) = (z, 0)$, $\sigma_2(z) = (z, 1)$, $\sigma_3(z) = (z, \infty)$ and $\sigma_4(z) = (z, z)$.

This concrete description of the universal family over $\overline{\mathcal{M}}_{0,5}$ allows us to calculate its cohomology explicitly. It is generated by the five divisors H, F, E_0, E_1, E_∞, where $H = \mathbb{CP}^1 \times \{h\}$ for some $h \notin \{0, 1, \infty\}$, $F = \{f\} \times \mathbb{CP}^1$ for some $f \notin \{0, 1, \infty\}$ and E_0, E_1, E_∞ are the exceptional divisors of the blow-ups. For ease of notation, we will use these curves to represent their divisor classes, homology classes and Poincaré dual cohomology classes. The intersection matrix with respect to the ordered basis $(H, F, E_0, E_1, E_\infty)$ has the following simple form.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Since this matrix is invertible, we can deduce the value of a cohomology class X from the intersection numbers $X \cdot H$, $X \cdot F$, $X \cdot E_0$, $X \cdot E_1$ and $X \cdot E_\infty$. This allows us to calculate $D_1 = H - E_0$.

Now if we denote by $\mathcal{T}$ the vertical tangent bundle, then a section of $\sigma_1^* \mathcal{T}$ corresponds to a choice of vertical vector for each point in D_1. This is precisely a section of the normal bundle to $D_1 \subseteq \overline{\mathcal{M}}_{0,5}$. The degree of this normal bundle is the self-intersection of D_1 in $\overline{\mathcal{M}}_{0,5}$, so we have

$$\int_{\overline{\mathcal{M}}_{0,4}} c_1(\sigma_1^* \mathcal{T}) = D_1 \cdot D_1.$$

Taking the dual gives

$$\int_{\overline{\mathcal{M}}_{0,4}} \psi_1 = -D_1 \cdot D_1 = -(H - E_0) \cdot (H - E_0) = 1,$$

from which we conclude that $\langle \tau_0^3 \tau_1 \rangle = 1$.

Example 1.16 (Psi-class intersection numbers on $\overline{\mathcal{M}}_{1,1}$)**.** Let $f(x, y)$ and $g(x, y)$ be generic cubic polynomials and consider the family of cubic curves

$$F = \{(x, y, t) \mid f(x, y) - tg(x, y) = 0\} \subseteq \mathbb{CP}^2 \times \mathbb{CP}^1$$

over the base $B = \mathbb{CP}^1$, parametrised by t. The cubic curves $f(x, y) = 0$ and $g(x, y) = 0$ intersect in nine points $p_1, p_2, \ldots, p_9$ and we may choose the point p_1 as the marked point in our family. This family then induces a map $\phi : B \to \overline{\mathcal{M}}_{1,1}$

It turns out that F is the blow-up of $\mathbb{CP}^2$ at the points $p_1, p_2, \ldots, p_9$. By a similar reasoning to the previous example,

$$\int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = \frac{1}{\deg \phi} \int_B \phi^* \psi_1 = -\frac{1}{\deg \phi} S_1 \cdot S_1,$$

where S_1 denotes the image of the section associated to the marked point. In this case, S_1 is precisely the exceptional divisor over p_1, so $S_1 \cdot S_1 = -1$.

Note that the unique singular curve in $\overline{\mathcal{M}}_{1,1}$ will appear in the family F precisely when the discriminant vanishes. The discriminant of a cubic curve is a polynomial of degree 12 in its coefficients, hence a polynomial of degree 12 in t. The singular curve is generic enough to deduce that the degree of the map $\phi : B \to \overline{\mathcal{M}}_{1,1}$ is 12. However, since the generic point in $\overline{\mathcal{M}}_{1,1}$ is an orbifold point of order two, the true degree of the map ϕ is actually $2 \times 12 = 24$. Therefore, we have

$$\langle \tau_1 \rangle = \int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = \frac{1}{24}.$$

The psi-class intersection numbers contain a great deal of structure, as hinted by the following fact.

Proposition 1.17 (String equation)**.** *For $2g - 2 + n > 0$, the psi-class intersection numbers satisfy the relation*

$$\langle \tau_0 \tau_{\alpha_1} \tau_{\alpha_2} \ldots \tau_{\alpha_n} \rangle = \sum_{k=1}^{n} \langle \tau_{\alpha_1} \ldots \tau_{\alpha_k - 1} \ldots \tau_{\alpha_n} \rangle.$$

Observe that the string equation reduces a psi-class intersection number on $\overline{\mathcal{M}}_{g,n+1}$ which has a ψ_k appearing with exponent zero to a sum of psi-class intersection numbers on $\overline{\mathcal{M}}_{g,n}$. In fact, from the base case $\langle \tau_0^3 \rangle = 1$ and the string equation, all psi-class intersection numbers in genus 0 can be uniquely determined. To see this, note that every non-zero psi-class intersection number on $\overline{\mathcal{M}}_{0,n}$ must be of the form $\langle \tau_{\alpha_1} \tau_{\alpha_2} \ldots \tau_{\alpha_n} \rangle$, where $|\boldsymbol{\alpha}| = n - 3$. So at least one of $\alpha_1, \alpha_2, \ldots, \alpha_n$ must be equal to 0 and the string equation reduces the calculation to a sum of intersection numbers on $\overline{\mathcal{M}}_{0,n-1}$. Therefore, these numbers can be calculated inductively, starting from the base case $\langle \tau_0^3 \rangle = 1$. The particularly pleasant case of genus 0 yields a particularly pleasant answer. In fact, when $|\boldsymbol{\alpha}| = n - 3$ we have

$$\langle \tau_{\alpha_1} \tau_{\alpha_2} \ldots \tau_{\alpha_n} \rangle = \frac{(n-3)!}{\boldsymbol{\alpha}!},$$

which is tantalisingly combinatorial.

Proposition 1.18 (Dilaton equation)**.** *For $2g-2+n>0$, the psi-class intersection numbers satisfy the relation*

$$\langle \tau_1 \tau_{\alpha_1} \tau_{\alpha_2} \ldots \tau_{\alpha_n} \rangle = (2g-2+n) \langle \tau_{\alpha_1} \tau_{\alpha_2} \ldots \tau_{\alpha_n} \rangle.$$

Observe that the dilaton equation reduces a psi-class intersection number on $\overline{\mathcal{M}}_{g,n+1}$ which has a ψ_k appearing with exponent one to a psi-class intersection number on $\overline{\mathcal{M}}_{g,n}$. In fact, from the base case $\langle \tau_1 \rangle = \frac{1}{24}$, as well as the string and dilaton equations, all psi-class intersection numbers in genus 1 can be uniquely determined. To see this, note that every non-zero psi-class intersection number on $\overline{\mathcal{M}}_{1,n}$ must be of the form $\langle \tau_{\alpha_1} \tau_{\alpha_2} \ldots \tau_{\alpha_n} \rangle$, where $|\boldsymbol{\alpha}| = n$. So at least one of $\alpha_1, \alpha_2, \ldots, \alpha_n$ must be equal to 0 or 1. In the former case, the string equation reduces the calculation to a sum of intersection numbers on $\overline{\mathcal{M}}_{1,n-1}$ while in the latter case, the dilaton equation reduces the calculation to an intersection number on $\overline{\mathcal{M}}_{1,n-1}$. Therefore, these numbers can be calculated inductively, starting from the base case $\langle \tau_1 \rangle = \frac{1}{24}$. Unfortunately — or perhaps fortunately, depending on one's outlook — there exists no simple closed formula for psi-class intersection numbers for the case of genus $g \geq 1$.

Witten's conjecture

One of the landmark results concerning the intersection theory on moduli spaces of curves is Witten's conjecture, now Kontsevich's theorem. In his foundational paper [57], Witten conjectured that a particular generating function for the psi-class intersection numbers satisfies the Korteweg–de Vries hierarchy, often abbreviated to the KdV hierarchy. This sequence of partial differential equations begins with the KdV equation, which originally arose in classical physics to model waves in shallow water. It is now well-known as the prototypical example of an exactly solvable model, whose soliton solutions have attracted tremendous mathematical interest over the past few decades.

Interestingly, Witten was led to his conjecture from the analysis of a particular model of two-dimensional quantum gravity, where one encounters infinite-dimensional integrals over the space of Riemannian metrics on a surface. Arguing on physical grounds, such a calculation can be reduced to finitely many dimensions in two distinct ways. First, the integral can be localised to the space of conformal classes of metrics, which leads directly to computations on moduli spaces of curves. Second, one can produce singular metrics on a surface by tiling it with triangles and declaring them to be equilateral. As the number of triangles tends to infinity, these singular metrics begin to approximate random metrics and the infinite-dimensional integrals reduce to asymptotic enumerations of such triangulations. Enumerations of this kind are performed using Feynman diagram and matrix model techniques and are known to be governed by the KdV hierarchy.

In order to describe Witten's conjecture explicitly, let $\mathbf{t} = (t_0, t_1, t_2, \ldots)$ and $\boldsymbol{\tau} = (\tau_0, \tau_1, \tau_2, \ldots)$

and consider the generating function $F(\mathbf{t}) = \langle \exp(\mathbf{t} \cdot \boldsymbol{\tau}) \rangle$. Here, the expression is to be expanded as a Taylor series using multilinearity in the variables $t_0, t_1, t_2, \ldots$. Equivalently, define

$$F(t_0, t_1, t_2, \ldots) = \sum_{\mathbf{d}} \prod_{k=0}^{\infty} \frac{t_k^{d_k}}{d_k!} \langle \tau_0^{d_0} \tau_1^{d_1} \tau_2^{d_2} \ldots \rangle$$

where the summation is over all sequences $\mathbf{d} = (d_0, d_1, d_2, \ldots)$ of non-negative integers with finitely many non-zero terms. Witten conjectured that the formal series $U = \frac{\partial^2 F}{\partial t_0^2}$ satisfies the KdV hierarchy of partial differential equations. More explicitly, Witten's conjecture can be stated as follows.

Theorem 1.19 (Witten's conjecture)**.** *The generating function F satisfies the following partial differential equation for every non-negative integer n.*

$$(2n+1)\frac{\partial^3 F}{\partial t_n \partial t_0^2} = \left(\frac{\partial^2 F}{\partial t_{n-1} \partial t_0}\right)\left(\frac{\partial^3 F}{\partial t_0^3}\right) + 2\left(\frac{\partial^3 F}{\partial t_{n-1} \partial t_0^2}\right)\left(\frac{\partial^2 F}{\partial t_0^2}\right) + \frac{1}{4}\frac{\partial^5 F}{\partial t_{n-1} \partial t_0^4}$$

Given Witten's conjecture, the string equation and the base case $\langle \tau_0^3 \rangle = 1$, every intersection number of psi-classes can be obtained. The following example demonstrates this via the calculation of $\langle \tau_1 \rangle$.

Example 1.20. Observe that

$$\left.\frac{\partial^n F}{\partial t_{\alpha_1} \partial t_{\alpha_2} \cdots \partial t_{\alpha_n}}\right|_{\mathbf{t}=\mathbf{0}} = \langle \tau_{\alpha_1} \tau_{\alpha_2} \ldots \tau_{\alpha_n} \rangle,$$

and consider the equation in Witten's conjecture with $n = 3$ evaluated at $\mathbf{t} = \mathbf{0}$. This yields the equality $7\langle \tau_0^2 \tau_3 \rangle = \langle \tau_0 \tau_2 \rangle \langle \tau_0^3 \rangle + 2\langle \tau_0^2 \tau_2 \rangle \langle \tau_0^2 \rangle + \frac{1}{4}\langle \tau_0^4 \tau_2 \rangle$. Now use the fact that $\langle \tau_0^2 \rangle = 0$ and the base case $\langle \tau_0^3 \rangle = 1$ to reduce the relation to $7\langle \tau_0^2 \tau_3 \rangle = \langle \tau_0 \tau_2 \rangle + \frac{1}{4}\langle \tau_0^4 \tau_2 \rangle$. Applying the string equation to each term, we obtain $7\langle \tau_1 \rangle = \langle \tau_1 \rangle + \frac{1}{4}\langle \tau_0^3 \rangle$ from which it follows that $\langle \tau_1 \rangle = \frac{1}{24}\langle \tau_0^3 \rangle = \frac{1}{24}$. Note that this is in agreement with the calculation of $\langle \tau_1 \rangle$ in Example 1.16.

A thorough analysis of the KdV hierarchy allows Witten's conjecture to be stated in an alternative way. Define the sequence of Virasoro operators by

$$V_{-1} = -\frac{1}{2}\frac{\partial}{\partial t_0} + \frac{1}{2}\sum_{k=0}^{\infty} t_{k+1}\frac{\partial}{\partial t_k} + \frac{t_0^2}{4}, \qquad V_0 = -\frac{3}{2}\frac{\partial}{\partial t_1} + \frac{1}{2}\sum_{k=0}^{\infty}(2k+1)t_k\frac{\partial}{\partial t_k} + \frac{1}{48},$$

and for positive integers n,

$$V_n = -\frac{(2n+3)!!}{2}\frac{\partial}{\partial t_{n+1}} + \sum_{k=0}^{\infty}\frac{(2k+2n+1)!!}{2(2k-1)!!}t_k\frac{\partial}{\partial t_{k+n}} + \sum_{k_1+k_2=n-1}\frac{(2k_1+1)!!(2k_2+1)!!}{4}\frac{\partial^2}{\partial t_{k_1}\partial t_{k_2}}.$$

The operators are named so because they span a subalgebra of the Virasoro Lie algebra. It is relatively straightforward to verify that they satisfy the relation $[V_m, V_n] = (m - n)V_{m+n}$ for all m and n. One can state Witten's conjecture in terms of the Virasoro operators in a rather succinct fashion.

Theorem 1.21 (Witten's conjecture — Virasoro version)**.** *For every integer $n \geq -1$,*

$$V_n(\exp F) = 0.$$

Witten originally provided evidence for his conjecture in the form of the string and dilaton equations as well as low genus results [57]. In particular, the string and dilaton equations correspond precisely to the annihilation of $\exp F$ by the operators V_{-1} and V_0, respectively. Since then, several proofs of Witten's conjecture have emerged, three of which we briefly discuss here.

- Kontsevich [26]
  The year after Witten announced his conjecture, Kontsevich produced a proof as part of his doctoral thesis. He used a cell decomposition of $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$ arising from results concerning quadratic differentials on a Riemann surface known as Jenkins–Strebel differentials. This allowed him to deduce a combinatorial formula which equates a generating function for the psi-class intersection numbers with a particular enumeration of combinatorial objects known as trivalent ribbon graphs. Kontsevich carried out this enumeration using Feynman diagram techniques and a certain matrix model, from which Witten's conjecture followed. His proof will be discussed in greater detail in Section 3.1.

- Okounkov and Pandharipande [44]
  The main tool in their proof of Witten's conjecture was the ELSV formula. Originally proven by Ekedahl, Lando, Shapiro and Vainshtein [11], this formula relates intersection numbers of psi-classes and Hodge classes — also known as Hodge integrals — with Hurwitz numbers. Hurwitz numbers enumerate topological types of branched covers of the sphere or, equivalently, factorisations of permutations into transpositions. Okounkov and Pandharipande reproduced the ELSV formula using a technique known as virtual localisation and proceeded to show that Kontsevich's combinatorial formula was a consequence, using asymptotic combinatorial methods.

- Mirzakhani [33, 34]
  More recently, Mirzakhani has produced a proof of Witten's conjecture quite distinct from those before her. In particular, she adopts a hyperbolic geometric approach and considers the symplectic geometry of moduli spaces of hyperbolic surfaces. We will have a lot more to say about Mirzakhani's work in Section 1.4.

We point out that there are also other proofs — such as that by Kazarian and Lando [24] or by Kim and Liu [25] — but these have a lesser bearing on the work contained in this thesis.

1.3 Moduli spaces of hyperbolic surfaces

The uniformisation theorem

One of the most important foundational results in algebraic geometry asserts that the category of irreducible projective algebraic curves and the category of compact Riemann surfaces are equivalent. Due to this equivalence, the boundary between these two fields is rather porous, with techniques from complex analysis flowing into algebraic geometry and vice versa. In addition, the following theorem allows us to adopt a geometric viewpoint when dealing with algebraic curves or Riemann surfaces.

Theorem 1.22 (The uniformisation theorem)**.** *Every metric on a surface is conformally equivalent to a complete constant curvature metric. Furthermore, the sign of the curvature is equal to the sign of the Euler characteristic of the surface.*

From the previous discussion, a smooth genus g algebraic curve with n marked points corresponds to a genus g Riemann surface with n marked points, which we think of as punctures.

$$\left\{ \begin{array}{c} \text{smooth algebraic curves with} \\ \text{genus } g \text{ and } n \text{ marked points} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Riemann surfaces with} \\ \text{genus } g \text{ and } n \text{ punctures} \end{array} \right\}$$

The complex structure defines a conformal class of metrics which, by the uniformisation theorem, contains a hyperbolic metric when $2g - 2 + n > 0$. Furthermore, if we demand that the resulting surface has finite area, then this hyperbolic metric is unique and endows each puncture with the structure of a hyperbolic cusp. So we have the following one-to-one correspondence.

$$\left\{ \begin{array}{c} \text{smooth algebraic curves with} \\ \text{genus } g \text{ and } n \text{ marked points} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{hyperbolic surfaces with} \\ \text{genus } g \text{ and } n \text{ cusps} \end{array} \right\}$$

Moduli spaces of hyperbolic surfaces can be given a natural topology. The correspondence described above defines a map from the moduli space of smooth algebraic curves to the moduli space of hyperbolic surfaces which respects not only this topology, but also the structure-preserving automorphism group of the surface. In short, the map is a homeomorphism of orbifolds. Therefore, we can and will use the notation $\mathcal{M}_{g,n}$ to denote the moduli space of smooth genus g curves with n marked points as well as the moduli space of genus g hyperbolic surfaces with n cusps — the particular meaning should be clear from the context.

If the finite area condition is relaxed, then there exist hyperbolic metrics which endow each puncture with the structure of a hyperbolic flare. Each such flare has a unique geodesic waist curve which, after being cut along, leaves a compact hyperbolic surface with geodesic boundaries. Furthermore, the only moduli of the removed flare is the length of the geodesic waist curve. So for $\mathbf{L} = (L_1, L_2, \ldots, L_n)$ an n-tuple of positive real numbers, we may define

$$\mathcal{M}_{g,n}(\mathbf{L}) = \left\{ (X, \beta_1, \beta_2, \ldots, \beta_n) \;\middle|\; \begin{array}{l} X \text{ is a genus } g \text{ hyperbolic surface with } n \text{ boundary} \\ \text{components } \beta_1, \beta_2, \ldots, \beta_n \text{ of lengths } L_1, L_2, \ldots, L_n \end{array} \right\} \Big/ \sim$$

where $(S, \beta_1, \beta_2, \ldots, \beta_n) \sim (T, \gamma_1, \gamma_2, \ldots, \gamma_n)$ if and only if there is an isometry from S to T which sends β_k to γ_k for all k. Note that when a boundary length approaches zero, we recover the cusp case in the limit, so $\mathcal{M}_{g,n}(\mathbf{0}) = \mathcal{M}_{g,n}$.

Teichmüller theory

Teichmüller theory will enable us to construct the moduli space $\mathcal{M}_{g,n}(\mathbf{L})$ and endow it with a natural symplectic structure. Fix a smooth orientable surface $S_{g,n}$ with genus g and n boundary components labelled from 1 up to n, where $\chi(S_{g,n}) = 2 - 2g - n$ is negative. Now define a marked hyperbolic surface of type (g, n) to be a pair (X, f) where X is a hyperbolic surface and $f : S_{g,n} \to X$ is a diffeomorphism. We call f the marking of the hyperbolic surface and define the Teichmüller space

$$\mathcal{T}_{g,n}(\mathbf{L}) = \left\{ (X, f) \;\middle|\; \begin{array}{l} (X, f) \text{ is a marked hyperbolic surface of type } (g, n) \\ \text{with geodesic boundaries of lengths } L_1, L_2, \ldots, L_n \end{array} \right\} \Big/ \sim$$

where $(X, f) \sim (Y, g)$ if and only if there exists an isometry $\phi : X \to Y$ such that $\phi \circ f$ is isotopic to g. In essence, Teichmüller space is the space of all deformations of the hyperbolic structure on a surface.

We now define global coordinates on Teichmüller space, known as Fenchel–Nielsen coordinates. Start by considering a pair of pants decomposition of the surface $S_{g,n}$ — in other words, a collection of disjoint simple closed curves whose complement is a disjoint union of genus 0 surfaces with 3 boundary components. Alternatively, a pair of pants decomposition is a maximal collection of disjoint simple closed curves such that no curve is parallel to the boundary and no two are homotopic. Since the Euler characteristic is additive on surfaces glued along circles, the number of pairs of pants in any such decomposition must be $-\chi(S_{g,n}) = 2g - 2 + n$. It follows that every pair of pants decomposition of $S_{g,n}$ must consist of precisely $3g - 3 + n$ simple closed curves.

Note that a marking $f : S_{g,n} \to X$ maps a pair of pants decomposition to a collection of simple closed curves, each of which has a unique geodesic representative in its homotopy class. Denote these simple closed geodesics by $\gamma_1, \gamma_2, \ldots, \gamma_{3g-3+n}$ and let their lengths be $\ell_1, \ell_2, \ldots, \ell_{3g-3+n}$, respectively. Cutting X along $\gamma_1, \gamma_2, \ldots, \gamma_{3g-3+n}$ leaves a disjoint union of $2g - 2 + n$ hyperbolic pairs of pants. The following simple lemma guarantees that the lengths $\ell_1, \ell_2, \ldots, \ell_{3g-3+n}$ are sufficient to determine the hyperbolic structure on each pair of pants.

Lemma 1.23. *Given three non-negative real numbers L_1, L_2, L_3, there exists a unique hyperbolic pair of pants up to isometry with geodesic boundaries of lengths L_1, L_2, L_3. We refer to the three geodesic arcs perpendicular to the boundary components and joining them in pairs as the seams. Every hyperbolic pair of pants can be decomposed into two congruent right-angled hexagons by cutting along the seams. As per usual, a boundary of length 0 corresponds to a hyperbolic cusp.*

Note that the lengths $\ell_1, \ell_2, \ldots, \ell_{3g-3+n}$ provide insufficient information to reconstruct the hyperbolic structure on all of X, since there are infinitely many ways to glue together the hyperbolic pairs of pants. This extra gluing information is stored in the twist parameters, which we denote by $\tau_1, \tau_2, \ldots, \tau_{3g-3+n}$. To construct them, fix a collection C of disjoint curves on $S_{g,n}$ which are either closed or have endpoints on the boundary. We require that C meets the pair of pants decomposition transversely, and such that its restriction to any particular pair of pants consists of three disjoint arcs, connecting the boundary components pairwise. Now to construct the twist parameter τ_k, note that there are either one or two curves $\gamma \in C$ such that $f(\gamma)$ meets γ_k. Homotopic to $f(\gamma)$, relative to the boundary of X, is a unique length-minimising piecewise geodesic curve which is entirely contained in the seams of the hyperbolic pairs of pants and the curves $\gamma_1, \gamma_2, \ldots, \gamma_{3g-3+n}$. The twist parameter τ_k is the signed distance that this curve travels along γ_k, according to the following sign convention.

positive twist parameter negative twist parameter

For further details, one can consult Thurston's book [54], where he notes the following.

> *"That a twist parameter takes values in $\mathbb{R}$, rather than S^1, tends to be a confusing issue, because twist parameters that are the same modulo 1 result in surfaces that are isometric. But, remember, to determine a point in Teichmüller space we need to consider how many times the leg of the pajama suit is twisted before it fits onto the baby's foot."*

More prosaically, the length parameters and the twist parameters modulo 1 are sufficient to reconstruct the hyperbolic structure on X. However, to recover the marking as well, it is necessary to consider the twist parameters as elements of $\mathbb{R}$. So there is a one-to-one correspondence between marked hyperbolic surfaces and their associated length and twist parameters, which we refer to as Fenchel–Nielsen coordinates.

Theorem 1.24. *The Fenchel–Nielsen map $\mathcal{FN} : \mathcal{T}_{g,n}(\mathbf{L}) \to \mathbb{R}_+^{3g-3+n} \times \mathbb{R}^{3g-3+n}$, which associates to a marked hyperbolic surface the length and twist parameters, is a bijection. In fact, if Teichmüller space is endowed with its natural topology, then the map is a homeomorphism.*

There is clearly a projection map $\mathcal{T}_{g,n}(\mathbf{L}) \to \mathcal{M}_{g,n}(\mathbf{L})$ given by forgetting the marking. In fact, we obtain the moduli space as a quotient of Teichmüller space by a group action. Consider the mapping class group

$$\mathrm{Mod}_{g,n} = \mathrm{Diff}^+(S_{g,n})/\mathrm{Diff}_0^+(S_{g,n}),$$

where $\mathrm{Diff}^+(S_{g,n})$ is the group of orientation preserving diffeomorphisms fixing the boundaries and $\mathrm{Diff}_0^+(S_{g,n})$ is the normal subgroup consisting of those diffeomorphisms isotopic to the identity. There is a natural action of the mapping class group on Teichmüller space described as follows: if $[\phi]$ is an element of $\mathrm{Mod}_{g,n}$, then $[\phi]$ sends the marked hyperbolic surface (X, f) to the marked hyperbolic surface $(X, f \circ \phi)$.

Proposition 1.25. *The action of $Mod_{g,n}$ on $\mathcal{T}_{g,n}(\mathbf{L})$ is properly discontinuous, though not necessarily free. Therefore, the quotient $\mathcal{M}_{g,n}(\mathbf{L}) = \mathcal{T}_{g,n}(\mathbf{L}) / Mod_{g,n}$ is an orbifold.*

Compactification and symplectification

Earlier, we described the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$, obtained by considering stable algebraic curves. It should be noted that there is an analogous construction in the hyperbolic setting, where a node of an algebraic curve corresponds to degenerating the length of a simple closed curve on a hyperbolic surface to zero. In fact, one can construct $\overline{\mathcal{T}}_{g,n}(\mathbf{L})$, the Teichmüller space of marked stable hyperbolic surfaces, in the following way. Define a stable hyperbolic surface of type (g, n) to be a pair (X, M) where X is a surface of genus g with n punctures, M is a collection of disjoint simple closed curves on X and $X \setminus M$ is endowed with a finite area hyperbolic metric. Again, we refer to a diffeomorphism $f : S_{g,n} \to X$ as a marking and define the compactified Teichmüller space

$$\overline{\mathcal{T}}_{g,n}(\mathbf{L}) = \left\{ (X, M, f) \;\middle|\; \begin{array}{l} f \text{ is a marking of a stable hyperbolic surface } (X, M) \text{ of} \\ \text{genus } g \text{ with } n \text{ boundaries of lengths } L_1, L_2, \ldots, L_n \end{array} \right\} \Big/ \sim$$

where $(X, M, f) \sim (Y, N, g)$ if and only if there exists a homeomorphism $\phi : X \to Y$ such that $\phi(M) = N$, ϕ restricted to $X \setminus M$ is an isometry, and $\phi \circ f$ is isotopic to g on each connected component of $S_{g,n} \setminus f^{-1}(M)$. Once again, the mapping class group acts on the compactified Teichmüller space and one may define a compactification of the moduli space $\mathcal{M}_{g,n}(\mathbf{L})$ as

$$\overline{\mathcal{M}}_{g,n}(\mathbf{L}) = \overline{\mathcal{T}}_{g,n}(\mathbf{L})/\mathrm{Mod}_{g,n}.$$

The moduli space $\overline{\mathcal{M}}_{g,n}(\mathbf{0})$ can be canonically identified with $\overline{\mathcal{M}}_{g,n}$ by the uniformisation theorem. When $\mathbf{L} \neq \mathbf{0}$, the moduli space $\overline{\mathcal{M}}_{g,n}(\mathbf{L})$ does not possess a natural complex structure. However, by the work of Wolpert [61], the Fenchel–Nielsen coordinates do induce a real analytic structure. For more information on the compactification and real analytic structure of moduli spaces of hyperbolic surfaces, the reader is encouraged to consult the references [1, 3].

The Teichmüller space $\mathcal{T}_{g,n}(\mathbf{L})$ can be endowed with the canonical symplectic form

$$\omega = \sum_{k=1}^{3g-3+n} d\ell_k \wedge d\tau_k$$

using the Fenchel–Nielsen coordinates. Although this is a rather trivial statement, a deep fact is that this form is invariant under the action of the mapping class group. Therefore, ω descends to a symplectic form on the quotient, namely the moduli space $\mathcal{M}_{g,n}(\mathbf{L})$. This is referred to as the Weil–Petersson symplectic form and we will also denote it by ω. Its existence allows us to use the techniques of symplectic geometry in the study of moduli spaces. Wolpert [61] used the real analytic structure on $\mathcal{M}_{g,n}(\mathbf{L})$ to show that the Weil–Petersson form extends smoothly to a closed form on $\overline{\mathcal{M}}_{g,n}(\mathbf{L})$. In the particular case $\mathbf{L} = \mathbf{0}$, he showed that this extension defines a cohomology class $[\omega] \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{R})$ which satisfies the following.

Theorem 1.26. *The de Rham cohomology class of the Weil–Petersson symplectic form and the characteristic class κ_1 are related by the equation $[\omega] = 2\pi^2\kappa_1 \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{R})$.*

Note that for all values of $\mathbf{L}$, the spaces $\mathcal{M}_{g,n}(\mathbf{L})$ are diffeomorphic to each other, but not necessarily symplectomorphic, when endowed with the Weil–Petersson symplectic structure. It is therefore natural to ask how the symplectic structure varies as $\mathbf{L}$ varies, a topic which we discuss in the next section.

1.4 Volumes of moduli spaces

Weil–Petersson volumes

Powering up the Weil–Petersson symplectic form, one obtains the corresponding volume form

$$\frac{\omega^{3g-3+n}}{(3g-3+n)!} = d\ell_1 \wedge d\tau_1 \wedge d\ell_2 \wedge d\tau_2 \wedge \ldots \wedge d\ell_{3g-3+n} \wedge d\tau_{3g-3+n}.$$

Of course, Teichmüller space has infinite volume with respect to this form. However, the action of the mapping class group is such that the volume of the moduli space is finite. This follows from the fact that the Weil–Petersson symplectic form can be extended smoothly to the compactification. Therefore, define $V_{g,n}(\mathbf{L})$ to be the Weil–Petersson volume of $\mathcal{M}_{g,n}(\mathbf{L})$. Note that when dealing with volumes, one need not worry about the compactification of the moduli space since, by Theorem 1.2, the boundary divisor has positive codimension. In particular, it is not necessary for us to consider an extension of the Weil–Petersson symplectic form to the boundary.

The following is a brief selection of the early results concerning Weil–Petersson volumes.[6]

- Wolpert [58, 59] proved that $V_{0,4}(\mathbf{0}) = 2\pi^2$, $V_{1,1}(0) = \frac{\pi^2}{12}$ and $V_{g,n}(\mathbf{0}) = q(2\pi^2)^{3g-3+n}$ for some rational number q. This last fact is a corollary of Theorem 1.26, from which we deduce that $q = \int_{\overline{\mathcal{M}}_{g,n}} \kappa_1^{3g-3+n}$.
- Penner [50] proved that $V_{1,2}(\mathbf{0}) = \frac{\pi^4}{4}$.
- Zograf [63, 62] proved that $V_{0,5}(\mathbf{0}) = 10\pi^4$ and that $V_{0,n}(\mathbf{0}) = \frac{(2\pi^2)^{n-3}}{(n-3)!} a_n$, where $a_3 = 1$ and for $n \geq 4$,
$$a_n = \frac{1}{2} \sum_{k=1}^{n-3} \frac{k(n-k-2)}{n-1} \binom{n-4}{k-1} \binom{n}{k+1} a_{k+2} a_{n-k}.$$
- Näätänen and Nakanishi [40, 41] proved that
$$V_{0,4}(L_1, L_2, L_3, L_4) = \frac{1}{2}(L_1^2 + L_2^2 + L_3^2 + L_4^2 + 4\pi^2) \text{ and } V_{1,1}(L_1) = \frac{1}{48}(L_1^2 + 4\pi^2).$$

Much of the early work in this direction was rooted in algebraic geometry, where there is no analogue to the length of a boundary component. Hence, the results often concerned the constant $V_{g,n}(\mathbf{0})$ rather than the more general volume function $V_{g,n}(\mathbf{L})$. However, Näätänen and Nakanishi's work suggests that this latter problem yields nice results, at least for moduli spaces of complex dimension one. In fact, they showed that $V_{1,1}(L_1)$ and $V_{0,4}(L_1, L_2, L_3, L_4)$ are both polynomials in the boundary lengths. That this is the case for all of the Weil–Petersson volumes $V_{g,n}(\mathbf{L})$ was proven by Mirzakhani in two distinct ways [33, 34]. The remainder of this section is dedicated to giving the essential ideas, results and proofs involved in Mirzakhani's work.

Mirzakhani's recursion

One of the main obstacles in calculating the volume of the moduli space is the fact that the Fenchel–Nielsen coordinates do not behave nicely under the action of the mapping class group. In particular, there is no concrete description for a fundamental domain of $\mathcal{M}_{g,n}(\mathbf{L})$ in $\mathcal{T}_{g,n}(\mathbf{L})$ for general values of g and n. Mirzakhani had the idea of unfolding the integral required to calculate the volume of the moduli space to a cover over the moduli space. In general, consider a covering $\pi : X_1 \to X_2$, let dv_2 be a volume form on X_2, and let $dv_1 = \pi^* dv_2$ be the pull-back

[6] When comparing these results with the original sources, there may be some discrepancy due to two issues. First, there are distinct normalisations of the Weil–Petersson symplectic form which differ by a factor of two. We have scaled the results, where appropriate, to correspond with the Weil–Petersson symplectic form defined earlier. Second, one must treat the special cases of $V_{1,1}(L_1)$ and $V_{2,0}$ with some care. This is due to the fact that every point on $\mathcal{M}_{1,1}(L_1)$ and $\mathcal{M}_{2,0}$ is an orbifold point, generically with orbifold group $\mathbb{Z}_2$. As a result, the statement of certain theorems holds true only if one considers $V_{1,1}(L_1)$ and $V_{2,0}$ as orbifold volumes — in other words, half of the true volumes. The upshot is that one should not be alarmed if results concerning Weil–Petersson volumes from different sources differ by a factor which is a power of two.

volume form on X_1. If π is a finite covering, then for a function $f : X_1 \to \mathbb{R}$, one can consider the push-forward function $\pi_* f : X_2 \to \mathbb{R}$ defined by

$$(\pi_* f)(x) = \sum_{y \in \pi^{-1}(x)} f(y).$$

In fact, even if π is an infinite covering, then the push-forward may still exist provided f is sufficiently well-behaved. The main reason for considering this setup is the fact that

$$\int_{X_1} f \, dv_1 = \int_{X_2} (\pi_* f) \, dv_2.$$

Therefore, we wish to find a covering $\pi : \widetilde{\mathcal{M}}_{g,n}(\mathbf{L}) \to \mathcal{M}_{g,n}(\mathbf{L})$ such that the former space is easier to integrate over than the latter. A natural candidate is $\mathcal{T}_{g,n}(\mathbf{L})/G$, where G is a subgroup of the mapping class group $\mathrm{Mod}_{g,n}$. So, to calculate the volume $V_{g,n}(\mathbf{L})$ in this way, it is desirable to express the constant function on $\mathcal{M}_{g,n}(\mathbf{L})$ as the sum of push-forward functions of the type described above. The main tool used by Mirzakhani to carry out this integration scheme was a generalisation of the following "remarkable identity" discovered by McShane [30].

Theorem 1.27 (McShane's Identity). *On a cusped hyperbolic torus,*

$$\sum_{\gamma} \frac{1}{1 + e^{-\ell(\gamma)/2}} = \frac{1}{2},$$

where the sum is over all simple closed geodesics and $\ell(\gamma)$ denotes the length of γ.

Mirzakhani [33] was able to extend McShane's identity to arbitrary hyperbolic surfaces with geodesic boundary components in the following way.

Theorem 1.28 (Generalised McShane's Identity). *On a hyperbolic surface with n geodesic boundary components $\beta_1, \beta_2, \ldots, \beta_n$ of lengths $L_1, L_2, \ldots, L_n$, respectively,*

$$\sum_{(\alpha_1, \alpha_2)} \mathcal{D}(L_1, \ell(\alpha_1), \ell(\alpha_2)) + \sum_{k=2}^{n} \sum_{\gamma} \mathcal{R}(L_1, L_k, \ell(\gamma)) = L_1.$$

Here, the first summation is over unordered pairs (α_1, α_2) of simple closed geodesics which bound a pair of pants with β_1, while the second summation is over simple closed geodesics γ which bound a pair of pants with β_1 and β_k. The functions $\mathcal{D} : \mathbb{R}^3 \to \mathbb{R}$ and $\mathcal{R} : \mathbb{R}^3 \to \mathbb{R}$ are defined as follows.

$$\mathcal{D}(x, y, z) = 2 \log \left(\frac{e^{\frac{x}{2}} + e^{\frac{y+z}{2}}}{e^{\frac{-x}{2}} + e^{\frac{y+z}{2}}} \right) \qquad \mathcal{R}(x, y, z) = x - \log \left(\frac{\cosh \frac{y}{2} + \cosh \frac{x+z}{2}}{\cosh \frac{y}{2} + \cosh \frac{x-z}{2}} \right)$$

This identity allows one to express the constant function on $\mathcal{M}_{g,n}(\mathbf{L})$ as a sum of push-forward functions on coverings of the form $\mathcal{M}^{\gamma}_{g,n}(\mathbf{L}) \to \mathcal{M}_{g,n}(\mathbf{L})$. Here, γ is a multicurve on the surface $S_{g,n}$ and $\mathcal{M}^{\gamma}_{g,n}(\mathbf{L}) = \mathcal{T}_{g,n}(\mathbf{L})/G$, where $G \subseteq \mathrm{Mod}_{g,n}$ is the stabiliser of the multicurve γ. Since these coverings are inherently easier to integrate over, it is possible to unfold the integral required to calculate the volume of $\mathcal{M}_{g,n}(\mathbf{L})$. The final result is the following recursive formula due to Mirzakhani [33].

Theorem 1.29 (Mirzakhani's recursion)**.** *The Weil–Petersson volumes satisfy the following formula.*

$$\begin{aligned} 2\frac{\partial}{\partial L_1} L_1 V_{g,n}(\mathbf{L}) &= \int_0^\infty \int_0^\infty xy\, H(x+y, L_1)\, V_{g-1,n+1}(x, y, \widehat{\mathbf{L}})\, dx\, dy \\ &+ \sum_{\substack{g_1+g_2=g \\ \mathcal{I}_1 \sqcup \mathcal{I}_2 = [2,n]}} \int_0^\infty \int_0^\infty xy\, H(x+y, L_1)\, V_{g_1,|\mathcal{I}_1|+1}(x, \mathbf{L}_{\mathcal{I}_1})\, V_{g_2,|\mathcal{I}_2|+1}(y, \mathbf{L}_{\mathcal{I}_2})\, dx\, dy \\ &+ \sum_{k=2}^{n} \int_0^\infty x[H(x, L_1 + L_k) + H(x, L_1 - L_k)]\, V_{g,n-1}(x, \widehat{\mathbf{L}}_k)\, dx \end{aligned}$$

We have used $\widehat{\mathbf{L}} = (L_2, L_3, \ldots, L_n)$, $\widehat{\mathbf{L}}_k = (L_2, \ldots, \widehat{L}_k, \ldots, L_n)$ *and* $\mathbf{L}_{\mathcal{I}} = (L_{i_1}, L_{i_2}, \ldots, L_{i_m})$ *for* $\mathcal{I} = \{i_1, i_2, \ldots, i_m\}$. *Furthermore, the function* $H : \mathbb{R}^2 \to \mathbb{R}$ *is defined by*

$$H(x, y) = \frac{1}{1 + e^{\frac{x+y}{2}}} + \frac{1}{1 + e^{\frac{x-y}{2}}}.$$

Mirzakhani's recursion expresses $V_{g,n}(\mathbf{L})$ in terms of certain integral transforms of volumes corresponding to moduli spaces with smaller negative Euler characteristic. Therefore, the calculation of any Weil–Petersson volume can be reduced to the base cases

$$V_{0,1}(L_1) = 0, \quad V_{0,2}(L_1, L_2) = 0, \quad V_{0,3}(L_1, L_2, L_3) = 1 \quad \text{and} \quad V_{1,1}(L_1) = \frac{1}{48}(L_1^2 + 4\pi^2).$$

Without going into the details of the proof of Mirzakhani's recursion, we can at least understand how the three terms arise. Philosophically, the mechanism behind Mirzakhani's recursion is based on removing pairs of pants from $S_{g,n}$ which contain the boundary component β_1. The three types of surface which can result are

- $S_{g-1,n+1}$ when the pair of pants is bound by β_1 and two interior simple closed curves and its removal leaves a connected surface;
- $S_{g_1,n_1+1} \cup S_{g_2,n_2+1}$ when the pair of pants is bound by β_1 and two interior simple closed curves and its removal leaves a disconnected surface; or
- $S_{g,n-1}$ when the pair of pants is bound by β_1, another boundary component β_k and an interior simple closed curve.

These correspond precisely to the three terms on the right hand side of Mirzakhani's recursion.

In practice, Mirzakhani's recursion requires one to be able to integrate expressions of the form

$$\int_0^\infty x^{2k-1} H(x,t)\, dx \quad \text{and} \quad \int_0^\infty \int_0^\infty x^{2a-1} y^{2b-1} H(x+y,t)\, dx\, dy,$$

where k, a and b are positive integers. It is straightforward to prove that

$$\int_0^\infty x^{2k-1} H(x,t)\, dx = F_{2k-1}(t)$$

$$\int_0^\infty \int_0^\infty x^{2a-1} y^{2b-1} H(x+y,t)\, dx\, dy = \frac{(2a-1)!(2b-1)!}{(2a+2b-1)!} F_{2a+2b-1}(t)$$

where

$$F_{2k-1}(t) = (2k-1)! \sum_{i=0}^{k} \frac{\zeta(2i)(2^{2i+1}-4)}{(2k-2i)!} t^{2k-2i}.$$

The following equations give the explicit computations of $F_1(t)$, $F_3(t)$, $F_5(t)$ and $F_7(t)$.

$$\begin{aligned}
F_1(t) &= \frac{t^2}{2} + \frac{2\pi^2}{3} \\
F_3(t) &= \frac{t^4}{4} + 2\pi^2 t^2 + \frac{28\pi^4}{15} \\
F_5(t) &= \frac{t^6}{6} + \frac{10\pi^2 t^4}{3} + \frac{56\pi^4 t^2}{3} + \frac{992\pi^6}{63} \\
F_7(t) &= \frac{t^8}{8} + \frac{14\pi^2 t^6}{3} + \frac{196\pi^4 t^4}{3} + \frac{992\pi^6 t^2}{3} + \frac{4064\pi^8}{15}
\end{aligned}$$

Observe that $F_{2k-1}(t)$ is an even polynomial of degree $2k$ in t and that the coefficient of t^{2m} is a rational multiple of π^{2k-2m}. The following calculation of $V_{1,2}(L_1, L_2)$ demonstrates how Mirzakhani's recursion can be used to compute Weil–Petersson volumes.

Example 1.30. Only two of the three terms on the right hand side are non-zero, one depending on $V_{0,3}$ and the other on $V_{1,1}$. This corresponds to the fact that removing a pair of pants from the surface $S_{1,2}$ which contains at least one of the boundary components must leave either $S_{0,3}$ or $S_{1,1}$.

$$\begin{aligned}
&2\frac{\partial}{\partial L_1} L_1 V_{1,2}(L_1, L_2) \\
&= \int_0^\infty \int_0^\infty xy\, H(x+y, L_1)\, V_{0,3}(x,y,L_2)\, dx\, dy + \int_0^\infty x[H(x, L_1+L_2) + H(x, L_1-L_2)]\, V_{1,1}(x)\, dx \\
&= \int_0^\infty \int_0^\infty xy\, H(x+y, L_1)\, dx\, dy + \int_0^\infty x[H(x, L_1+L_2) + H(x, L_1-L_2)] \left(\frac{x^2+4\pi^2}{48}\right) dx \\
&= \frac{1}{6} F_3(L_1) + \frac{1}{48} F_3(L_1+L_2) + \frac{1}{48} F_3(L_1-L_2) + \frac{\pi^2}{12} F_1(L_1+L_2) + \frac{\pi^2}{12} F_1(L_1-L_2) \\
&= \frac{5L_1^4}{96} + \frac{L_1^2 L_2^2}{16} + \frac{L_2^4}{96} + \frac{\pi^2 L_1^2}{2} + \frac{\pi^2 L_2^2}{6} + \frac{\pi^4}{2}
\end{aligned}$$

Now integrate with respect to L_1 and divide by $2L_1$ to obtain the desired volume.

$$\begin{aligned} V_{1,2}(L_1, L_2) &= \frac{L_1^4}{192} + \frac{L_1^2 L_2^2}{96} + \frac{L_2^4}{192} + \frac{\pi^2 L_1^2}{12} + \frac{\pi^2 L_2^2}{12} + \frac{\pi^4}{4} \\ &= \frac{1}{192}(L_1^2 + L_2^2 + 4\pi^2)(L_1^2 + L_2^2 + 12\pi^2) \end{aligned}$$

A simple though important corollary of Mirzakhani's recursion is the following result.

Corollary 1.31. *The volume $V_{g,n}(\mathbf{L})$ is an even symmetric polynomial of degree $6g - 6 + 2n$ in the boundary lengths $L_1, L_2, \ldots, L_n$. Furthermore, the coefficient of $L_1^{2\alpha_1} L_2^{2\alpha_2} \ldots L_n^{2\alpha_n}$ is a rational multiple of $\pi^{6g-6+2n-2|\boldsymbol{\alpha}|}$.*

The symmetry of $V_{g,n}(\mathbf{L})$ follows immediately from the symmetry of the boundary labels. The remainder of the statement can be proven by induction on the negative Euler characteristic. That the Weil–Petersson volumes are polynomials is a rather amazing fact. Actually, we will see that the technique of symplectic reduction can be used to prove that their coefficients store interesting information.

Volumes and symplectic reduction

Symplectic geometry has its origins in the mathematical formulation and generalisation of the phase space of a classical mechanical system.[7] For a long time, physicists have taken advantage of the fact that when a symmetry group of dimension n acts on a system, then the number of degrees of freedom for the positions and momenta can be reduced by $2n$. The analogous mathematical phenomenon is known as symplectic reduction. More precisely, consider a symplectic manifold (M, ω) of dimension $2d$ with a

$$T^n = \underbrace{S^1 \times S^1 \times \ldots \times S^1}_{n \text{ times}}$$

action that preserves the symplectic form. Furthermore, suppose that this action is the Hamiltonian flow for the moment map $\mu : M \to \mathbb{R}^n$ and that $\mathbf{0}$ is a regular value of the moment map. Note that T^n must act on the level sets of μ, so one can define

$$M_{\mathbf{0}} = \mu^{-1}(\mathbf{0})/T^n.$$

The main theorem of symplectic reduction states that $M_{\mathbf{0}}$ is a symplectic manifold of dimension $2d - 2n$ with respect to the unique 2-form $\omega_{\mathbf{0}}$ which satisfies $i^*\omega = \pi^*\omega_{\mathbf{0}}$, where $\pi : \mu^{-1}(\mathbf{0}) \to M_{\mathbf{0}}$ and $i : \mu^{-1}(\mathbf{0}) \to M$ are the natural projection and inclusion maps. In fact, since $\mathbf{0}$ is a

[7] For a reasonably elementary introduction to symplectic geometry, we recommend the text [7].

regular value, there exists an $\varepsilon > 0$ such that all $\mathbf{a} \in \mathbb{R}^n$ satisfying $|\mathbf{a}| < \varepsilon$ are also regular values. So it is possible to define symplectic manifolds $(M_{\mathbf{a}}, \omega_{\mathbf{a}})$ for all such $\mathbf{a}$.

If we think of the T^n action as n commuting circle actions, then the kth copy of S^1 induces a circle bundle $\mathcal{S}_k$ on $M_{\mathbf{0}}$. Let the first Chern class of this circle bundle be denoted by $c_1(\mathcal{S}_k) = \phi_k$. Then the variation of the symplectic form $\omega_{\mathbf{0}}$ is described by the following result [19].

Theorem 1.32. *For $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ sufficiently close to $\mathbf{0}$, $(M_{\mathbf{a}}, \omega_{\mathbf{a}})$ is symplectomorphic to $M_{\mathbf{0}}$ equipped with a symplectic form whose cohomology class is equal to $[\omega_{\mathbf{0}}] + a_1\phi_1 + a_2\phi_2 + \cdots + a_n\phi_n$.*

This then allows us to consider the variation of the volume.

Corollary 1.33. *For $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ sufficiently close to $\mathbf{0}$, the volume of $(M_{\mathbf{a}}, \omega_{\mathbf{a}})$ is a polynomial in $a_1, a_2, \ldots, a_n$ of degree $d = \frac{1}{2}dim(M_{\mathbf{a}})$ given by the formula*

$$\sum_{|\boldsymbol{\alpha}|+m=d} \frac{\int_{M_0} \phi_1^{\alpha_1}\phi_2^{\alpha_2}\ldots\phi_n^{\alpha_n}\omega^m}{\boldsymbol{\alpha}!m!} a_1^{\alpha_1}a_2^{\alpha_2}\ldots a_n^{\alpha_n}.$$

We now retrace Mirzakhani's steps in constructing a setup in which these techniques will produce Weil–Petersson volumes. Doing so will provide a second proof of the fact that $V_{g,n}(\mathbf{L})$ is a polynomial and, furthermore, show that its coefficients store interesting information — namely, intersection numbers on the moduli space $\overline{\mathcal{M}}_{g,n}$.

Consider the space

$$\widehat{\mathcal{M}}_{g,n} = \left\{ (X, p_1, p_2, \ldots, p_n) \;\middle|\; \begin{array}{l} X \text{ is a genus } g \text{ hyperbolic surface with } n \text{ geodesic boundary} \\ \text{components } \beta_1, \beta_2, \ldots, \beta_n \text{ and } p_k \in \beta_k \text{ for all } k \end{array} \right\}$$

and note that there is a $T^n = S^1 \times S^1 \times \ldots \times S^1$ action on the space, where the kth copy of S^1 moves the point p_k along the boundary β_k. This action is the Hamiltonian flow for the moment map $\mu : \widehat{\mathcal{M}}_{g,n} \to \mathbb{R}^n$ defined by $\mu(X, p_1, p_2, \ldots, p_n) = (\frac{1}{2}L_1^2, \frac{1}{2}L_2^2, \ldots, \frac{1}{2}L_n^2)$, where L_k denotes the length of the geodesic boundary component β_k.

Fix a tuple $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ of disjoint simple closed curves on $S_{g,2n}$ such that γ_k bounds a pair of pants with the boundaries labelled $2k-1$ and $2k$. Then $g \in \mathrm{Mod}_{g,2n}$ acts on γ by $g\gamma = (g\gamma_1, g\gamma_2, \ldots, g\gamma_n)$. Now define

$$\mathcal{M}_{g,2n}^{\gamma} = \{(X, \eta_1, \eta_2, \ldots, \eta_n) \mid X \in \mathcal{M}_{g,2n}(\mathbf{0}) \text{ and } (\eta_1, \eta_2, \ldots, \eta_n) \in \mathrm{Mod}_{g,2n} \cdot \gamma\}$$

or, equivalently, consider $\mathcal{M}_{g,2n}^{\gamma} = \mathcal{T}_{g,2n}(\mathbf{0})/G$ where $G = \bigcap \mathrm{Stab}(\gamma_k) \subseteq \mathrm{Mod}_{g,2n}$. Since the Weil–Petersson symplectic form on $\mathcal{T}_{g,2n}(\mathbf{0})$ is invariant under the action of $\mathrm{Mod}_{g,2n}$, it is also invariant under G. Therefore, it descends to a symplectic form on $\mathcal{M}_{g,2n}^{\gamma}$.

There is a natural map $f : \widehat{\mathcal{M}}_{g,n} \to \mathcal{M}_{g,2n}^{\gamma}$. Simply take $(X, p_1, p_2, \ldots, p_n)$ where $X \in \mathcal{M}_{g,n}(\mathbf{L})$

and, to the kth boundary component, glue in a pair of pants with two cusps labelled $2k-1$ and $2k$ and a boundary component of length L_k. Of course, this can be done in infinitely many ways and we choose the unique way such that the seam from the cusp labelled $2k$ meets the point p_k. The map f can be used to pull back the symplectic form to $\widehat{\mathcal{M}}_{g,n}$, where it is invariant under the T^n-action. Furthermore, the canonical map $\ell^{-1}(\mathbf{L})/T^n \to \mathcal{M}_{g,n}(\mathbf{L})$ is a symplectomorphism, where $\ell : \widehat{\mathcal{M}}_{g,n} \to \mathbb{R}^n$ sends a hyperbolic surface to its boundary lengths.

As mentioned earlier, the T^n action gives rise to n circle bundles $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n$ on the symplectic reduction $\mathcal{M}_{g,n}(\mathbf{0}) = \mu^{-1}(\mathbf{0})/T^n$. Mirzakhani managed to prove the following fact concerning the Chern classes of these circle bundles.

Proposition 1.34. *For $k = 1, 2, \ldots, n$, $c_1(\mathcal{S}_k) = \psi_k \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$.*

We are now ready to state and prove one of the most important results underlying this thesis.

Theorem 1.35 (Mirzakhani's theorem)**.** *The volume polynomial $V_{g,n}(\mathbf{L})$ is given by the formula*

$$\sum_{|\boldsymbol{\alpha}|+m=3g-3+n} \frac{(2\pi^2)^m \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \psi_2^{\alpha_2} \cdots \psi_n^{\alpha_n} \kappa_1^m}{2^{|\boldsymbol{\alpha}|} \boldsymbol{\alpha}! \, m!} L_1^{2\alpha_1} L_2^{2\alpha_2} \cdots L_n^{2\alpha_n}.$$

Proof. We simply apply Corollary 1.33 to the symplectic manifold $\widehat{\mathcal{M}}_{g,n}$ with the moment map μ defined earlier. This shows that the Weil–Petersson volume of $\mathcal{M}_{g,n}(\mathbf{L})$ in a neighbourhood of $\mathbf{0}$ is a polynomial in $L_1^2, L_2^2, \ldots, L_n^2$ whose coefficients are given by integrating products of Chern classes of the circle bundles $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n$ alongside powers of the reduced symplectic form. By Theorem 1.34, these Chern classes are precisely the psi-classes. Furthermore, the reduced symplectic form coincides with the Weil–Petersson form, whose cohomology class we can write as $2\pi^2\kappa_1$, by Theorem 1.26. □

We have purposefully glossed over some of the more technical details in the proof of Theorem 1.35. The three main issues are as follows.

- In actual fact, $\mathbf{0}$ is not a regular value of the moment map, so we cannot legitimately construct the symplectic reduction as proposed. However, all values away from $\mathbf{0}$ are regular, so we can form the reduced manifold $M_{\boldsymbol{\epsilon}}$ for $\boldsymbol{\epsilon} \neq \mathbf{0}$. We recover the desired result in the $\boldsymbol{\epsilon} \to \mathbf{0}$ limit.

- The literature on symplectic reduction generally does not consider the case of symplectic orbifolds. However, in analogy with Theorem 1.1 and Theorem 1.2, one can get around such problems by lifting to a manifold cover. This takes a little extra care, but essentially causes no problems.

- Proposition 1.34 claims that $c_1(\mathcal{S}_k)$ is an element of $H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ when it is apparent that $\mathcal{S}_k$ is a circle bundle over the uncompactified space $\mathcal{M}_{g,n}(\mathbf{0})$. However, it is straightforward to extend $\mathcal{S}_k$ to the compactification.

Mirzakhani's theorem shows that $V_{g,n}(\mathbf{L})$ is a polynomial whose coefficients store information about the intersection numbers on $\overline{\mathcal{M}}_{g,n}$. In fact, all psi-class intersection numbers on $\overline{\mathcal{M}}_{g,n}$ can be recovered from the top degree part of $V_{g,n}(\mathbf{L})$ alone. On the other hand, Mirzakhani's recursion shows that the Weil–Petersson volumes can be calculated recursively. Therefore, these two theorems together provide an effective recursive algorithm to determine all psi-class intersection numbers on $\overline{\mathcal{M}}_{g,n}$. So it should come as little surprise that Mirzakhani was able to give a new proof of Witten's conjecture using these results. However, there are three remarkable aspects of her proof. First, she proved Witten's conjecture by directly verifying the Virasoro constraints. Second, Mirzakhani's proof was the first to appear which did not require the use of a matrix model. Third, and most importantly for us, her proof of Witten's conjecture is deeply rooted in hyperbolic geometry.

The basis for the new results appearing in this thesis is Mirzakhani's theorem. In particular, it allows us to adopt the philosophy that any meaningful statement about the volume $V_{g,n}(\mathbf{L})$ gives a meaningful statement about the intersection theory on $\overline{\mathcal{M}}_{g,n}$, and vice versa.

Chapter 2

Weil–Petersson volume relations and hyperbolic cone surfaces

In this chapter, we prove the generalised string and dilaton equations, which take the form of relations between Weil–Petersson volumes. Two distinct proofs are supplied — one arising from algebraic geometry and the other from Mirzakhani's recursion. One striking feature of these results is that they are highly suggestive of a third proof, involving the geometry of hyperbolic cone surfaces. As applications of the generalised string and dilaton equations, we show how to efficiently calculate genus 0 and genus 1 Weil–Petersson volumes and give a formula for the volume $V_{g,0}$, a case not dealt with by Mirzakhani's recursion. The chapter concludes with some ideas on how this work may be extended.

2.1 Volume polynomial relations

Generalised string and dilaton equations

Mirzakhani's theorem — Theorem 1.35 — asserts that the Weil–Petersson volumes are polynomials whose coefficients store interesting information. In particular, any meaningful statement about $V_{g,n}(\mathbf{L})$ gives a meaningful statement about the intersection theory on $\overline{\mathcal{M}}_{g,n}$, and vice versa. Mirzakhani's recursion — Theorem 1.29 — provides an effective algorithm to calculate these Weil–Petersson volumes. Therefore, it is possible to adopt a numerological approach and simply search for interesting patterns among the volume polynomials. These, in turn, should translate into relations concerning the intersection theory on moduli spaces of curves. The hope is that packaging the intersection numbers on $\overline{\mathcal{M}}_{g,n}$ into the polynomial $V_{g,n}(\mathbf{L})$ may reveal useful information which is otherwise obscured.

For example, consider $V_{g,1}(L)$ for small values of g.

$$\begin{aligned}
V_{1,1}(L) &= \frac{1}{48}(L^2+4\pi^2)\\
V_{2,1}(L) &= \frac{1}{2211840}(L^2+4\pi^2)(L^2+12\pi^2)(5L^4+384\pi^2L^2+6960\pi^4)\\
V_{3,1}(L) &= \frac{1}{267544166400}(L^2+4\pi^2)(5L^{12}+2136\pi^2L^{10}+\cdots+152253906944\pi^{12})\\
V_{4,1}(L) &= \frac{1}{1035588555767808000}(L^2+4\pi^2)(35L^{18}+\cdots+24243263955499483136\pi^{18})
\end{aligned}$$

From this evidence alone, it is natural to conjecture that $V_{g,1}(L)$ always possesses a factor of $L^2+4\pi^2$. This is indeed true, but also suggests that the volume polynomials exhibit interesting behaviour when one of the arguments is evaluated at $2\pi i$. A more thorough investigation uncovers the following results.

Theorem 2.1 (Generalised string equation). *For $2g-2+n>0$, the Weil–Petersson volumes satisfy the following relation.*[1]

$$V_{g,n+1}(\mathbf{L},2\pi i) = \sum_{k=1}^{n}\int L_k V_{g,n}(\mathbf{L})\,dL_k$$

Theorem 2.2 (Generalised dilaton equation). *For $2g-2+n>0$, the Weil–Petersson volumes satisfy the following relation.*

$$\frac{\partial V_{g,n+1}}{\partial L_{n+1}}(\mathbf{L},2\pi i) = 2\pi i(2g-2+n)V_{g,n}(\mathbf{L})$$

These two statements relate the volume $V_{g,n+1}(\mathbf{L},L_{n+1})$ to the volume $V_{g,n}(\mathbf{L})$ and, therefore, also relate the intersection theory on $\overline{\mathcal{M}}_{g,n+1}$ to the intersection theory on $\overline{\mathcal{M}}_{g,n}$. In fact, equating coefficients of the top degree terms, one recovers the string and dilaton equations, thereby explaining our choice of nomenclature. At first glance, there are two notable observations one can make about these relations. First, their concise nature indicates that the Weil–Petersson volume $V_{g,n}(\mathbf{L})$ is a useful way to package intersection numbers on $\overline{\mathcal{M}}_{g,n}$. Second, the intriguing appearance of the number $2\pi i$ suggests that there is interesting geometry lurking behind these statements.

Three viewpoints

There are at least three possible approaches to proving the generalised string and dilaton equations.

[1] Here, we have used the notation $\int L_k V_{g,n}(\mathbf{L})\,dL_k$ to refer to the unique antiderivative of $L_k V_{g,n}(\mathbf{L})$ with respect to L_k which has zero constant term. A more accurate, though also more cumbersome, notation would have been to express this as $\int_0^{L_k} x V_{g,n}(L_1,\ldots,L_{k-1},x,L_{k+1},\ldots,L_n)\,dx$.

- In the world of algebraic geometry, one can pass from a curve in $\overline{\mathcal{M}}_{g,n+1}$ to a curve in $\overline{\mathcal{M}}_{g,n}$ simply by forgetting the last marked point and stabilising, if necessary. This process is formalised by the forgetful morphism $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$. Witten's proofs of the string and dilaton equations arise from the analysis of psi-classes under pull-back via the forgetful morphism [57]. Naturally, one would expect their generalisations to succumb to a similar approach. The first step is to invoke Mirzakhani's theorem in order to translate the generalised string and dilaton equations into equivalent statements concerning intersection numbers. For our purposes, the analysis of kappa-classes under pull-back via the forgetful morphism is also required. This approach is presented in Section 2.2.

- Mirzakhani's recursion allows one to determine all of the polynomials $V_{g,n,}(\mathbf{L})$ from a small number of base cases. Therefore, any relation between the Weil–Petersson volumes is, in some sense, encapsulated in her recursive formula. So it should come as little surprise that the generalised string and dilaton equations follow from Mirzakhani's results. What is interesting, however, is the nature of these proofs and the fact that they rely on certain interesting identities among the Bernoulli numbers. This approach is presented in Section 2.3.

- In both the generalised string and dilaton equations, one of the arguments in the volume polynomial is evaluated at $2\pi i$. It would be desirable to ascribe geometric meaning to this formal evaluation. Indeed, a general phenomenon in hyperbolic geometry is the fact that a purely imaginary length often corresponds to an angle. In this way, one is motivated to consider hyperbolic cone surfaces with a cone angle approaching 2π. Therefore, we have a tantalising connection between the intersection theory on moduli spaces of curves and the geometry of hyperbolic cone surfaces.

  Unfortunately, it is a difficult task to make such a connection explicit, since the geometry of hyperbolic cone surfaces is not very well understood. We conclude this chapter with some discussion of these difficulties and how they may possibly be overcome. This approach largely remains work in progress but the hope is that the ideas presented may be the germ for future results.

Note that these three viewpoints naturally correspond to three ways in which one can pass from a surface of type $(g, n+1)$ to a surface of type (g, n). When the surfaces are endowed with the structure of algebraic curves, then the forgetful morphism allows one to essentially remove one of the marked points. In the hyperbolic setting, there is no analogous map which forgets a boundary component. On the other hand, one can decrease the number of boundary components by simply removing a pair of pants. As noted earlier, this is precisely the mechanism by which Mirzakhani's recursion inductively reduces the calculation of $V_{g,n}(\mathbf{L})$. Another way to decrease the number of boundary components is to degenerate one of them — first, from a geodesic boundary to a cusp, then from a cusp to a cone point with cone angle 2π.

2.2 Proofs via algebraic geometry

Pull-back relations

In this section, we describe the behaviour of the psi-classes and kappa-classes under pull-back via the forgetful morphism. These results, which appear in various references such as Arbarello and Cornalba's paper [2], are then used to deduce the generalised string and dilaton equations. For clarification, we will occasionally use a tilde over a symbol to distinguish between analogous constructions on $\overline{\mathcal{M}}_{g,n}$ and those on $\overline{\mathcal{M}}_{g,n+1}$. For example, ψ_k denotes the kth psi-class on $\overline{\mathcal{M}}_{g,n}$ while $\widetilde{\psi}_k$ denotes the kth psi-class on $\overline{\mathcal{M}}_{g,n+1}$.

One might naively expect that $\widetilde{\mathcal{L}}_k = \pi^*\mathcal{L}_k$ or, at the level of Chern classes, that $\widetilde{\psi}_k = \pi^*\psi_k$ for $k = 1, 2, \ldots, n$. However, this is not the case due to the fact that the morphism $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ not only forgets the last marked point, but also stabilises the resulting curve, if necessary. So the discrepancy is, in some sense, caused by the geometry occurring at the boundary divisor of the moduli space. The actual behaviour of the psi-classes under pull-back via the forgetful morphism is given by the following result.

Lemma 2.3 (Pull-back relation for psi-classes)**.** *For $k = 1, 2, \ldots, n$, the cohomology classes $\widetilde{\psi}_k \in H^2(\overline{\mathcal{M}}_{g,n+1}, \mathbb{Q})$ and $\psi_k \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ are related by the equation*

$$\widetilde{\psi}_k = \pi^*\psi_k + D_k.$$

Proof. Recall that we are using the notation D_k to represent the divisor $\sigma_k(\overline{\mathcal{M}}_{g,n})$ as well as the corresponding homology class and Poincaré dual cohomology class. Away from D_k, the line bundles $\widetilde{\mathcal{L}}_k$ and $\pi^*\mathcal{L}_k$ are identical. Therefore, we have the following isomorphism of line bundles for some value of m.

$$\widetilde{\mathcal{L}}_k \cong \pi^*\mathcal{L}_k \otimes \mathcal{O}(mD_k)$$

For curves corresponding to points in D_k, the kth marked point lies on a rigid object — namely, a copy of $\mathbb{CP}^1$ with three special points. Therefore, the pull-back of $\widetilde{\mathcal{L}}_k$ under σ_k is trivial. So we have

$$\mathcal{O} = \sigma_k^*\widetilde{\mathcal{L}}_k = \sigma_k^*\pi^*\mathcal{L}_k \otimes \sigma_k^*\mathcal{O}(mD_k) = \mathcal{L}_k \otimes (\mathcal{L}_k^*)^m,$$

from which it follows that $m = 1$. Here, we have used $\sigma_k^*\mathcal{O}(D_k) = \mathcal{L}_k^*$, which holds because both line bundles are isomorphic to N_{σ_k}, the normal bundle to the embedding $\sigma_k : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$. Finally, we deduce that $\widetilde{\mathcal{L}}_k \cong \pi^*\mathcal{L}_k \otimes \mathcal{O}(D_k)$ and, after taking Chern classes, one obtains the desired result. □

In the previous proof, we observed that the pull-back of $\widetilde{\mathcal{L}}_k$ along σ_k is trivial. It follows that $\widetilde{\psi}_k \cdot D_k = 0$, and we can combine this with the pull-back relation for psi-classes to obtain the following equation.

$$\widetilde{\psi}_k^{m+1} = \widetilde{\psi}_k \cdot (\pi^*\psi_k + D_k)^m = \widetilde{\psi}_k \cdot \pi^*\psi_k^m = (\pi^*\psi_k + D_k) \cdot \pi^*\psi_k^m = \pi^*\psi_k^{m+1} + D_k \cdot \pi^*\psi_k^m$$

Another consequence of $\widetilde{\psi}_k \cdot D_k = 0$ and the pull-back relation for psi-classes is the equation $(\pi^*\psi_k + D_k) \cdot D_k = 0$ or equivalently, $D_k^2 = (-\pi^*\psi_k) \cdot D_k$. By induction on this identity, we deduce that $D_k^{m+1} = (-\pi^*\psi_k)^m \cdot D_k$ for all non-negative integers m. Combining this with the equation above yields

$$\widetilde{\psi}_k^{m+1} = \pi^*\psi_k^{m+1} + (-1)^m D_k^{m+1}.$$

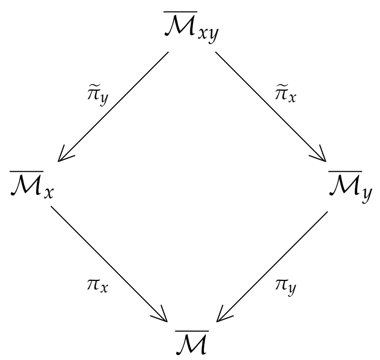
Lemma 2.4 (Pull-back relation for kappa-classes). *The cohomology classes* $\widetilde{\kappa}_m \in H^2(\overline{\mathcal{M}}_{g,n+1}, \mathbb{Q})$ *and* $\kappa_m \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ *are related by the equation*

$$\widetilde{\kappa}_m = \pi^*\kappa_m + \psi_{n+1}^m.$$

Proof. Consider the following commutative diagram, where

$$\overline{\mathcal{M}} \cong \overline{\mathcal{M}}_{g,n}, \quad \overline{\mathcal{M}}_x \cong \overline{\mathcal{M}}_y \cong \overline{\mathcal{M}}_{g,n+1} \quad \text{and} \quad \overline{\mathcal{M}}_{xy} \cong \overline{\mathcal{M}}_{g,n+2}.$$

All are moduli spaces of genus g curves with marked points labelled $1, 2, \ldots, n$. The space $\overline{\mathcal{M}}_x$ has an extra point labelled x, the space $\overline{\mathcal{M}}_y$ has an extra point labelled y and the space $\overline{\mathcal{M}}_{xy}$ has two extra points labelled x and y. All maps are forgetful morphisms with the subscript denoting the label of the forgotten point.



From the previous discussion, we have the relation $\widetilde{\psi}_x^{m+1} = \widetilde{\pi}_y^*\psi_x^{m+1} + (-1)^m D_{xy}^{m+1}$ on $\overline{\mathcal{M}}_{xy}$. Here, D_{xy} denotes the divisor corresponding to the image of the section $\sigma_x : \overline{\mathcal{M}}_x \to \overline{\mathcal{M}}_{xy}$, which coincides with the image of the section $\sigma_y : \overline{\mathcal{M}}_y \to \overline{\mathcal{M}}_{xy}$. Now use the forgetful morphism $\widetilde{\pi}_x$ to push this relation down to $\overline{\mathcal{M}}_y$ and obtain $\widetilde{\pi}_{x*}\widetilde{\psi}_x^{m+1} = \widetilde{\pi}_{x*}\widetilde{\pi}_y^*\psi_x^{m+1} + (-1)^m \widetilde{\pi}_{x*} D_{xy}^{m+1}$. From the equality of ψ_{n+1} and the twisted Euler class, we have the following.

$$\widetilde{\kappa}_m = \widetilde{\pi}_{x*}\widetilde{\pi}_y^*\psi_x^{m+1} + (-1)^m \sigma_y^* D_{xy}^m = \pi_y^*\pi_{x*}\psi_x^{m+1} + \sigma_y^*(-D_{xy})^m = \pi_y^*\kappa_m + \psi_y^m$$

The first equality is a result of the fact that intersecting with D_k followed by pushing down to $\overline{\mathcal{M}}_{g,n}$ is the same as pulling back along σ_k. To obtain the second equality, we have interchanged the order of the push-forward and pull-back operations — for justification of this step, see §6.2 in [15]. The third equality uses the fact that $\psi_k = \sigma_k^*(-D_k)$, which follows from our observation in the proof of Lemma 2.3 that $\sigma_k^*\mathcal{O}(D_k) = \mathcal{L}_k^*$. □

In the algebro-geometric proofs of the generalised string and dilaton equations, we will make particular use of Lemma 2.4 in the case $m = 1$ — in other words, $\kappa_1 = \pi^*\kappa_1 + \psi_{n+1}$. We also require the following straightforward evaluations of the Gysin map $\pi_* : H^*(\overline{\mathcal{M}}_{g,n+1}, \mathbb{Q}) \to H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$.

$$\pi_*1 = 0 \qquad \pi_*D_k = 1 \text{ for } k = 1, 2, \ldots, n \qquad \pi_*\psi_{n+1} = 2g - 2 + n$$

The first two evaluations can be directly deduced from the Poincaré duality description of the Gysin map. The third uses the fact that ψ_{n+1} coincides with the twisted Euler class e, which satisfies $\langle e, \Sigma\rangle = -\chi(\Sigma - \cup D_k) = 2g - 2 + n$ on every fibre Σ.

Proof of the generalised string equation

Recall that the generalised string equation states that the following relation holds.

$$V_{g,n+1}(\mathbf{L}, 2\pi i) = \sum_{k=1}^{n} \int L_k V_{g,n}(\mathbf{L})\, dL_k$$

By Mirzakhani's theorem, the left hand side can be written as

$$\sum_{|\boldsymbol{\alpha}|+j+k=3g-2+n} \frac{(2\pi^2)^k \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{\alpha_1}\psi_2^{\alpha_2}\ldots\psi_n^{\alpha_n}\psi_{n+1}^{j}\kappa_1^k}{2^{|\boldsymbol{\alpha}|}2^j\boldsymbol{\alpha}!j!k!} L_1^{2\alpha_1}L_2^{2\alpha_2}\ldots L_n^{2\alpha_n}(2\pi i)^{2j}.$$

Set $m = 3g - 2 + n - |\boldsymbol{\alpha}|$ and consider the coefficient of $L_1^{2\alpha_1}L_2^{2\alpha_2}\ldots L_n^{2\alpha_n}$ in this expression.

$$\begin{aligned}
&\sum_{j=0}^{m} \frac{(2\pi i)^{2j}(2\pi^2)^{m-j}}{2^{|\boldsymbol{\alpha}|}2^j\boldsymbol{\alpha}!j!(m-j)!} \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{\alpha_1}\psi_2^{\alpha_2}\ldots\psi_n^{\alpha_n}\psi_{n+1}^{j}\kappa_1^{m-j} \\
&= \frac{(2\pi^2)^m}{2^{|\boldsymbol{\alpha}|}\boldsymbol{\alpha}!m!} \sum_{j=0}^{m}(-1)^j\binom{m}{j} \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{\alpha_1}\psi_2^{\alpha_2}\ldots\psi_n^{\alpha_n}\psi_{n+1}^{j}\kappa_1^{m-j} \\
&= \frac{(2\pi^2)^m}{2^{|\boldsymbol{\alpha}|}\boldsymbol{\alpha}!m!} \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{\alpha_1}\psi_2^{\alpha_2}\ldots\psi_n^{\alpha_n}(\kappa_1 - \psi_{n+1})^m
\end{aligned}$$

Now consider the coefficient of $L_1^{2\alpha_1}L_2^{2\alpha_2}\ldots L_n^{2\alpha_n}$ on the right hand side of the generalised string equation. Invoking Mirzakhani's theorem again, we can express this as

$$\frac{(2\pi^2)^m}{2^{|\boldsymbol{\alpha}|}\boldsymbol{\alpha}!m!}\sum_{k=1}^{n}\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{\alpha_1}\ldots\psi_k^{\alpha_k-1}\ldots\psi_n^{\alpha_n}\kappa_1^m.$$

Therefore, it suffices to show that

$$\int_{\overline{\mathcal{M}}_{g,n+1}}\psi_1^{\alpha_1}\psi_2^{\alpha_2}\ldots\psi_n^{\alpha_n}(\kappa_1-\psi_{n+1})^m=\sum_{k=1}^{n}\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{\alpha_1}\ldots\psi_k^{\alpha_k-1}\ldots\psi_n^{\alpha_n}\kappa_1^m.$$

However, this is a direct result of the following chain of equalities.

$$\begin{aligned}\pi_*\left[\psi_1^{\alpha_1}\psi_2^{\alpha_2}\ldots\psi_n^{\alpha_n}(\kappa_1-\psi_{n+1})^m\right]&=\pi_*\left[\pi^*\kappa_1^m\prod_{k=1}^{n}\left(\pi^*\psi_k^{\alpha_k}+D_k\cdot\pi^*\psi_k^{\alpha_k-1}\right)\right]\\&=\kappa_1^m\sum_{k=1}^{n}\psi_1^{\alpha_1}\ldots\psi_k^{\alpha_k-1}\ldots\psi_n^{\alpha_n}\end{aligned}$$

The first equality comes about from substituting the pull-back relations stated as Lemma 2.3 and Lemma 2.4. The second equality follows from the push-pull formula, the straightforward fact that $D_i\cdot D_j=0$ for $i\neq j$, as well as the evaluations $\pi_*1=0$ and $\pi_*D_k=1$.

Proof of the generalised dilaton equation

Recall that the generalised dilaton equation states that the following relation holds.

$$\frac{\partial V_{g,n+1}}{\partial L_{n+1}}(\mathbf{L},2\pi i)=2\pi i(2g-2+n)V_{g,n}(\mathbf{L})$$

By Mirzakhani's theorem, the left hand side can be written as

$$\sum_{|\boldsymbol{\alpha}|+j+k=3g-3+n}\frac{(2\pi^2)^k\int_{\overline{\mathcal{M}}_{g,n+1}}\psi_1^{\alpha_1}\psi_2^{\alpha_2}\ldots\psi_n^{\alpha_n}\psi_{n+1}^{j+1}\kappa_1^k}{2^{|\boldsymbol{\alpha}|}2^j\boldsymbol{\alpha}!j!k!}L_1^{2\alpha_1}L_2^{2\alpha_2}\ldots L_n^{2\alpha_n}(2\pi i)^{2j+1}.$$

Set $m=3g-3+n-|\boldsymbol{\alpha}|$ and consider the coefficient of $L_1^{2\alpha_1}L_2^{2\alpha_2}\ldots L_n^{2\alpha_n}$ in this expression.

$$\begin{aligned}&\sum_{j=0}^{m}\frac{(2\pi i)^{2j+1}(2\pi^2)^{m-j}}{2^{|\boldsymbol{\alpha}|}2^j\boldsymbol{\alpha}!j!(m-j)!}\int_{\overline{\mathcal{M}}_{g,n+1}}\psi_1^{\alpha_1}\psi_2^{\alpha_2}\ldots\psi_n^{\alpha_n}\psi_{n+1}^{j+1}\kappa_1^{m-j}\\&=\frac{2\pi i(2\pi^2)^m}{2^{|\boldsymbol{\alpha}|}\boldsymbol{\alpha}!m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\int_{\overline{\mathcal{M}}_{g,n+1}}\psi_1^{\alpha_1}\psi_2^{\alpha_2}\ldots\psi_n^{\alpha_n}\psi_{n+1}^{j+1}\kappa_1^{m-j}\\&=\frac{2\pi i(2\pi^2)^m}{2^{|\boldsymbol{\alpha}|}\boldsymbol{\alpha}!m!}\int_{\overline{\mathcal{M}}_{g,n+1}}\psi_1^{\alpha_1}\psi_2^{\alpha_2}\ldots\psi_n^{\alpha_n}\psi_{n+1}(\kappa_1-\psi_{n+1})^m\end{aligned}$$

Now consider the coefficient of $L_1^{2\alpha_1} L_2^{2\alpha_2} \dots L_n^{2\alpha_n}$ on the right hand side of the generalised dilaton equation. Invoking Mirzakhani's theorem again, we can express this as

$$\frac{2\pi i(2g-2+n)(2\pi^2)^m}{2^{|\boldsymbol{\alpha}|}\boldsymbol{\alpha}!m!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1}\psi_2^{\alpha_2}\dots\psi_n^{\alpha_n}\kappa_1^m.$$

Therefore, it suffices to show that

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{\alpha_1}\psi_2^{\alpha_2}\dots\psi_n^{\alpha_n}\psi_{n+1}(\kappa_1-\psi_{n+1})^m = (2g-2+n)\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1}\psi_2^{\alpha_2}\dots\psi_n^{\alpha_n}\kappa_1^m.$$

However, this is a direct result of the following chain of equalities.

$$\begin{aligned}\pi_*\left[\psi_1^{\alpha_1}\psi_2^{\alpha_2}\dots\psi_n^{\alpha_n}\psi_{n+1}(\kappa_1-\psi_{n+1})^m\right] &= \pi_*\left[\psi_{n+1}\cdot\pi^*\kappa_1^m\prod_{k=1}^n\left(\pi^*\psi_k^{\alpha_k}+D_k\cdot\pi^*\psi_k^{\alpha_k-1}\right)\right]\\ &= (2g-2+n)\kappa_1^m\psi_1^{\alpha_1}\psi_2^{\alpha_2}\dots\psi_n^{\alpha_n}\end{aligned}$$

The first equality comes about from substituting the pull-back relations stated as Lemma 2.3 and Lemma 2.4. The second equality follows from the push-pull formula, the straightforward fact that $D_i \cdot D_j = 0$ for $i \neq j$, as well as the evaluation $\pi_*\psi_{n+1} = 2g-2+n$.

2.3 Proofs via Mirzakhani's recursion

Bernoulli numbers

The Bernoulli numbers $B_0, B_1, B_2, \dots$ can be defined by the generating function

$$\frac{x}{e^x-1} = \sum_{k=0}^{\infty} B_k\frac{x^k}{k!}.$$

They have a habit of appearing in various disparate branches of mathematics, often unexpectedly, and the theory of moduli spaces of curves is no exception. One of the first results enunciating such a connection was the calculation of the orbifold Euler characteristic of $\mathcal{M}_{g,n}$, performed by Harer and Zagier [21] and also by Penner [49]. They essentially proved that

$$\chi(\mathcal{M}_{g,n}) = (-1)^n\frac{(2g-3+n)!}{2g(2g-2)!}B_{2g}.$$

Proposition 1.12 gives an elegant formula relating Hodge classes to kappa-classes which also features the Bernoulli numbers. They make an appearance in our work through the evaluations of the integrals arising from Mirzakhani's recursion. These involve values of the Riemann zeta function at non-negative even integers, which are directly related to the Bernoulli numbers by the well-known formula

$$\zeta(2k) = \frac{(-1)^{k+1}(2\pi)^{2k}B_{2k}}{2(2k)!}.$$

Another elementary result is the fact that, for odd k greater than 1, $B_k = 0$. The Bernoulli numbers satisfy a myriad of other interesting relations, though we will only require the following facts.

Lemma 2.5. *The Bernoulli numbers obey the following two relations.*

(i) For every positive integer n,

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0.$$

(ii) For every even positive integer n,

$$\sum_{k=0}^{n} 2^k \binom{n+1}{k} B_k = 0.$$

Proof.

(i) From the definition of the Bernoulli numbers, we obtain the following chain of equalities.

$$x = \left(\sum_{k=0}^{\infty} B_k \frac{x^k}{k!}\right)(e^x - 1) = \left(\sum_{k=0}^{\infty} B_k \frac{x^k}{k!}\right)\left(\sum_{m=1}^{\infty} \frac{x^m}{m!}\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{B_k}{k!(n+1-k)!}\right) x^{n+1}$$

Equating the coefficients of x^{n+1} on both sides for positive n yields the desired relation.

$$\sum_{k=0}^{n} \frac{B_k}{k!(n+1-k)!} = 0 \quad \Rightarrow \quad \sum_{k=0}^{n} \binom{n+1}{k} B_k = 0$$

(ii) Again, from the definition of the Bernoulli numbers, we obtain the following.

$$\frac{2x}{e^{2x}-1} = \sum_{k=0}^{\infty} B_k \frac{(2x)^k}{k!} \Rightarrow \frac{2x}{e^x - 1} = \left(\sum_{k=0}^{\infty} 2^k B_k \frac{x^k}{k!}\right)(e^x + 1)$$

Therefore, we have the generating function identity

$$\sum_{n=0}^{\infty} 2B_n \frac{x^n}{n!} = \left(\sum_{k=0}^{\infty} 2^k B_k \frac{x^k}{k!}\right)\left(2 + \sum_{m=1}^{\infty} \frac{x^m}{m!}\right),$$

which implies

$$\sum_{n=1}^{\infty} (2 - 2^{n+1}) B_n \frac{x^n}{n!} = \sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1} \frac{2^k B_k}{k!(n-k)!}\right) x^n.$$

After shifting the index of the summations, we have

$$\sum_{n=0}^{\infty}(2-2^{n+2})B_{n+1}\frac{x^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{2^k B_k}{k!(n+1-k)!}\right)x^{n+1},$$

and equating the coefficients of x^{n+1} on both sides for n positive and even yields the desired relation.

$$\sum_{k=0}^{n}\frac{2^k B_k}{k!(n+1-k)!} = 0 \quad \Rightarrow \quad \sum_{k=0}^{n} 2^k \binom{n+1}{k} B_k = 0$$

□

Lemma 2.6. *For every non-negative integer n,*

$$\sum_{i+j\leq n}\frac{(2^{2i}-2)(2^{2j}-2)B_{2i}B_{2j}}{(2i)!(2j)!(2n-2i-2j)!} = (2n-1)\frac{(2^{2n}-2)B_{2n}}{(2n)!},$$

where the summation is over ordered pairs of non-negative integers (i, j).

Proof. For every non-negative integer k, define $A_k = \frac{(2^{2k}-2)B_{2k}}{(2k)!}$ and $E_k = \frac{1}{(2k)!}$. Given this notation, the result we wish to prove takes the form

$$\sum_{i+j+k=n} A_i A_j E_k = (2n-1)A_n.$$

Identities involving convolutions like this are particularly amenable to a generating function approach. In fact, if we define $a(x) = \sum A_k x^{2k}$ and $e(x) = \sum E_k x^{2k}$, then the problem is equivalent to the identity $a(x)^2 e(x) = xa'(x) - a(x)$. This can be easily verified from the explicit expressions

$$a(x) = \frac{-2x}{e^x - e^{-x}} \quad \text{and} \quad e(x) = \frac{e^x + e^{-x}}{2}.$$

□

Proof of the generalised dilaton equation

Armed with the Bernoulli identities above, we will prove that the generalised dilaton equation is a consequence of Mirzakhani's recursion. Let us start by introducing the following operators.

$$\begin{aligned}
D[\cdot] &= 2\frac{\partial}{\partial L_1}L_1[\cdot] \\
\partial[\cdot] &= \frac{\partial}{\partial L_{n+1}}[\cdot]_{L_{n+1}=2\pi i} \\
H_1[\cdot] &= \int_0^\infty\int_0^\infty xy\ H(x+y, L_1)\ [\cdot]\ dx\ dy \\
H_k[\cdot] &= \int_0^\infty x[H(x, L_1+L_k) + H(x, L_1-L_k)]\ [\cdot]\ dx
\end{aligned}$$

Here, the function $H : \mathbb{R}^2 \to \mathbb{R}$ is defined as in the statement of Theorem 1.29. These operators can be considered as linear endomorphisms of the complex vector space with basis the set of monomials in $x^2, y^2, L_1^2, L_2^2, L_3^2, \ldots$. With this notation in place, Mirzakhani's recursion can be stated as

$$\begin{aligned} D[V_g(\mathbf{L}, L_{n+1})] =& H_1[V_{g-1}(x,y,\widehat{\mathbf{L}},L_{n+1})] + \sum_{\substack{g_1+g_2=g \\ \mathcal{I}_1 \sqcup \mathcal{I}_2 = [2,n+1]}} H_1[V_{g_1}(x,\mathbf{L}_{\mathcal{I}_1}) V_{g_2}(y,\mathbf{L}_{\mathcal{I}_2})] \\ &+ \sum_{k=2}^{n} H_k[V_g(x,\widehat{\mathbf{L}}_k,L_{n+1})] + H_{n+1}[V_g(x,\widehat{\mathbf{L}})]. \end{aligned}$$

Furthermore, the generalised dilaton equation can be stated as

$$\partial[V_g(\mathbf{L}, L_{n+1})] = 2\pi i(2g-2+n)V_g(\mathbf{L}).$$

Recall the notation $\widehat{\mathbf{L}} = (L_2, L_3, \ldots, L_n)$, $\widehat{\mathbf{L}}_k = (L_2, \ldots, \widehat{L_k}, \ldots, L_n)$ and $\mathbf{L}_{\mathcal{I}} = (L_{i_1}, L_{i_2}, \ldots, L_{i_m})$ for $\mathcal{I} = \{i_1, i_2, \ldots, i_m\}$. Note that we have omitted the second subscript of the volume polynomials where it is clear how many arguments are involved. For example, $V_{g,n}(L_1, L_2, \ldots, L_n)$ may be abbreviated to $V_g(\mathbf{L})$, and we will continue this practice for notational economy.

Observe that the action of H_k on the monomial x^{2a-2} for a positive integer a can be explicitly expressed as follows.

$$\begin{aligned} H_k[x^{2a-2}] &= \int_0^\infty x^{2a-1}[H(x, L_1+L_k) + H(x, L_1-L_k)]\, dx \\ &= (2a-1)! \sum_{i=0}^{a} \frac{\zeta(2i)(2^{2i+1}-4)}{(2a-2i)!}[(L_1+L_k)^{2a-2i} + (L_1-L_k)^{2a-2i}] \\ &= (2a-1)! \sum_{i=0}^{a} \frac{\zeta(2i)(2^{2i+1}-4)}{(2a-2i)!} \sum_{j=0}^{a-i} 2\binom{2a-2i}{2j} L_1^{2j} L_k^{2a-2i-2j} \\ &= 2(2a-1)! \sum_{i+j\le a} \frac{\zeta(2i)(2^{2i+1}-4)}{(2j)!(2a-2i-2j)!} L_1^{2j} L_k^{2a-2i-2j} \end{aligned}$$

It is a simple matter to verify the generalised dilaton equation when (g,n) is equal to $(0,3)$ or $(1,1)$. The proof now proceeds by induction on the negative Euler characteristic $2g-2+n$, and we start by applying the operator ∂ to both sides of Mirzakhani's recursion. Since ∂ and D commute, the left hand side becomes

$$\partial \circ D[V_g(\mathbf{L}, L_{n+1})] = D \circ \partial[V_g(\mathbf{L}, L_{n+1})].$$

The right hand side breaks up naturally as the sum of four terms, which we will deal with in order. The first term gives the following, where we have made use of the fact that ∂ and H_1

commute, as well as the inductive assumption.

$$\begin{aligned}\partial \circ H_1[V_{g-1}(x,y,\widehat{\mathbf{L}},L_{n+1})] &= H_1 \circ \partial[V_{g-1}(x,y,\widehat{\mathbf{L}},L_{n+1})] \\ &= 2\pi i(2g-3+n)H_1[V_{g-1}(x,y,\widehat{\mathbf{L}})]\end{aligned}$$

The second term can be taken care of by simply concentrating on the summation over $\mathcal{I}_1 \sqcup \mathcal{I}_2$. Once again, we have made use of the fact that ∂ and H_1 commute, as well as the inductive assumption.

$$\begin{aligned}&\partial\left[\sum_{\mathcal{I}_1\sqcup\mathcal{I}_2=[2,n+1]} H_1[V_{g_1}(x,\mathbf{L}_{\mathcal{I}_1})V_{g_2}(y,\mathbf{L}_{\mathcal{I}_2})]\right] \\ &= \sum_{\mathcal{I}_1\sqcup\mathcal{I}_2=[2,n]} H_1\circ\partial[V_{g_1}(x,\mathbf{L}_{\mathcal{I}_1},L_{n+1})V_{g_2}(y,\mathbf{L}_{\mathcal{I}_2})] + H_1\circ\partial[V_{g_1}(x,\mathbf{L}_{\mathcal{I}_1})V_{g_2}(y,\mathbf{L}_{\mathcal{I}_2},L_{n+1})] \\ &= \sum_{\mathcal{I}_1\sqcup\mathcal{I}_2=[2,n]} [2\pi i(2g_1-1+|\mathcal{I}_1|)+2\pi i(2g_2-1+|\mathcal{I}_2|)]\ H_1[V_{g_1}(x,\mathbf{L}_{\mathcal{I}_1})V_{g_2}(y,\mathbf{L}_{\mathcal{I}_2})] \\ &= 2\pi i(2g-3+n)\sum_{\mathcal{I}_1\sqcup\mathcal{I}_2=[2,n]} H_1[V_{g_1}(x,\mathbf{L}_{\mathcal{I}_1})V_{g_2}(y,\mathbf{L}_{\mathcal{I}_2})]\end{aligned}$$

The third term can be dealt with in a similar manner, using the fact that ∂ and H_k commute.

$$\begin{aligned}\partial\left[\sum_{k=2}^{n} H_k[V_g(x,\widehat{\mathbf{L}}_k,L_{n+1})\right] &= \sum_{k=2}^{n} H_k\circ\partial[V_g(x,\widehat{\mathbf{L}}_k,L_{n+1})] \\ &= 2\pi i(2g-3+n)\sum_{k=2}^{n} H_k[V_g(x,\widehat{\mathbf{L}}_k)]\end{aligned}$$

Putting all of this together and invoking Mirzakhani's recursion again, we obtain the following.

$$D\circ\partial[V_g(\mathbf{L},L_{n+1})] = 2\pi i(2g-3+n)D[V_g(\mathbf{L})] + \partial\circ H_{n+1}[V_g(x,\widehat{\mathbf{L}})]$$

Now since D is invertible, the generalised dilaton equation will follow if we can prove that

$$\partial\circ H_{n+1}[V_g(x,\widehat{\mathbf{L}})] = 2\pi i D[V_g(\mathbf{L})].$$

However, this is a direct corollary of the following result.

Lemma 2.7. *For every even polynomial P, the following relation holds.*

$$\partial\circ H_{n+1}[P(x)] = 2\pi i D[P(L_1)]$$

Proof. By linearity, it suffices to prove the lemma for monomials of the form x^{2a}, where a is a non-negative integer.

$$\partial \circ H_{n+1}[x^{2a}] = 2\pi i D[L_1^{2a}]$$

We can use the explicit expressions for the operators ∂, D, H_{n+1} to state this as

$$(2a)! \sum_{i+j \leq a} \frac{\zeta(2i)(2^{2i+1}-4)}{(2j)!(2a+1-2i-2j)!} L_1^{2j} (2\pi i)^{2a-2i-2j} = L_1^{2a}.$$

Note that both sides of this equation are polynomials in L_1 and that the coefficient of L_1^d on the left hand side can be written as

$$\frac{(2a)!}{(2d)!} \sum_{i=0}^{a-d} \frac{\zeta(2i)(2^{2i+1}-4)}{(2a-2d+1-2i)!} (2\pi i)^{2a-2d-2i}. \tag{2.1}$$

Set $m = a - d$ and replace the values of the Riemann zeta function with Bernoulli numbers to obtain

$$\frac{(-1)^{m+1}(2\pi)^{2m}}{2m+1} \binom{2a}{2d} \sum_{i=0}^{m} B_{2i}(2^{2i}-2) \binom{2m+1}{2i}.$$

Since $B_i(2^i - 2) = 0$ for all odd positive integers i, this is equivalent to

$$\begin{aligned} &\frac{(-1)^{m+1}(2\pi)^{2m}}{2m+1} \binom{2a}{2d} \sum_{i=0}^{2m} B_i(2^i-2) \binom{2m+1}{i} \\ = &\frac{(-1)^{m+1}(2\pi)^{2m}}{2m+1} \binom{2a}{2d} \left[\sum_{i=0}^{2m} 2^i \binom{2m+1}{i} B_i - 2 \sum_{i=0}^{2m} \binom{2m+1}{i} B_i \right]. \end{aligned}$$

The two summations in the latter expression are equal to zero by Lemma 2.5 unless $m = 0$. Therefore, the only contribution to equation (2.1) occurs when $d = a$. In this case, it is easy to verify that the coefficient of L_1^{2d} is indeed 1 and the result follows. □

Proof of the generalised string equation

The method of proof for the generalised string equation is very similar in nature to that used for the generalised dilaton equation, though even more unwieldy. Rather than provide the lengthy and tedious algebraic manipulations, we will content ourselves by stating the important steps of the proof. We will need some further notation before we begin.

$$I_k[\cdot] = \int L_k[\cdot]\, dL_k \qquad I_x[\cdot] = \int x[\cdot]\, dx \qquad I_y[\cdot] = \int y[\cdot]\, dy$$

$$K[\cdot] = \frac{1}{2L_1} \int [\cdot]\, dL_1 \qquad H_{2\pi i}[\cdot] = H_k[\cdot]_{L_k = 2\pi i}$$

Once again, these can be considered as linear endomorphisms of the complex vector space with basis the set of monomials in $x^2, y^2, L_1^2, L_2^2, L_3^2, \ldots$. With this notation, the generalised string equation can be stated as

$$V_g(\mathbf{L}, 2\pi i) = \sum_{k=1}^{n} I_k[V_g(\mathbf{L})].$$

The action of H_1 on the monomial $x^{2a-2}y^{2b-2}$ for positive integers a and b can be explicitly expressed as follows.

$$\begin{aligned} H_1[x^{2a-2}y^{2b-2}] &= \int_0^\infty \int_0^\infty x^{2a-1}y^{2b-1} H(x+y, L_1)\, dx\, dy \\ &= (2a-1)!(2b-1)! \sum_{i=0}^{a+b} \frac{\zeta(2i)(2^{2i+1}-4)}{(2a+2b-2i)!} L_1^{2a+2b-2i} \end{aligned}$$

Similarly, the action of $H_{2\pi i}$ on the monomial x^{2a-2} for a positive integer a takes the following form.

$$\begin{aligned} H_{2\pi i}[x^{2a-2}] &= H_k[x^{2a-2}]_{L_k = 2\pi i} \\ &= 2(2a-1)! \sum_{i+j \le a} \frac{\zeta(2i)(2^{2i+1}-4)(2\pi i)^{2a-2i-2j}}{(2j)!(2a-2i-2j)!} L_1^{2j} \end{aligned}$$

It is a simple matter to verify the generalised string equation when (g, n) is equal to $(0, 3)$ or $(1, 1)$. The proof now proceeds by induction on the negative Euler characteristic $2g - 2 + n$, and our starting point is Mirzakhani's recursion. Applying the inductive hypothesis allows us to write the left hand side of the generalised string equation in the following lengthy manner.

$$\begin{aligned} V_g(\mathbf{L}, 2\pi i) = 2K \circ H_1 \circ I_x[V_{g-1}(x, y, \widehat{\mathbf{L}})] &+ 2 \sum_{\substack{g_1+g_2=g \\ \mathcal{I}_1 \sqcup \mathcal{I}_2 = [2,n]}} K \circ H_1 \circ I_x[V_{g_1}(x, \mathbf{L}_{\mathcal{I}_1}) V_{g_2}(y, \mathbf{L}_{\mathcal{I}_2})] \\ &+ \sum_{i=2}^{n} K_i \circ H_i \circ I_x[V_g(x, \widehat{\mathbf{L}}_i)] + \sum_{k=2}^{n} K \circ H_1 \circ I_k[V_{g-1}(x, y, \widehat{\mathbf{L}})] \\ &+ \sum_{\substack{g_1+g_2=g \\ \mathcal{I}_1 \sqcup \mathcal{I}_2 = [2,n]}} \sum_{k=2}^{n} K \circ H_1 \circ I_k[V_{g_1}(x, \mathbf{L}_{\mathcal{I}_1}) V_{g_2}(y, \mathbf{L}_{\mathcal{I}_2})] + 2 \sum_{k=2}^{n} K \circ H_1[V_g(x, \widehat{\mathbf{L}}_k)] \\ &+ \sum_{i=2}^{n} \sum_{k=2, k\neq i}^{n} K \circ H_i \circ I_k[V_g(x, \widehat{\mathbf{L}}_i)] + K \circ H_{2\pi i}[V_g(x, \widehat{\mathbf{L}})] \end{aligned}$$

Our approach from here will be to use Mirzakhani's recursion and the inductive hypothesis to express the right hand side of the generalised string equation in a similar manner. Equating the two sides and performing certain cancellations and simplifications, we will arrive at a statement involving only Bernoulli numbers.

The right hand side of the generalised string equation can be expressed as follows.

$$\begin{aligned}\sum_{k=1}^{n} I_k[V_g(\mathbf{L})] = I_1 \circ K \circ H_1[V_{g-1}(x,y,\widehat{\mathbf{L}})] + \sum_{\substack{g_1+g_2=g\\ \mathcal{I}_1\sqcup\mathcal{I}_2=[2,n]}} I_1 \circ K \circ H_1[V_{g_1}(x,\mathbf{L}_{\mathcal{I}_1})V_{g_2}(y,\mathbf{L}_{\mathcal{I}_2})]\\ + \sum_{i=2}^{n} I_1 \circ K \circ H_i[V_g(x,\widehat{\mathbf{L}}_i)] + \sum_{k=2}^{n} K \circ H_1 \circ I_k[V_{g-1}(x,y,\widehat{\mathbf{L}})]\\ + \sum_{\substack{g_1+g_2=g\\ \mathcal{I}_1\sqcup\mathcal{I}_2=[2,n]}} \sum_{k=2}^{n} K \circ H_1 \circ I_k[V_{g_1}(x,\mathbf{L}_{\mathcal{I}_1})V_{g_2}(y,\mathbf{L}_{\mathcal{I}_2})]\\ + \sum_{i,k=2,i\neq k}^{n} K \circ H_i \circ I_k[V_g(x,\widehat{\mathbf{L}}_i)] + \sum_{k=2}^{n} I_k \circ K \circ H_k[V_g(x,\widehat{\mathbf{L}}_k)]\end{aligned}$$

After equating the previous two expressions and performing some mild cancellation, the generalised string equation boils down to proving the following equality.

$$\begin{aligned}&2K \circ H_1 \circ I_x[V_{g-1}(x,y,\widehat{\mathbf{L}})] + 2\sum_{\substack{g_1+g_2=g\\ \mathcal{I}_1\sqcup\mathcal{I}_2=[2,n]}} K \circ H_1 \circ I_x[V_{g_1}(x,\mathbf{L}_{\mathcal{I}_1})V_{g_2}(y,\mathbf{L}_{\mathcal{I}_2})]\\ &+ 2\sum_{k=2}^{n} K \circ H_1[V_g(x,\widehat{\mathbf{L}}_k)] + \sum_{i=2}^{n} K_i \circ H_i \circ I_x[V_g(x,\widehat{\mathbf{L}}_i)] + K \circ H_{2\pi i}[V_g(x,\widehat{\mathbf{L}})]\\ =\ &I_1 \circ K \circ H_1[V_{g-1}(x,y,\widehat{\mathbf{L}})] + \sum_{\substack{g_1+g_2=g\\ \mathcal{I}_1\sqcup\mathcal{I}_2=[2,n]}} I_1 \circ K \circ H_1[V_{g_1}(x,\mathbf{L}_{\mathcal{I}_1})V_{g_2}(y,\mathbf{L}_{\mathcal{I}_2})]\\ &+ \sum_{k=2}^{n} I_k \circ K \circ H_k[V_g(x,\widehat{\mathbf{L}}_k)] + \sum_{i=2}^{n} I_1 \circ K \circ H_i[V_g(x,\widehat{\mathbf{L}}_i)]\end{aligned}$$

At this stage, we invoke Mirzakhani's recursion yet again, after which we arrive at the following rather cumbersome equation.

$$\begin{aligned}&K \circ H_{2\pi i}[K \circ H_1[V_{g-1}(x,y,\widehat{\mathbf{L}}_i)]]_{L_1=x} + 2\sum_{\substack{g_1+g_2=g\\ \mathcal{I}_1\sqcup\mathcal{I}_2=[2,n]}} K \circ H_1 \circ I_x[V_{g_1}(x,\mathbf{L}_{\mathcal{I}_1})V_{g_2}(y,\mathbf{L}_{\mathcal{I}_2})]\\ &+ 2\sum_{k=2}^{n} K \circ H_1[V_g(x,\widehat{\mathbf{L}}_k)] + \sum_{i=2}^{n} K_i \circ H_i \circ I_x[V_g(x,\widehat{\mathbf{L}}_i)] + \sum_{i=2}^{n} K \circ H_{2\pi i}[K \circ H_i[V_g(x,\widehat{\mathbf{L}}_i)]]_{L_1=x}\\ &+ \sum_{\substack{g_1+g_2=g\\ \mathcal{I}_1\sqcup\mathcal{I}_2=[2,n]}} K \circ H_{2\pi i}[K \circ H_1[V_{g_1}(x,\mathbf{L}_{\mathcal{I}_1})V_{g_2}(y,\mathbf{L}_{\mathcal{I}_2})]]_{L_1=x} + 2K \circ H_1 \circ I_x[V_{g-1}(x,y,\widehat{\mathbf{L}})]\\ =\ &I_1 \circ K \circ H_1[V_{g-1}(x,y,\widehat{\mathbf{L}})] + \sum_{\substack{g_1+g_2=g\\ \mathcal{I}_1\sqcup\mathcal{I}_2=[2,n]}} I_1 \circ K \circ H_1[V_{g_1}(x,\mathbf{L}_{\mathcal{I}_1})V_{g_2}(y,\mathbf{L}_{\mathcal{I}_2})]\\ &+ \sum_{k=2}^{n} I_k \circ K \circ H_k[V_g(x,\widehat{\mathbf{L}}_k)] + \sum_{i=2}^{n} I_1 \circ K \circ H_i[V_g(x,\widehat{\mathbf{L}}_i)]\end{aligned}$$

However, by inspection, this equality is a direct corollary of the following two lemmas.

Lemma 2.8. *For every symmetric even polynomial $P(x,y)$, the following relation holds.*

$$2K \circ H_1 \circ I_x[P] + K \circ H_{2\pi i}[K \circ H_1[P]]_{L_1=x} = I_1 \circ K \circ H_1[P]$$

Lemma 2.9. *For every even polynomial $P(x)$, the following relation holds.*

$$K \circ H_k \circ I_x[P] + 2K \circ H_1[P] + K \circ H_{2\pi i}[K \circ H_k[P]]_{L_1=x} = I_k \circ K \circ H_k[P] + I_1 \circ K \circ H_k[P]$$

Proof of Lemma 2.8. By linearity, it suffices to prove the lemma for polynomials of the form $x^{2a}y^{2b} + x^{2b}y^{2a}$, where a and b are non-negative integers. If we set $m = a + b + 1$, then the first term on the left hand side is

$$2m(2a-1)!(2b-1)!\sum_{i=0}^{m} \frac{\zeta(2i)(2^{2i+1}-4)}{(2m+1-2i)!} L_1^{2m-2i},$$

the second term on the left hand side is

$$(2a-1)!(2b-1)!\sum_{i=0}^{m}\sum_{j+k\leq m-i} \frac{\zeta(2i)(2^{2i+1}-4)\zeta(2j)(2^{2j+1}-4)}{(2m-2i-2j-2k)!(2k+1)!}(-4\pi^2)^{m-i-j-k}L_1^{2k},$$

and the right hand side is

$$(2a-1)!(2b-1)!\sum_{i=0}^{m-1} \frac{\zeta(2i)(2^{2i+1}-4)}{(2m-2i)!} L_1^{2m-2i}.$$

Therefore, what we wish to prove can be equivalently expressed as

$$\sum_{i+j+k+l=m} \frac{\zeta(2i)(2^{2i+1}-4)\zeta(2j)(2^{2j+1}-4)}{(2l)!(2k+1)!}(-4\pi^2)^l L_1^{2k} = \sum_{i=0}^{m} \frac{\zeta(2i)(2^{2i+1}-4)(1-2i)}{(2m+1-2i)!} L_1^{2m-2i}.$$

Note that both sides are even polynomials in L_1 of degree at most $2m$, and equating the coefficients of L_1^{2m-2d} on both sides yields

$$\sum_{i+j\leq d} \frac{\zeta(2i)(2^{2i+1}-4)\zeta(2j)(2^{2j+1}-4)}{(2d-2i-2j)!}(-4\pi^2)^{d-i-j} = \zeta(2d)(2^{2d+1}-4)(1-2d).$$

Replacing the zeta values with Bernoulli numbers leads to

$$\sum_{i+j\leq d} \frac{(2^{2i}-2)(2^{2j}-2)B_{2i}B_{2j}}{(2i)!(2j)!(2d-2i-2j)!} = (2d-1)\frac{(2^{2d}-2)B_{2d}}{(2d)!},$$

which is true by Lemma 2.6. □

The following is very similar in nature to the previous proof and, perhaps surprisingly, relies on the very same identity involving Bernoulli numbers.

Proof of Lemma 2.9. By linearity, it suffices to prove the lemma for monomials of the form x^{2a}, where a is a non-negative integer. The first term on the left hand side is

$$(2a+1)(2a-1)! \sum_{i+j\leq a+1} \frac{\zeta(2i)(2^{2i+1}-4)}{(2a+2-2i-2j)!(2j+1)!} L_1^{2j} L_k^{2a+2-2i-2j},$$

the second term on the left hand side is

$$(2a-1)! \sum_{i=0}^{a+1} \frac{\zeta(2i)(2^{2i+1}-4)}{(2a+3-2i)!} L_1^{2a+2-2i},$$

and the third term on the left hand side is

$$(2a-1)! \sum_{i+j\leq a} \sum_{k+l\leq j+1} \frac{\zeta(2i)(2^{2i+1}-4)\zeta(2k)(2^{2k+1}-4)(2\pi i)^{2j+2-2k-2l}}{(2a-2i-2j)!(2j+2-2k-2l)!(2l+1)!} L_1^{2l} L_k^{2a-2i-2j}.$$

The first term on the right hand side is

$$(2a-1)! \sum_{i+j\leq a} \frac{\zeta(2i)(2^{2i+1}-4)}{(2a-2i-2j)!(2j+1)!} \frac{L_1^{2j} L_k^{2a+2-2i-2j}}{2a+2-2i-2j},$$

and the second term on the right hand side is

$$(2a-1)! \sum_{i+j\leq a} \frac{\zeta(2i)(2^{2i+1}-4)}{(2a-2i-2j)!(2j+2)!} L_1^{2j+2} L_k^{2a-2i-2j}.$$

Therefore, what we are required to prove amounts to the following, after some tedious though mild simplification.

$$\begin{aligned}
&\sum_{i+j\leq a+1} \frac{(2i+2j)\zeta(2i)(2^{2i+1}-4)}{(2a+2-2i-2j)!(2j+1)!} L_1^{2j} L_k^{2a+2-2i-2j} \\
&+ \sum_{i+j\leq a} \sum_{k+l\leq j+1} \frac{\zeta(2i)(2^{2i+1}-4)\zeta(2k)(2^{2k+1}-4)(2\pi i)^{2j+2-2k-2l}}{(2a-2i-2j)!(2j+2-2k-2l)!(2l+1)!} L_1^{2l} L_k^{2a-2i-2j} \\
=& \sum_{i+j\leq a} \frac{\zeta(2i)(2^{2i+1}-4)}{(2a-2i-2j)!(2j+2)!} L_1^{2j+2} L_k^{2a-2i-2j}
\end{aligned}$$

Note that both sides of the equation are even polynomials in L_1 and L_k with degree at most $2a+2$. Now equate the coefficient of $L_1^{2m-2d} L_k^{2a+2-2m}$ and note that to obtain a non-zero contribution on both sides, we require $0 \leq d \leq m$.

$$\frac{2m\zeta(2d)(2^{2d+1}-4)}{(2a+2-2m)!(2m-2d+1)!} + \sum_{i+k\leq d} \frac{\zeta(2i)(2^{2i+1}-4)\zeta(2k)(2^{2k+1}-4)(2\pi i)^{2d-2i-2k}}{(2a+2-2m)!(2d-2i-2k)!(2m-2d+1)!}$$
$$= \frac{\zeta(2d)(2^{2d+1}-4)}{(2a+2-2m)!(2m-2d)!}$$

A little rearrangement yields the equation

$$\sum_{i+k\leq d} \frac{\zeta(2i)(2^{2i+1}-4)\zeta(2k)(2^{2k+1}-4)(2\pi i)^{2d-2i-2k}}{(2d-2i-2k)!} = \zeta(2d)(2^{2d+1}-4)(1-2d).$$

Once again, replacing the zeta values with Bernoulli numbers leads to

$$\sum_{i+j\leq d} \frac{(2^{2i}-2)(2^{2j}-2)B_{2i}B_{2j}}{(2i)!(2j)!(2d-2i-2j)!} = (2d-1)\frac{(2^{2d}-2)B_{2d}}{(2d)!},$$

which is true by Lemma 2.6. □

2.4 Applications and extensions

Small genus Weil–Petersson volumes

The generalised string and dilaton equations are insufficient to recover all of the Weil–Petersson volumes. However, they do uniquely determine the volume polynomials in genus 0 and 1, given the base cases $V_{0,3}(L_1, L_2, L_3) = 1$ and $V_{1,1}(L_1) = \frac{1}{48}(L_1^2 + 4\pi^2)$. The proof of this fact hinges on the following lemma.

Lemma 2.10.

(i) A symmetric polynomial $P(x_1, x_2, \ldots, x_n)$ of degree less than n is uniquely determined by the evaluation $P(x_1, x_2, \ldots, x_{n-1}, c)$ for any $c \in \mathbb{C}$.

(ii) A symmetric polynomial $P(x_1, x_2, \ldots, x_n)$ of degree less than or equal to n is uniquely determined by the evaluations $P(x_1, x_2, \ldots, x_{n-1}, c)$ and $\frac{\partial P}{\partial x_n}(x_1, x_2, \ldots, x_{n-1}, d)$ for any $c, d \in \mathbb{C}$.

Proof.

(i) Suppose that $P(x_1, x_2, \ldots, x_n)$ and $Q(x_1, x_2, \ldots, x_n)$ are symmetric polynomials of degree less than n which have the same evaluation at $x_n = c$. Then $P - Q$ vanishes on the hyperplane defined by the equation $x_n = c$ and it follows that $x_n - c \mid P - Q$. By symmetry, we deduce that $(x_1 - c)(x_2 - c) \dots (x_n - c) \mid P - Q$, and since $P - Q$ has degree less than n, it must be the case that $P = Q$.

(ii) Suppose that $P(x_1, x_2, \dots, x_n)$ and $Q(x_1, x_2, \dots, x_n)$ are symmetric polynomials of degree less than or equal to n which have the same evaluation at $x_n = c$. From the previous discussion, we have $(x_1 - c)(x_2 - c) \dots (x_n - c) \mid P - Q$, and it follows that

$$P(x_1, x_2, \dots, x_n) = Q(x_1, x_2, \dots, x_n) + a(x_1 - c)(x_2 - c) \dots (x_n - c)$$

for some $a \in \mathbb{C}$. Taking the partial derivative of this equation with respect to x_n and substituting $x_n = d$ yields

$$\frac{\partial P}{\partial x_n}(x_1, x_2, \dots, x_{n-1}, d) = \frac{\partial Q}{\partial x_n}(x_1, x_2, \dots, x_{n-1}, d) + a(x_1 - c)(x_2 - c) \dots (x_{n-1} - c).$$

So if $\frac{\partial P}{\partial x_n}$ and $\frac{\partial Q}{\partial x_n}$ have the same evaluation at $x_n = d$, then $a = 0$ and it must be the case that $P = Q$. □

Theorem 2.11.

(i) The volume $V_{0,n+1}(\mathbf{L}, L_{n+1})$ is uniquely determined from $V_{0,n}(\mathbf{L})$ and the generalised string equation for $n \geq 3$.

(ii) The volume $V_{1,n+1}(\mathbf{L}, L_{n+1})$ is uniquely determined from $V_{1,n}(\mathbf{L})$, the generalised string equation, and the generalised dilaton equation for $n \geq 1$.

Proof.

(i) Note that $V_{0,n+1}(\mathbf{L}, L_{n+1})$ is a symmetric polynomial in the variables $L_1^2, L_2^2, \dots, L_{n+1}^2$ of degree $n - 2$. Therefore, by Lemma 2.10, it is uniquely determined from its evaluation at $L_{n+1} = -4\pi^2$ or equivalently,

$$V_{0,n+1}(\mathbf{L}, 2\pi i).$$

However, this is given by the generalised string equation in terms of $V_{0,n}(\mathbf{L})$.

(ii) Note that $V_{1,n+1}(\mathbf{L}, L_{n+1})$ is a symmetric polynomial in the variables $L_1^2, L_2^2, \dots, L_{n+1}^2$ of degree $n + 1$. Therefore, by Lemma 2.10, it is uniquely determined from its evaluation at $L_{n+1}^2 = -4\pi^2$ as well as the evaluation of its partial derivative at $L_{n+1} = -4\pi^2$. However, these can be expressed as

$$V_{1,n+1}(\mathbf{L}, 2\pi i) \quad \text{and} \quad \frac{1}{4\pi i} \frac{\partial V_{1,n+1}}{\partial L_{n+1}}(\mathbf{L}, 2\pi i)$$

respectively, both of which are given by the generalised string and dilaton equations in terms of $V_{1,n}(\mathbf{L})$. □

Theorem 2.11 can be used to produce an effective algorithm which computes $V_{0,n+1}(\mathbf{L}, L_{n+1})$ from $V_{0,n}(\mathbf{L})$ and $V_{1,n+1}(\mathbf{L}, L_{n+1})$ from $V_{1,n}(\mathbf{L})$. For example, see Appendix A.2 for a simple Maple routine which performs the former of these two tasks. Empirically, these algorithms seem to compute the Weil–Petersson volumes in genus 0 and 1 much faster than implementing Mirzakhani's recursive formula. This is to be expected, since the computation of $V_{0,n+1}(\mathbf{L}, L_{n+1})$ using Mirzakhani's recursion requires knowledge of the volumes $V_{0,m}(L_1, L_2, \ldots, L_m)$ for all $m \leq n$. Similarly, the computation of $V_{1,n+1}(\mathbf{L}, L_{n+1})$ using Mirzakhani's recursion requires knowledge of the volumes $V_{0,m}(L_1, L_2, \ldots, L_m)$ for all $m \leq n+1$ and $V_{1,m}(L_1, L_2, \ldots, L_m)$ for all $m \leq n$. Therefore, we see that one of the strengths of the generalised string and dilaton equations is their inherent simplicity.

Closed hyperbolic surfaces

There is a variety of constructions, calculations and theorems concerning the moduli spaces $\mathcal{M}_{g,n}$ or $\overline{\mathcal{M}}_{g,n}$ which do not apply when $n = 0$. For example, the cell decomposition of the decorated moduli space $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$ used in Kontsevich's proof of Witten's conjecture has no analogue in the case without marked points, punctures or boundaries. In fact, it remains an important open problem to find a natural cell decomposition for $\mathcal{M}_{g,0}$. As another example, there is no McShane-type identity for closed hyperbolic surfaces of arbitrary genus. Thus, Mirzakhani's recursion has nothing to say about the volume $V_{g,0}$. However, the generalised string and dilaton equations do allow us to calculate $V_{g,0}$ for all values of g using the following.

Theorem 2.12.

(i) *When $n = 1$, the volume factorises as $V_{g,1}(L) = (L^2 + 4\pi^2)P_g(L)$ for some polynomial P_g.*

(ii) *For $g \geq 2$, we have the following formula.*

$$V_{g,0} = \frac{1}{4\pi i(g-1)} \frac{\partial V_{g,1}}{\partial L}(2\pi i) = \frac{P_g(2\pi i)}{g-1}$$

Proof.

(i) In the case $g = 1$, we have $V_{1,1}(L) = \frac{1}{48}(L^2 + 4\pi^2)$, so the result is true by inspection. For $g \geq 2$, $V_{g,1}(L)$ is a real polynomial which satisfies $V_{g,1}(2\pi i) = 0$ by the generalised string equation. It follows that $V_{g,1}(L)$ must possess a factor of $(L^2 + 4\pi^2)$.

(ii) By the generalised dilaton equation, we have $\frac{\partial V_{g,1}}{\partial L}(2\pi i) = 2\pi i(2g-2)V_{g,0}$. Substituting $V_{g,1}(L) = (L^2 + 4\pi^2)P_g(L)$, we obtain $4\pi i P_g(2\pi i) = 2\pi i(2g-2)V_{g,0}$, which can be rearranged to give the desired result. □

For example, one can calculate the following Weil–Petersson volumes corresponding to moduli spaces of closed hyperbolic surfaces.

$$V_{2,0} = \frac{43\pi^6}{2160}$$
$$V_{3,0} = \frac{176557\pi^{12}}{1209600}$$
$$V_{4,0} = \frac{1959225867017\pi^{18}}{493807104000}$$
$$V_{5,0} = \frac{84374265930915479\pi^{24}}{355541114880000}$$

The case of genus two is particularly special since every genus two closed hyperbolic surface possesses a hyperelliptic involution. McShane [31] and the team of Tan, Wong and Zhang [53] have independently capitalised on this extra symmetry to produce the following McShane-type identity.

Proposition 2.13. *If S is a genus 2 closed hyperbolic surface, then*

$$\sum_{(\alpha,\beta)} \tan^{-1} \exp\left(-\frac{\ell(\alpha)}{4} - \frac{\ell(\beta)}{2}\right) = \frac{3\pi}{2}.$$

Here, the sum is over all ordered pairs (α, β) of disjoint simple closed geodesics on S such that α is separating and β is non-separating.

It is possible to use this result in conjunction with Mirzakhani's integration scheme to unfold the integral required to calculate $V_{2,0}$. The result is the following expression for the volume.

$$V_{2,0} = \frac{1}{144\pi} \int_0^\infty \int_0^\infty xy(x^2 + 4\pi^2) \tan^{-1} \exp\left(-\frac{x}{4} - \frac{y}{2}\right) dx\, dy$$

Although we have not calculated the integral explicitly, we have computationally verified that it does agree with the predicted value of $\frac{43\pi^6}{2160}$ to 12 significant figures. It would certainly be interesting to generalise the McShane-type identity for closed surfaces and this volume calculation to the case of arbitrary genus.

Further volume polynomial relations

Given the nature of the generalised string and dilaton equations, one would expect further relations involving the higher order derivatives of $V_{g,n+1}(\mathbf{L}, L_{n+1})$ evaluated at $L_{n+1} = 2\pi i$. Evidence comes from the fact that the Virasoro constraints are a sequence of relations for the top degree terms of $V_{g,n}(\mathbf{L})$ whose first two terms precisely encode the string and dilaton equations. In recent work, Mulase and Safnuk [38] have extended the Virasoro relations to the full volume polynomials. It would be desirable to express their results in a form similar to the generalised string and dilaton equations. This would allow one to determine the Weil–Petersson volumes recursively, relying on the elementary fact that the derivatives of a polynomial evaluated at a point completely determine the polynomial. Further evidence is given by the following volume polynomial relation involving the second derivative of $V_{g,n+1}(\mathbf{L}, L_{n+1})$.

Proposition 2.14. *For $2g - 2 + n > 0$, the Weil–Petersson volumes satisfy the following relation.*

$$\frac{\partial^2 V_{g,n+1}}{\partial L_{n+1}^2}(\mathbf{L}, 2\pi i) = \left[\sum_{k=1}^{n} L_k \frac{\partial}{\partial L_k} - (4g - 4 + n)\right] V_{g,n}(\mathbf{L})$$

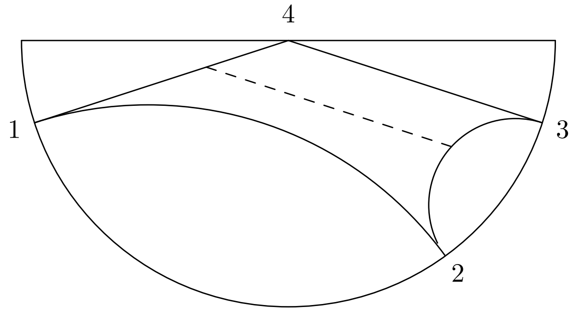
This was obtained by differentiating Mirzakhani's recursion and then using the generalised string and dilaton equations. By taking higher derivatives of Mirzakhani's recursion, one can hope to recursively obtain further volume polynomial relations. However, the strength of the generalised string and dilaton equations lies in their simplicity. It is not clear that any higher derivative relations obtained in this way may possess this same strength. Therefore, one would like to know when the expression $\frac{\partial^k V_{g,n+1}}{\partial L_{n+1}^k}(\mathbf{L}, 2\pi i)$ depends only on $V_{g,n}(\mathbf{L})$.

Hyperbolic cone surfaces

In hyperbolic geometry, one encounters the general phenomenon that a purely imaginary length corresponds to an angle. For example, the elements of $PSL_2(\mathbb{R})$ representing translations of distance d have trace equal to $2\cosh\frac{d}{2}$ while those representing rotations of angle θ have trace equal to $2\cos\frac{\theta}{2} = 2\cosh\frac{i\theta}{2}$. Note that the generalised string and dilaton equations involve evaluation of Weil–Petersson volumes with one of the lengths equal to $2\pi i$. The natural geometric interpretation for this is that the boundary component has degenerated to a cone point with cone angle 2π. In this way, the generalised string and dilaton equations provide a tantalising connection between the intersection theory on $\overline{\mathcal{M}}_{g,n}$ and hyperbolic cone surfaces.

Unfortunately, the geometry of hyperbolic cone surfaces is not very well understood. Many of the results concerning hyperbolic surfaces with geodesic boundary do not translate in any straightforward manner to the case of hyperbolic surfaces with cone points. The following example shows that there is not always a simple closed geodesic in every isotopy class of simple closed curves.

Example 2.15. The following diagram shows a quadrilateral with three ideal vertices in the Poincaré disk model of the hyperbolic plane. It can be doubled along its boundary to create a genus 0 hyperbolic surface with three cusps labelled 1, 2, 3 and a cone point labelled 4. The dashed curve lifts to a simple closed curve γ on the surface. However, if we treat the cone point as a puncture, then there is no simple closed geodesic in the same isotopy class as γ.



The example above demonstrates that a topological pair of pants decomposition cannot always be isotoped to a geometric decomposition involving hyperbolic pairs of pants with geodesic boundary. Since such decompositions are fundamental in the existing theory, this obstruction is a major stumbling block in the analysis of hyperbolic cone surfaces. However, note that the surface described in the example does actually allow at least one geometric pair of pants decomposition, so not all hope is lost.

Many of the results concerning hyperbolic surfaces which generalise to the case of cone surfaces do so only when the cone angles are at most π. As an example, consider the following generalisation of Theorem 1.28 to hyperbolic cone surfaces due to Tan, Wong and Zhang [53]. They refer to geodesic boundary components, cusps and cone points as geometric boundary components. Furthermore, they define the complex length of a geodesic boundary component of length L to be L, of a cusp to be 0 and of a cone point with cone angle θ to be $i\theta$.

Theorem 2.16 (Generalised McShane's identity)**.** *Consider a hyperbolic cone surface with n geometric boundary components $\beta_1, \beta_2, \ldots, \beta_n$ of complex lengths $L_1, L_2, \ldots, L_n$, respectively. If all cone angles lie in the interval $(0, \pi]$, then*

$$\sum_{(\alpha_1,\alpha_2)} \mathcal{D}(L_1, \ell(\alpha_1), \ell(\alpha_2)) + \sum_{k=2}^{n} \sum_{\gamma} \mathcal{R}(L_1, L_k, \ell(\gamma)) = L_1.$$

Here, the first summation is over unordered pairs (α_1, α_2) of simple closed geodesics which bound a pair of pants with β_1, while the second summation is over simple closed geodesics γ which bound a pair of pants with β_1 and β_k. The functions $\mathcal{D} : \mathbb{R}^3 \to \mathbb{R}$ and $\mathcal{R} : \mathbb{R}^3 \to \mathbb{R}$ are defined as follows.

$$\mathcal{D}(x, y, z) = 4 \tanh^{-1} \left(\frac{\sinh \frac{x}{2}}{\cosh \frac{x}{2} + \exp \frac{y+z}{2}} \right) \qquad \mathcal{R}(x, y, z) = 2 \tanh^{-1} \left(\frac{\sinh \frac{x}{2} \sinh \frac{y}{2}}{\cosh \frac{z}{2} + \cosh \frac{x}{2} \cosh \frac{y}{2}} \right)$$

In an ideal world, Mirzakhani's results on Weil–Petersson volumes would generalise to the case of hyperbolic surfaces with cone points. Consequently, one would be able to use the intermediate moduli spaces $\mathcal{M}_{g,n+1}(\mathbf{L}, i\theta)$ for $0 \leq \theta \leq 2\pi$ in order to give relations between the intersection theory on $\overline{\mathcal{M}}_{g,n+1}$ and the intersection theory on $\overline{\mathcal{M}}_{g,n}$. This approach to proving the generalised string and dilaton equations sheds light on these relations and, furthermore, predicts that there are others. As pointed out by Norbury, the idea of using these intermediate moduli spaces is reminiscent of work by Kronheimer and Mrowka [27]. In their paper, they use moduli spaces of anti-self-dual connections with cone singularities around an embedded surface in a four-manifold to obtain information about intersection numbers on instanton moduli spaces.

Unfortunately, Mirzakhani's results simply do not extend to the case of hyperbolic cone surfaces when the cone angles are as large as 2π. In fact, it is presently unclear how to even define a moduli space of hyperbolic cone surfaces, a corresponding Weil–Petersson symplectic form and, hence, a volume polynomial. Perhaps the correct point of view might be to consider the algebraic analogue of the geometric picture. For example, a promising avenue is to analyse representations of the fundamental group of $S_{g,n}$ into $PSL_2(\mathbb{R})$ rather than hyperbolic metrics on $S_{g,n}$, à la Goldman [16, 17].

Chapter 3

A new approach to Kontsevich's combinatorial formula

In this chapter, we provide a new approach to Kontsevich's combinatorial formula via hyperbolic geometry. This formula relates psi-class intersection numbers with combinatorial objects known as ribbon graphs. The starting point is Mirzakhani's theorem, which motivates one to consider the asymptotics of the Weil–Petersson volume $V_{g,n}(\mathbf{L})$. The first step in our journey will be to prove that the moduli space of hyperbolic surfaces is homeomorphic as an orbifold to the moduli space of metric ribbon graphs. One advantage of working with this latter space is the fact that it possesses a natural user-friendly system of local coordinates. Next, we use this result to determine the asymptotic behaviour of the Weil–Petersson symplectic form. This part of the proof involves a careful analysis of the behaviour of hyperbolic surfaces as their boundary lengths approach infinity. The key geometric intuition involved is the fact that, in the limit, the hyperbolic surface resembles a ribbon graph after appropriate rescaling of the metric. The final piece of the puzzle is a combinatorial problem relating two determinants associated to a trivalent ribbon graph. These results essentially show that the information stored in the asymptotics of $V_{g,n}(\mathbf{L})$ is precisely Kontsevich's combinatorial formula. As a whole, this work draws together Kontsevich's combinatorial approach and Mirzakhani's hyperbolic approach to Witten's conjecture into a coherent narrative.

> *"I think it is said that Gauss had ten different proofs for the law of quadratic reciprocity. Any good theorem should have several proofs, the more the better. For two reasons: usually, different proofs have different strengths and weaknesses, and they generalise in different directions: they are not just repetitions of each other."*
>
> *Sir Michael Atiyah [51]*

3.1 Kontsevich's combinatorial formula

Ribbon graphs and intersection numbers

One of the fundamental problems concerning moduli spaces of curves is the calculation of intersection numbers, particularly those involving the psi-classes. Witten [57] conjectured that a certain generating function for these numbers satisfies the KdV hierarchy, while Kontsevich [26] supplied the proof. The main result in Kontsevich's paper is a formula which relates psi-class intersection numbers with combinatorial objects known as ribbon graphs.

A ribbon graph is essentially the 1-skeleton of a finite cell decomposition of a compact, connected, oriented surface. Note that such a graph may not necessarily be simple — although it must be finite, it may have multiple edges or loops. The orientation of the surface endows each vertex of the graph with a cyclic orientation of the half-edges meeting there. Conversely, given the graph and the cyclic orientation of the half-edges meeting at each vertex, the topological type of the surface and its cell decomposition may be recovered. This is accomplished by using the extra structure to thicken the graph into a surface with boundaries which may be filled with disks to produce a closed surface. More precisely, we have the following definition.

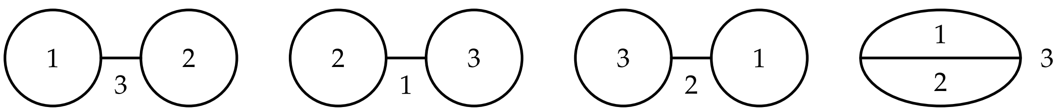
Definition 3.1. A *ribbon graph of type* (g, n) is a graph such that every vertex has degree at least three, there is a cyclic ordering of the half-edges meeting at each vertex, and the thickening of the graph is a genus g connected surface with n boundary components labelled from 1 up to n.

Given a ribbon graph Γ, let X denote the set of its half-edges and let s_0 be the permutation on X which cyclically permutes all half-edges adjacent to the same vertex in an anticlockwise manner. Also, let s_1 be the permutation on X which interchanges each pair of half-edges which together form an edge of the ribbon graph. The set $X_0 = X/\langle s_0\rangle$ is canonically equivalent to the set of vertices of Γ while the set $X_1 = X/\langle s_1\rangle$ is canonically equivalent to the set of edges of Γ. Furthermore, if we let $s_2 = s_0^{-1}s_1$, then the set $X_2 = X/\langle s_2\rangle$ is canonically equivalent to the set of boundary components of Γ. Therefore, one can alternatively consider a ribbon graph to be a triple (X, s_0, s_1) where X is a finite set, s_0 is a permutation on X without fixed points or transpositions and s_1 is an involution on X without fixed points. We also require a labelling of the boundary components of Γ, which is simply a bijection from $X/\langle s_2\rangle$ to $\{1, 2, \ldots, n\}$. Define two ribbon graphs (X, s_0, s_1) and (X', s_0', s_1') to be isomorphic if and only if there exists a bijection $f : X \to X'$ such that $f \circ s_0 = s_0' \circ f$ and $f \circ s_1 = s_1' \circ f$. We also impose the condition that f must preserve the labelling of the boundary components. A ribbon graph automorphism is, of course, an isomorphism from a ribbon graph to itself. The set of automorphisms of a ribbon graph Γ forms a group which is denoted by Aut Γ.

Example 3.2. On the left of the diagram below is a trivalent ribbon graph with two vertices and three edges. It has been drawn such that the cyclic ordering of the half-edges meeting at every vertex matches the orientation of the page, a convention which we will continue to adopt. On the right is the thickening of the ribbon graph, which plainly shows a surface with one boundary component. Substituting $V = 2$, $E = 3$ and $n = 1$ into the formula for the Euler characteristic $V - E = 2 - 2g - n$ yields the fact that $g = 1$. In fact, this is the only trivalent ribbon graph of type $(1, 1)$, and its automorphism group has precisely six elements. In order to see this, observe that there exists a unique automorphism which maps any fixed half-edge to any of the six half-edges in the graph.

The above example illustrates that, given a ribbon graph, it is possible to determine the value of n by thickening up the graph and then the value of g from an Euler characteristic calculation. It should also be clear from the diagram why ribbon graphs are named so.

Example 3.3. The diagram below depicts the four trivalent ribbon graphs of type $(0, 3)$. Note that the rightmost example is isomorphic as an abstract graph to the unique trivalent ribbon graph of type $(1, 1)$. However, the two graphs possess different ribbon structures and represent different topological surfaces.



The set of ribbon graphs of type (g, n) is denoted by $RG_{g,n}$. A particularly important subset of these is the set of trivalent ribbon graphs of type (g, n), which we denote by $TRG_{g,n}$. We are now in a position to state Kontsevich's combinatorial formula.

Theorem 3.4 (Kontsevich's combinatorial formula)**.** *The psi-class intersection numbers on $\overline{\mathcal{M}}_{g,n}$ satisfy the following formula.*

$$\sum_{|\boldsymbol{\alpha}|=3g-3+n} \langle \tau_{\alpha_1} \tau_{\alpha_2} \cdots \tau_{\alpha_n} \rangle \prod_{k=1}^{n} \frac{(2\alpha_k - 1)!!}{s_k^{2\alpha_k+1}} = \sum_{\Gamma \in TRG_{g,n}} \frac{2^{2g-2+n}}{|Aut\ \Gamma|} \prod_{e \in E(\Gamma)} \frac{1}{s_{\ell(e)} + s_{r(e)}}$$

Here, $E(\Gamma)$ denotes the set of edges of Γ and the expression $(2\alpha - 1)!!$ is a shorthand for $\frac{(2\alpha)!}{2^\alpha \alpha!}$. For an edge e, the terms $\ell(e)$ and $r(e)$ are the labels of the boundaries on its left and right.[1]

[1]Note that, although the left and right of an edge are not well-defined, the sum $s_{\ell(e)} + s_{r(e)}$ certainly is.

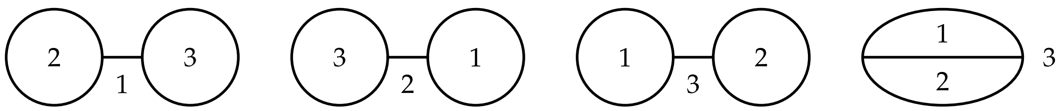
Kontsevich's combinatorial formula is a rather amazing result. In fact, a priori, it appears as if it could not even be true, since the left hand side is explicitly a polynomial in $\frac{1}{s_1}, \frac{1}{s_2}, \ldots, \frac{1}{s_n}$ while the right hand side only appears to be a rational function of $s_1, s_2, \ldots, s_n$. To see how the formula works in practice, consider the following examples.

Example 3.5. In the case $(g, n) = (1, 1)$, the only contribution to the left hand side is $\langle \tau_1 \rangle \frac{1}{s_1^3}$, resulting from $\alpha_1 = 1$. There is only one term on the right hand side, corresponding to the unique trivalent ribbon graph of type $(1, 1)$, discussed in Example 3.2. Its contribution is

$$2 \times \frac{1}{6} \times \frac{1}{(s_1 + s_1)(s_1 + s_1)(s_1 + s_1)} = \frac{1}{24 s_1^3}.$$

Hence, we conclude that $\langle \tau_1 \rangle = \frac{1}{24}$, which agrees with the calculation from Example 1.16.

Example 3.6. In the case $(g, n) = (0, 3)$, the only contribution to the left hand side is $\langle \tau_0^3 \rangle \frac{1}{s_1 s_2 s_3}$, resulting from $\alpha_1 = \alpha_2 = \alpha_3 = 0$. There are four terms on the right hand side, one for each trivalent ribbon graph of type $(0, 3)$, discussed in Example 3.3. None of these ribbon graphs possess non-trivial automorphisms, so their contributions are as follows.



$$\frac{2}{(s_1+s_1)(s_1+s_2)(s_1+s_3)} \qquad \frac{2}{(s_2+s_2)(s_2+s_3)(s_2+s_1)} \qquad \frac{2}{(s_3+s_3)(s_3+s_1)(s_3+s_2)} \qquad \frac{2}{(s_1+s_2)(s_2+s_3)(s_3+s_1)}$$

These four terms may be placed over a common denominator in order to simplify their sum.

$$\begin{aligned} & \frac{s_2 s_3 (s_2 + s_3) + s_3 s_1 (s_3 + s_1) + s_1 s_2 (s_1 + s_2) + 2 s_1 s_2 s_3}{s_1 s_2 s_3 (s_1 + s_2)(s_2 + s_3)(s_3 + s_1)} \\ = & \frac{(s_1 + s_2)(s_2 + s_3)(s_3 + s_1)}{s_1 s_2 s_3 (s_1 + s_2)(s_2 + s_3)(s_3 + s_1)} \\ = & \frac{1}{s_1 s_2 s_3} \end{aligned}$$

Hence, we conclude that $\langle \tau_0^3 \rangle = 1$, which agrees with the calculation from Example 1.14.

From Witten to Kontsevich

In Section 1.2, we discussed the intersection theory on moduli spaces of curves, noting that one of the landmark results in the area is Witten's conjecture. This can be used, with the help of the string equation and the base case $\langle \tau_0^3 \rangle = 1$, to effectively calculate any psi-class intersection number. Apart from its actual content, there are two notable aspects of Witten's conjecture.

First, it emerged from the analysis of a particular model of two-dimensional quantum gravity. This is one of the more striking instances of the symbiosis between pure mathematics and theoretical physics. Second, the statement of Witten's conjecture involves the KdV hierarchy of partial differential equations, whose origin lies in the analysis of shallow water waves from classical physics. The KdV hierarchy is now known as the prototypical example of an exactly solvable model, thereby providing a connection between moduli spaces of curves and the theory of integrable systems.

The first proof of Witten's conjecture is due to Kontsevich [26] and relies on a cell decomposition of the decorated moduli space $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$ previously noted by Harer, Mumford, Penner and Thurston. Kontsevich used the complex analytic formulation of this theorem, which relies on results concerning quadratic differentials on Riemann surfaces with punctures. These are holomorphic sections of the complex line bundle $T^* \otimes T^*$, where T denotes the tangent bundle of the Riemann surface. In a local coordinate z, a quadratic differential can be expressed in the form $\phi(z)\ dz^2$, where ϕ is a holomorphic function. The expression $\phi(z)\ dz^2$ transforms under change of coordinates to w as $\phi(z(w))(\frac{dz}{dw})^2\ dw^2$. A horizontal trajectory of a quadratic differential is a curve along which $\phi(z)\ dz^2$ is both real and positive. Among the quadratic differentials on a Riemann surface is the class of Jenkins–Strebel quadratic differentials, for which the union of non-closed horizontal trajectories has measure zero. This union of non-closed horizontal trajectories is then a ribbon graph embedded in the Riemann surface. Furthermore, if the Riemann surface has genus g and n punctures, then the ribbon graph has type (g, n). One can naturally associate a positive real number to each edge of the ribbon graph by integrating the 1-form $\sqrt{|\phi(z)|}\ dz$ along the edge. This motivates the following definition.

Definition 3.7. A *metric ribbon graph* is a ribbon graph with a positive real number assigned to every edge. We refer to this number as the length of the edge and the sum of the numbers around a boundary as the length of the boundary.[2]

Let $\mathcal{MRG}_{g,n}$ denote the set of all metric ribbon graphs of type (g, n) and observe that it has a natural topology. For every ribbon graph Γ of type (g, n), there is a subset $\mathcal{MRG}_\Gamma \subseteq \mathcal{MRG}_{g,n}$ consisting of those metric ribbon graphs whose underlying ribbon graph is Γ. The set $\mathcal{MRG}_\Gamma$ can be described as the quotient of an open cell canonically homeomorphic to $\mathbb{R}_+^{|E(\Gamma)|}$ by the action of $\mathrm{Aut}\ \Gamma$. These orbifold cells glue together via edge degenerations — in other words, when an edge length goes to zero, the edge contracts to give a ribbon graph with fewer edges. In this way, we have endowed $\mathcal{MRG}_{g,n}$ with not only a topology, but also an orbifold structure.

Strebel [52] proved that, given a Riemann surface C with distinct points $p_1, p_2, \ldots, p_n$ and positive real numbers $x_1, x_2, \ldots, x_n$, there exists a unique Jenkins–Strebel quadratic differential on $C \setminus \{p_1, p_2, \ldots, p_n\}$ whose associated ribbon graph has boundary lengths $x_1, x_2, \ldots, x_n$. One consequence of Strebel's work is the following result.

[2]Note that if an edge is incident to a boundary on both sides, then its length must be included in the sum twice.

Theorem 3.8. *The map $\mathcal{JS} : \mathcal{M}_{g,n} \times \mathbb{R}_+^n \to \mathcal{MRG}_{g,n}$ described above is a homeomorphism of orbifolds.*

Due to this theorem, the natural cell decomposition of $\mathcal{MRG}_{g,n}$ gives rise to a cell decomposition of $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$. Kontsevich used this fact to to combinatorialise the moduli space of curves and the psi-class intersection numbers. The end result was precisely Kontsevich's combinatorial formula. The right hand side of this formula can be interpreted as an enumeration of trivalent ribbon graphs which is, quite fortunately, amenable to Feynman diagram techniques. In particular, Kontsevich showed that this enumeration of trivalent ribbon graphs is governed by the asymptotic expansion of the following matrix model.

$$F_n(S) = \log \left[\frac{\int \exp\left(-\frac{1}{2}\text{tr } SX^2 + \frac{i}{6}\text{tr } X^3\right) \, dX}{\int \exp\left(-\frac{1}{2}\text{tr } SX^2\right) \, dX} \right]$$

Here, the integrals are over the space of $n \times n$ Hermitian matrices, dX denotes the standard volume form compatible with the metric $d(X, Y) = \sqrt{\text{tr } (X - Y)^2}$, and $S = \text{diag}(s_1, s_2, \ldots, s_n)$. Note that this idea of applying Feynman diagrams and matrix models to problems concerning moduli spaces of curves was not without precedent. These techniques were previously utilised in the calculation of the Euler characteristic of $\mathcal{M}_{g,n}$ by Harer and Zagier [21] and also by Penner [49]. Furthermore, the link between Hermitian matrix models and integrable systems had already been established, so it was a relatively straightforward matter for Kontsevich to then deduce Witten's conjecture.

It should be noted that Kontsevich's paper [26], containing his proof of Witten's conjecture, is very concise in nature. This is partly due to the fact that various technical results are stated without proof. These centre around the compactification of the moduli space, the cell decomposition of the decorated moduli space and the combinatorialisation of the psi-classes. Subsequently, Looijenga [29] provided rigorous proofs for Kontsevich's claims about the compactification of the moduli space. The remaining subtleties arising from Kontsevich's paper are discussed in full detail in a paper by Zvonkine [64].

The remainder of this chapter is dedicated to providing a new proof of Kontsevich's combinatorial formula, from the perspective of hyperbolic geometry. One strength of this proof is that it avoids the complications inherent in Kontsevich's groundbreaking work. Furthermore, it draws together the distinct proofs of Witten's conjecture by Kontsevich and Mirzakhani.

The proof of Kontsevich's combinatorial formula: Part 0

Recall our philosophy that any meaningful statement about the volume $V_{g,n}(\mathbf{L})$ gives a meaningful statement about the intersection theory on $\overline{\mathcal{M}}_{g,n}$, and vice versa. In more explicit terms, Mirzakhani's theorem — see Theorem 1.35 — asserts that the Weil–Petersson volume takes the form

$$V_{g,n}(\mathbf{L}) = \sum_{|\boldsymbol{\alpha}|+m=3g-3+n} \frac{(2\pi^2)^m \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1}\psi_2^{\alpha_2}\ldots\psi_n^{\alpha_n}\kappa_1^m}{2^{|\boldsymbol{\alpha}|}\boldsymbol{\alpha}!m!} L_1^{2\alpha_1}L_2^{2\alpha_2}\ldots L_n^{2\alpha_n}.$$

So the intersection numbers of psi-classes are stored in the top degree part of the volume polynomial. Therefore, information concerning psi-class intersection numbers can be accessed via the asymptotics of $V_{g,n}(\mathbf{L})$. A fact well-known amongst combinatorialists is that asymptotics are often more tractable than exact enumeration, and it will be advantageous to subscribe to this school of thought. Indeed, the Weil–Petersson volumes were calculated by Mirzakhani using an intricate integration scheme which only produces a recursive formula. In this chapter, we will calculate the asymptotics of $V_{g,n}(\mathbf{L})$ directly and, hence, obtain information about psi-class intersection numbers on $\overline{\mathcal{M}}_{g,n}$. In fact, this information turns out to be precisely Kontsevich's combinatorial formula.

The previous discussion motivates us to study the following expression for a fixed n-tuple of positive real numbers $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

$$\begin{aligned}\lim_{N\to\infty}\frac{V_{g,n}(N\mathbf{x})}{N^{6g-6+2n}} &= \lim_{N\to\infty}\sum_{|\boldsymbol{\alpha}|+m=3g-3+n}\frac{(2\pi^2)^m \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1}\psi_2^{\alpha_2}\ldots\psi_n^{\alpha_n}\kappa_1^m}{2^{|\boldsymbol{\alpha}|}\boldsymbol{\alpha}!m!}\frac{x_1^{2\alpha_1}x_2^{2\alpha_2}\ldots x_n^{2\alpha_n}}{N^{6g-6+2n-2|\boldsymbol{\alpha}|}} \\ &= \sum_{|\boldsymbol{\alpha}|=3g-3+n}\frac{1}{2^{3g-3+n}\boldsymbol{\alpha}!}\langle\tau_{\alpha_1}\tau_{\alpha_2}\ldots\tau_{\alpha_n}\rangle\, x_1^{2\alpha_1}x_2^{2\alpha_2}\ldots x_n^{2\alpha_n}\end{aligned}$$

This is a homogeneous symmetric polynomial whose coefficients store all of the psi-class intersection numbers on $\overline{\mathcal{M}}_{g,n}$. The left hand side of Kontsevich's combinatorial formula is also a homogeneous symmetric polynomial whose coefficients store the same information. Despite the fact that they are clearly distinct, they are related by a simple transformation — namely, the Laplace transform — as demonstrated by the following calculation.

$$\begin{aligned}\mathcal{L}\left\{\lim_{N\to\infty}\frac{V_{g,n}(N\mathbf{x})}{N^{6g-6+2n}}\right\} &= \mathcal{L}\left\{\sum_{|\boldsymbol{\alpha}|=3g-3+n}\frac{1}{2^{3g-3+n}\boldsymbol{\alpha}!}\langle\tau_{\alpha_1}\tau_{\alpha_2}\ldots\tau_{\alpha_n}\rangle\, x_1^{2\alpha_1}x_2^{2\alpha_2}\ldots x_n^{2\alpha_n}\right\} \\ &= \sum_{|\boldsymbol{\alpha}|=3g-3+n}\langle\tau_{\alpha_1}\tau_{\alpha_2}\ldots\tau_{\alpha_n}\rangle\prod_{k=1}^{n}\frac{(2\alpha_k-1)!!}{s_k^{2\alpha_k+1}}\end{aligned}$$

In short, we have shown that the Laplace transform of the asymptotics of $V_{g,n}(\mathbf{L})$ is precisely the left hand side of Kontsevich's combinatorial formula. It practically goes without saying that our goal now is to show that it is also equal to the right hand side.

$$\mathcal{L}\left\{\lim_{N\to\infty}\frac{V_{g,n}(N\mathbf{x})}{N^{6g-6+2n}}\right\} = \sum_{\Gamma\in TRG_{g,n}}\frac{2^{2g-2+n}}{|\mathrm{Aut}\,\Gamma|}\prod_{e\in E(\Gamma)}\frac{1}{s_{\ell(e)}+s_{r(e)}} \qquad (3.0)$$

To do this, we need to deal with $V_{g,n}(N\mathbf{x})$ in the $N \to \infty$ limit, which leads us to consider hyperbolic surfaces with long boundaries. The key geometric intuition involved is that, in the limit, the hyperbolic surface resembles a ribbon graph after appropriate rescaling of the metric. In the next section, we make this heuristic argument precise, thereby providing the desired relation between the asymptotics of Weil–Petersson volumes on the one hand and ribbon graphs on the other.

3.2 Hyperbolic surfaces and ribbon graphs

Combinatorial moduli space

For an n-tuple of positive real numbers $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, define $\mathcal{MRG}_{g,n}(\mathbf{x})$ to be the set of metric ribbon graphs of type (g, n), where the length of the boundary labelled k is x_k. As a subset of $\mathcal{MRG}_{g,n}$, it inherits not only a topology, but also an orbifold structure. The main reason for considering this space is the following theorem, which justifies referring to it as the combinatorial moduli space.

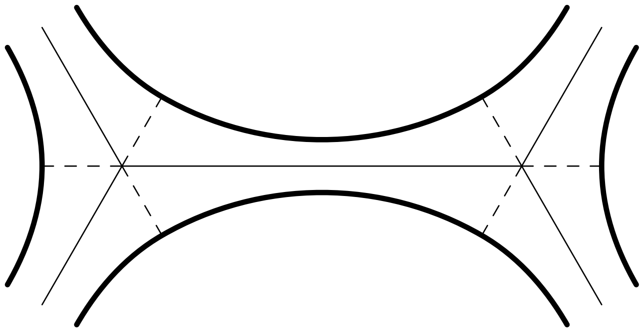
Theorem 3.9. *The spaces $\mathcal{M}_{g,n}(\mathbf{x})$ and $\mathcal{MRG}_{g,n}(\mathbf{x})$ are homeomorphic as orbifolds.*

Our proof of this theorem will essentially imitate the work of Bowditch and Epstein, who considered the case of cusped hyperbolic surfaces [5]. The main idea is to associate to a hyperbolic surface S with geodesic boundary its spine $\Gamma(S)$, otherwise referred to as its cut locus. For every point $p \in S$, let $n(p)$ denote the number of shortest paths from p to the boundary. Generically, we have $n(p) = 1$ and we define the spine as

$$\Gamma(S) = \{p \in S \mid n(p) \geq 2\}.$$

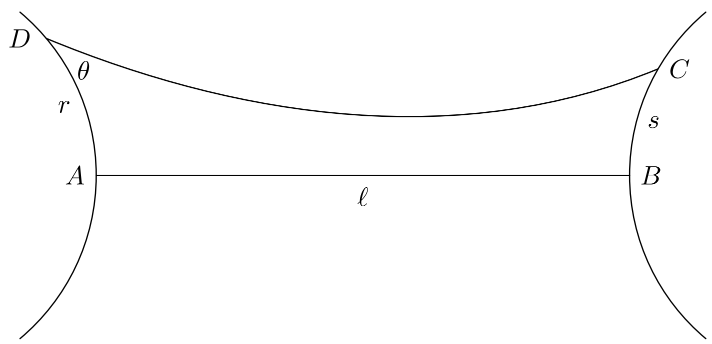
The locus of points with $n(p) = 2$ consists of a disjoint union of open geodesic segments. These correspond precisely to the edges of a graph embedded in S. The locus of points with $n(p) \geq 3$ forms a finite set which corresponds to the set of vertices of the aforementioned graph. In fact, if $n(p) \geq 3$, then the corresponding vertex will have degree $n(p)$. So $\Gamma(S)$ has the structure of a ribbon graph since the cyclic ordering of the half-edges meeting at every vertex can be derived from the orientation of the surface. Furthermore, it is a deformation retract of the original hyperbolic surface, so if S has genus g and n boundary components, then $\Gamma(S)$ will be a ribbon graph of type (g, n).

Now for each vertex p of $\Gamma(S)$, consider the $n(p)$ shortest paths from p to the boundary. We refer to these geodesic segments, which are perpendicular to the boundary of S, as ribs. The diagram below shows part of a hyperbolic surface S, along with its spine and ribs. Note that cutting S along its ribs leaves a collection of hexagons, each with four right angles and a reflective axis of symmetry along one of the diagonals. In fact, this diagonal is one of the edges of $\Gamma(S)$ and we assign to it the length of the side of the hexagon which lies along the boundary of S. Of course, there are two such sides; however, the reflective symmetry of the hexagon guarantees that they have the same length. In this way, $\Gamma(S)$ becomes a metric ribbon graph of type (g, n). By construction, the sum of the edge lengths around each boundary of $\Gamma(S)$ coincides precisely with the length of the corresponding boundary in S. In other words, we have constructed a map $\Gamma : \mathcal{M}_{g,n}(\mathbf{x}) \to \mathcal{MRG}_{g,n}(\mathbf{x})$. We now consider the non-trivial task of showing that this map is bijective by constructing the inverse map.



Fix a metric ribbon graph $\Gamma \in \mathcal{MRG}_{g,n}(\mathbf{x})$ whose vertex and edge sets are V and E, respectively. We wish to construct a hyperbolic surface $S(\Gamma)$ whose spine is precisely this metric ribbon graph. Given $\mathbf{r} \in \mathbb{R}_+^V$ which assigns a positive real number to each vertex, one can determine whether or not these constitute the set of rib lengths for such a hyperbolic surface $S(\Gamma)$. The data of Γ and $\mathbf{r}$ together uniquely determine the symmetric hexagons and the way that they glue together. This produces a surface with spine Γ which has a hyperbolic structure away from the vertices of Γ. At the vertices, the surface obtained may have cone points.

This construction yields a well-defined function $F : \mathbb{R}^V \to \mathbb{R}_+^V$ which takes the set of rib lengths to the set of cone angles around the vertices. Note that we are now allowing the rib lengths to be negative, a fact that will facilitate the proof to follow. We will prove that there exists a unique value of $\mathbf{r} \in \mathbb{R}^V$ such that $F(\mathbf{r}) = (2\pi, 2\pi, \ldots, 2\pi)$. In order to do this, define the function $\theta : \mathbb{R}^3 \to \mathbb{R}$, where $\theta(\ell, r, s)$ denotes the measure of the labelled angle in the following diagram.



To be more precise, construct the segment AB of length ℓ in the hyperbolic plane, as shown. The segment AD is constructed perpendicular to AB with length $|r|$, with D above or below the line AB according to whether r is positive or negative, respectively. The segment BC is constructed similarly, with length $|s|$. The angle $\theta(\ell, r, s)$ is the angle through which the line DA needs to be rotated in an anticlockwise manner about D so that it coincides with the line DC.

If $F_v(\mathbf{r})$ denotes the component of $F(\mathbf{r})$ corresponding to the vertex v, then we have the formula

$$F_v(\mathbf{r}) = 2\sum_{vw} \theta(\ell_{vw}, \mathbf{r}_v, \mathbf{r}_w).$$

Here, the summation is over the edges vw, ℓ_{vw} denotes the length of the edge and $\mathbf{r}_v$ denotes the component of $\mathbf{r}$ corresponding to the vertex v. We make the following simple observations about the behaviour of θ.

Lemma 3.10. *Define the function $\Theta : \mathbb{R}^3 \to \mathbb{R}$ by $\Theta(\ell, r, s) = \theta(\ell, r, s) + \theta(\ell, s, r)$. Then θ and Θ satisfy the following conditions.*

(i) $\frac{\partial\theta}{\partial r} < 0$

(ii) $\frac{\partial\theta}{\partial s} > 0$

(iii) $\frac{\partial\Theta}{\partial r} < 0$

(iv) $\frac{\partial\Theta}{\partial s} < 0$

(v) $r \to \infty \Rightarrow \theta \to 0$

(vi) $r \to -\infty \Rightarrow \theta \to \pi$

(vii) $r < 0$ *and* $r < s \Rightarrow \theta > \frac{\pi}{2}$

For every $W \subseteq V$, let $E_W \subseteq E$ denote the set of edges in Γ which are incident to at least one vertex in W. Define the open polytope $\mathcal{P} \subseteq \mathbb{R}_+^V$ by the condition: $\mathbf{p} \in \mathcal{P}$ if and only if for every non-empty $W \subseteq V$,

$$\sum_{w \in W} \mathbf{p}_w < 2\pi |E_W|.$$

Lemma 3.11. *For a fixed metric ribbon graph Γ, the function F*

(a) is injective;

(b) is an immersion;

(c) is proper; and

(d) is a bijection from $\mathbb{R}^V \to \mathcal{P}$.

Proof. The proof hinges on the facts stated in Lemma 3.10.

(a) In order to obtain a contradiction, suppose that $F(\mathbf{r}) = F(\mathbf{s})$ for $\mathbf{r} \neq \mathbf{s}$. Without loss of generality, assume that $\mathbf{r}_v < \mathbf{s}_v$ for some $v \in V$ and let $\mathbf{m} = \max(\mathbf{r}, \mathbf{s})$. Then from (iii), it follows that

$$\sum_{v\in V} F_v(\mathbf{r}) > \sum_{v\in V} F_v(\mathbf{m}).$$

So there must exist $w \in V$ such that $F_w(\mathbf{r}) > F_w(\mathbf{m})$. But by (i) and (ii), w must satisfy $\mathbf{r}_w < \mathbf{m}_w$ and it follows that $\mathbf{s}_w = \mathbf{m}_w$. Again by (ii), this implies that $F_w(\mathbf{m}) \geq F_w(\mathbf{s})$ and we have the contradiction $F_w(\mathbf{r}) > F_w(\mathbf{s})$.

(b) In order to obtain a contradiction, suppose that at some $\mathbf{r} \in \mathbb{R}^V$ we have $DF(\mathbf{x}) = \mathbf{0}$ for $\mathbf{x} \neq \mathbf{0}$. Without loss of generality, assume that at least one component of $\mathbf{x}$ is negative and let $\mathbf{y} = \max(\mathbf{x}, \mathbf{0})$. Then from (iii), it follows that

$$\sum_{v\in V} DF_v(\mathbf{y}) < \sum_{v\in V} DF_v(\mathbf{x}).$$

Now consider the non-empty set $W = \{v \in V \mid \mathbf{x}_v < 0\}$. Then (ii) implies that $DF_v(\mathbf{y}) \geq DF_v(\mathbf{x})$ for all $v \notin W$. So there must exist $w \in W$ such that $DF_w(\mathbf{y}) < DF_w(\mathbf{x})$. However, by (i) and the fact that $\mathbf{y}_w = 0$, $DF_w(\mathbf{y}) \geq 0$ and we have the contradiction $DF_w(\mathbf{x}) > 0$.

(c) Let $\mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2, \ldots$ be a sequence of points in $\mathbb{R}^V$ which converges to infinity. We will show that the sequence $F(\mathbf{r}_0), F(\mathbf{r}_1), F(\mathbf{r}_2), \ldots$ converges to the boundary of $\mathcal{P}$. First, write V as the disjoint union $V_- \sqcup V_+ \sqcup V_0$, where $\mathbf{r}_v \to -\infty$ for $v \in V_-$, $\mathbf{r}_v \to +\infty$ for $v \in V_+$, and $\mathbf{r}_v$ converges to a finite limit for $v \in V_0$. Then (v) implies that $F_v(\mathbf{r}) \to 0$ for $v \in V_+$. Hence, if V_+ is non-empty, then $F(\mathbf{r})$ converges to the boundary of $\mathcal{P}$. So let us assume that V_+ is empty while V_- is non-empty. In this case, we can use (v) and (vi) to deduce that

$$\sum_{v\in V_-} F_v(\mathbf{r}) \to 2\pi|E_{V_-}|.$$

So $F(\mathbf{r})$ converges to the boundary of $\mathcal{P}$, as desired.

(d) The fact that $F : \mathbb{R}^V \to \mathcal{P}$ is a bijection follows from the earlier parts and Hadamard's observation that a local homeomorphism $\mathbb{R}^N \to \mathbb{R}^N$ is bijective if and only if it is a proper map. □

We are now ready to prove Theorem 3.9.

Proof of Theorem 3.9. All that remains is to make the following three observations.

- First, we need to show that $(2\pi, 2\pi, \ldots, 2\pi) \in \mathcal{P}$. To see this, take some non-empty $W \subseteq V$ and observe that $3|W| \leq 2|E_W|$, since every vertex in W has degree at least 3. Therefore, we have the inequality $2\pi|W| \leq \frac{4\pi}{3}|E_W| < 2\pi|E_W|$, as desired.
- Thus far, we have allowed the rib lengths to be negative. In order to guarantee a bona fide hyperbolic structure, it is necessary to show that $F(\mathbf{r}) = (2\pi, 2\pi, \ldots, 2\pi)$ implies

that $\mathbf{r} \in \mathbb{R}_+^V$. However, note that if $\mathbf{r}_v$ is minimal and negative, then by (vii) we have $F_v(\mathbf{r}) > 3\pi$, a contradiction. Therefore, we are now able to associate to a metric ribbon graph Γ a hyperbolic surface $S(\Gamma)$ whose spine is precisely Γ. In other words, the map $S : \mathcal{MRG}_{g,n}(\mathbf{x}) \to \mathcal{M}_{g,n}(\mathbf{x})$ which we have constructed is the inverse of the map $\Gamma : \mathcal{M}_{g,n}(\mathbf{x}) \to \mathcal{MRG}_{g,n}(\mathbf{x})$ defined earlier.

- Finally, we need to show that the orbifold structure is preserved by the maps Γ and S. In other words, for every metric ribbon graph Γ, it should be true that $\text{Aut}\, S(\Gamma) \cong \text{Aut}\, \Gamma$. Certainly every automorphism of a hyperbolic surface gives rise to an automorphism of the corresponding metric ribbon graph, since it preserves the spine. Therefore, there is an injective homomorphism $\text{Aut}\, S(\Gamma) \to \text{Aut}\, \Gamma$. Now suppose that f is an automorphism of the metric ribbon graph Γ. The fact that the map F defined earlier is injective coupled with the fact that $\Gamma = f(\Gamma)$ implies that the rib lengths of $S(\Gamma)$ satisfy $\mathbf{r}_v = \mathbf{r}_{f(v)}$. Therefore, f extends to an automorphism of the metric ribbon graph as well as the associated rib lengths. However, this means that f extends to an automorphism of $S(\Gamma)$ itself. So the injective homomorphism described earlier is, in fact, an isomorphism of groups. □

The proof of Kontsevich's combinatorial formula: Part 1

Theorem 3.9 asserts that one may equivalently consider either $\mathcal{M}_{g,n}(\mathbf{x})$ or $\mathcal{MRG}_{g,n}(\mathbf{x})$. There are at least two distinct advantages in adopting the latter viewpoint. The moduli space of metric ribbon graphs possesses a tractable system of local coordinates provided by the edge lengths as well as a natural orbifold cell decomposition, which we now describe.

Proposition 3.12. *There is an orbifold cell decomposition*

$$\mathcal{MRG}_{g,n}(\mathbf{x}) = \bigcup_{\Gamma \in RG_{g,n}} \mathcal{MRG}_\Gamma(\mathbf{x}).$$

If E is the number of edges in Γ, then

$$\dim \mathcal{MRG}_\Gamma(\mathbf{x}) = \begin{cases} E - n & \text{if } \Gamma \text{ has at least one odd degree vertex,} \\ E - n + 1 & \text{if } \Gamma \text{ has no odd degree vertices.} \end{cases}$$

Proof. Simply let a metric ribbon graph lie in the set $\mathcal{MRG}_\Gamma(\mathbf{x})$ if its underlying ribbon graph coincides with Γ. In order to determine $\dim \mathcal{MRG}_\Gamma(\mathbf{x})$, label the edges of Γ from 1 up to E and denote the length of the edge labelled k by e_k. Then it follows from the proof of Theorem 3.9 that we can write

$$\mathcal{MRG}_\Gamma(\mathbf{x}) = \{\mathbf{e} \in \mathbb{R}_+^E \mid A\mathbf{e} = \mathbf{x}\} / \text{Aut}\, \Gamma,$$

where A denotes the $n \times E$ adjacency matrix between the boundaries and edges of Γ. The condition $A\mathbf{e} = \mathbf{x}$ simply captures the linear constraints that the edge lengths must satisfy in order for the boundary lengths to be prescribed by $\mathbf{x}$. Since Aut Γ is finite — each automorphism is a permutation on the half-edges of Γ — the dimension of $\mathcal{MRG}_\Gamma(\mathbf{x})$ is given by

$$\dim \mathcal{MRG}_\Gamma(\mathbf{x}) = E - \text{rank } A = E - n + \text{nullity } A^T.$$

Consider Γ to be the 1-skeleton of a cell decomposition of a genus g surface whose faces are denoted by $\{f_1, f_2, \ldots, f_n\}$. We think of A^T as a linear map from the real vector space with basis $\{f_1, f_2, \ldots, f_n\}$ to the real vector space with basis the set of edges of Γ. Now suppose that $a_1 f_1 + a_2 f_2 + \cdots + a_n f_n \in \ker A^T$ and note that if f_i and f_j are adjacent faces, then $a_i + a_j = 0$. Since our surface is connected, it must be the case that $|a_1| = |a_2| = \ldots = |a_n|$.

If Γ has a vertex of degree $2m+1$, then without loss of generality, suppose that the adjacent faces, in cyclic order, are $f_1, f_2, \ldots, f_{2m+1}$. It follows that

$$a_1 = -a_2 = a_3 = \ldots = -a_{2m} = a_{2m+1} = -a_1$$

from which we conclude that $a_1 = 0$. Therefore, $a_1 = a_2 = \ldots = a_n = 0$ and nullity $A^T = 0$.

The condition that Γ has no odd degree vertices is equivalent to the condition that the dual graph to Γ is bipartite. In other words, the faces can be coloured black and white so that adjacent faces are opposite in colour. If we denote the formal sum of the black faces by f_B and the formal sum of the white faces by f_W, then $\ker A^T$ is generated by the element $f_B - f_W$ and it follows that nullity $A^T = 1$. □

In particular, if Γ is trivalent, then the dimension of $\mathcal{MRG}_\Gamma(\mathbf{x})$ is equal to $6g - 6 + 2n$. Furthermore, if Γ is not trivalent, then the dimension of $\mathcal{MRG}_\Gamma(\mathbf{x})$ is strictly less than $6g - 6 + 2n$. By Theorem 3.9, we may consider the Weil–Petersson volume of $\mathcal{MRG}_\Gamma(\mathbf{x})$, which we denote by $V_\Gamma(\mathbf{x})$. Then, by Proposition 3.12, the volume of the moduli space can now be expressed as the following sum.

$$V_{g,n}(\mathbf{L}) = \sum_{\Gamma \in RG_{g,n}} V_\Gamma(\mathbf{L})$$

However, since the volume only cares about cells of top dimension, the sum need only be over the trivalent ribbon graphs. So equation (3.0) — which we wish to prove — becomes

$$\mathcal{L}\left\{\lim_{N\to\infty} \sum_{\Gamma \in TRG_{g,n}} \frac{V_\Gamma(N\mathbf{x})}{N^{6g-6+2n}}\right\} = \sum_{\Gamma \in TRG_{g,n}} \frac{2^{2g-2+n}}{|\text{Aut }\Gamma|} \prod_{e \in E(\Gamma)} \frac{1}{s_{\ell(e)} + s_{r(e)}}.$$

Therefore, Kontsevich's combinatorial formula will follow immediately once we are able to show that

$$\mathcal{L}\left\{\lim_{N\to\infty} \frac{V_\Gamma(N\mathbf{x})}{N^{6g-6+2n}}\right\} = \frac{2^{2g-2+n}}{|\text{Aut }\Gamma|} \prod_{e \in E(\Gamma)} \frac{1}{s_{\ell(e)} + s_{r(e)}}. \tag{3.1}$$

3.3 Hyperbolic surfaces with long boundaries

Preliminary geometric lemmas

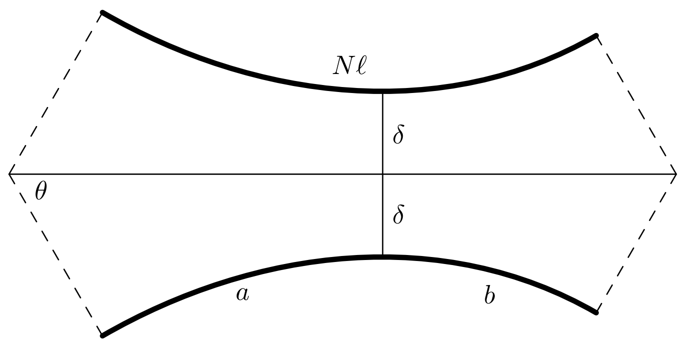
Since we are interested in $V_\Gamma(N\mathbf{x})$ in the large N limit, it is natural to consider hyperbolic surfaces with long boundaries. More precisely, we fix a trivalent metric ribbon graph $\Gamma \in \mathcal{MRG}_{g,n}(\mathbf{x})$ and let $N\Gamma \in \mathcal{MRG}_{g,n}(N\mathbf{x})$ denote the same underlying ribbon graph with the metric scaled by a factor of N. We will be interested in the geometry of the hyperbolic surfaces $S(N\Gamma)$ as N approaches infinity. It will be helpful to keep in mind the intuitive picture that, in the limit, the hyperbolic surface $S(N\Gamma)$ resembles the metric ribbon graph Γ after appropriate rescaling of the metric. In the following, this intuition will be made precise.

Recall that $N\Gamma$ occurs as the spine of a hyperbolic surface $S(N\Gamma)$. Every edge e of $N\Gamma$ is both a diagonal and the axis of symmetry of a hexagon obtained from cutting $S(N\Gamma)$ along its ribs. Lift a neighbourhood of the edge e which includes this hexagon to the hyperbolic plane. The two sides of the hexagon parallel to e lift to two geodesic segments, which lie on two lines. We refer to the common perpendicular between these two lines as the intercostal and denote its length by $2\delta(e)$.

Lemma 3.13. *For any edge e,*

$$\lim_{N\to\infty} \delta(e) = 0.$$

Proof. The diagram below shows an edge e and its corresponding hexagon. The boundary of the hexagon comprises four ribs which have been drawn as dotted lines and two boundary segments which have been drawn as solid lines. Note that the intercostal is perpendicular to the two boundary segments and hence, by symmetry, is perpendicular to the edge e. So there are four hyperbolic trirectangles which occur in the diagram.



Denote the length of the edge e in the metric ribbon graph Γ by ℓ. Consider the lower left trirectangle in the diagram and, without loss of generality, assume that the length marked a satisfies $a \geq \frac{N\ell}{2}$. By a standard hyperbolic trigonometric formula for trirectangles — for example, consider the reference [6] — we have the equation $\cos\theta = \sinh\delta \sinh a$. It follows that

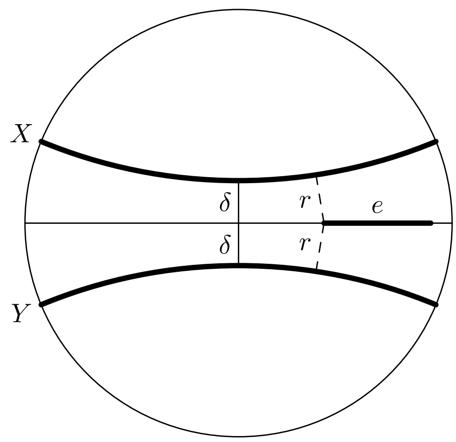
$$\sinh\delta = \frac{\cos\theta}{\sinh a} \leq \frac{1}{\sinh\frac{N\ell}{2}}.$$

So we have $\lim_{N\to\infty} \sinh\delta = 0$, and the desired result follows immediately. □

Although the proof of the previous lemma remains valid, the diagram used is somewhat misleading, since the intercostal may not actually intersect the edge e. However, the next lemma guarantees that this assumption is indeed correct, at least for N sufficiently large.

Lemma 3.14. *If N is sufficiently large, then the intercostal corresponding to an edge intersects that edge.*

Proof. Suppose that the intercostal of the edge e does not intersect it. Consider a lift of e to the hyperbolic plane, along with the associated intercostal of length 2δ, the adjacent boundary components and the two ribs from e which lie closer to the intercostal, as shown in the diagram below.



Suppose that the vertex adjacent to e closer to the intercostal is at distance r from the two adjacent boundary components. Then there must exist a third lift of a boundary component which is at distance r from this vertex. Furthermore, the endpoints of this lift must lie between the points X and Y in the diagram. By Lemma 3.13, as N approaches infinity, the length δ approaches zero. And as δ approaches zero, it is clear that e cannot remain equidistant from the three aforementioned lifts of boundaries without intersecting the intercostal. □

Lemma 3.15. *Let γ be a closed geodesic of length $m(\gamma)$ in $N\Gamma$. Since $N\Gamma$ is a deformation retract of the surface $S(N\Gamma)$, the curve γ defines a unique closed geodesic on the hyperbolic surface $S(N\Gamma)$ whose length we denote by $\ell(\gamma)$. Then*

$$\lim_{N\to\infty} [\ell(\gamma) - m(\gamma)] = 0.$$

Proof. The previous lemma allows us to consider N sufficiently large so that we may cut the surface $S(N\Gamma)$ along the intercostals to obtain a collection of right-angled hexagons. Note that if a closed curve enters a hexagon by one intercostal, it must leave by another. So we can consider the curve γ as the union

$$\gamma = \bigcup_{k=1}^{M} \gamma_k$$

where γ_k denotes the part of γ between entering and exiting one of these hexagons. Let $\ell(\gamma_k)$ denote the length of γ_k and let $m(\gamma_k)$ denote the length of its projection onto the adjacent boundary. Then by the triangle inequality, we have

$$m(\gamma_k) \leq \ell(\gamma_k) \leq m(\gamma_k) + 2\delta(e_k) + 2\delta(\overline{e}_k) \Rightarrow 0 \leq \ell(\gamma_k) - m(\gamma_k) \leq 2\delta(e_k) + 2\delta(\overline{e}_k),$$

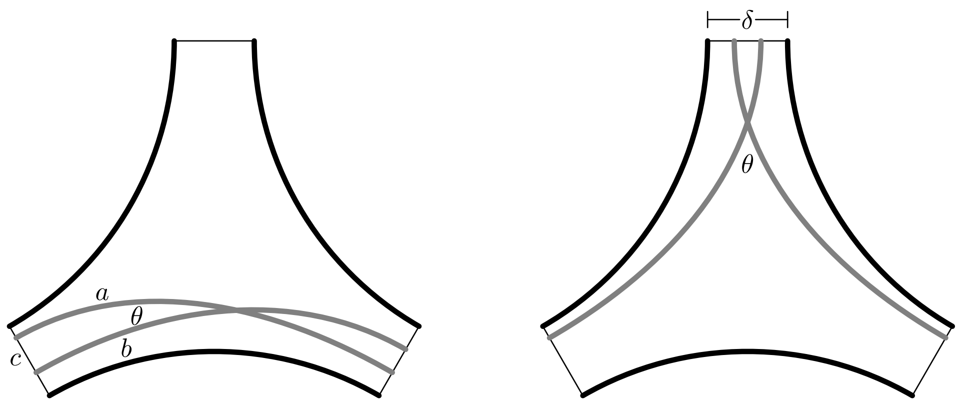
where the edges e_k and $\overline{e}_k$ correspond to the intercostals where the geodesic enters and exits the hexagon, respectively. After summing over $k = 1, 2, \ldots, M$, one obtains

$$0 \leq \ell(\gamma) - m(\gamma) \leq 4M \max \delta(e_k).$$

Lemma 3.13 can now be used to deduce that $\lim_{N\to\infty} [\ell(\gamma) - m(\gamma)] = 0$. □

Lemma 3.16. *Consider two distinct homotopy classes of closed curves in Γ. These define homotopy classes of curves in $S(N\Gamma)$ for all N. Let the geodesic representatives of these homotopy classes intersect in an angle $\theta \leq \frac{\pi}{2}$. Then as $N \to \infty$, we have $\theta \to 0$.*

Proof. Cutting the surface along the intercostals leaves a collection of right-angled hexagons. Each of these hexagons has three alternating sides which are intercostals and three alternating sides which are boundary segments. If a simple closed geodesic enters one of these hexagons via one intercostal, it must leave via another. Therefore, the intersection of two simple closed geodesics, restricted to one of these hexagons, must resemble one of the following two diagrams.



In the diagram on the left, as N approaches infinity, we may assume without loss of generality that the lengths denoted a and b also approach infinity. Furthermore, by Lemma 3.13, the length denoted c approaches zero. The hyperbolic cosine rule states that $\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos\theta$ or equivalently,

$$\cos\theta = \coth a \coth b - \frac{\cosh c}{\sinh a \sinh b}.$$

Therefore, as $N \to \infty$, we have $\cos\theta \to 1$ and $\theta \to 0$.

In the diagram on the right, as N approaches infinity, δ approaches zero by Lemma 3.13. Since the hexagon is right-angled, the two boundary segments adjacent to this intercostal limit to an ideal vertex. The two geodesic segments lie between them, so the acute angle they form — namely, θ — must approach zero. □

These results concerning hyperbolic surfaces confirm the intuition that, in the $N \to \infty$ limit, the hyperbolic surface $S(N\Gamma)$ resembles a metric ribbon graph. A precise statement of this fact can be made by rescaling the hyperbolic metric and making use of Gromov–Hausdorff convergence.

Proposition 3.17. *Given a metric ribbon graph Γ and a positive real number N, let $\widehat{S}_N$ denote the surface $S(N\Gamma)$, where the hyperbolic metric has been scaled by a factor of $\frac{1}{N}$. Then, in the Gromov–Hausdorff topology, we have*

$$\lim_{N\to\infty} \widehat{S}_N = \Gamma.$$

Asymptotic behaviour of the Weil–Petersson form

Now let $\Gamma \in TRG_{g,n}$ be a trivalent ribbon graph and fix an n-tuple of positive real numbers $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. We consider Γ to be embedded as the 1-skeleton of a cell decomposition of a genus g surface, so that the boundary labelled k corresponds to a face of the cell decomposition, which we also label k. Then for every positive real number N, we have the map

$$f : \mathcal{MRG}_\Gamma(\mathbf{x}) \to \mathcal{MRG}_\Gamma(N\mathbf{x}) \to \mathcal{M}_{g,n}(N\mathbf{x}),$$

which is a homeomorphism onto its image. This is the composition of two maps — the first scales the ribbon graph metric by a factor of N while the second uses the Bowditch–Epstein construction discussed in the proof of Theorem 3.9. Consider the normalised Weil–Petersson symplectic form $\frac{\omega}{N^2}$ on $\mathcal{M}_{g,n}(N\mathbf{x})$ and note that it pulls back via f to a symplectic form on $\mathcal{MRG}_\Gamma(\mathbf{x})$. We will be interested in the behaviour of this 2-form as N approaches infinity.

The Fenchel–Nielsen coordinates, which are canonical for the Weil–Petersson symplectic form, provide local coordinates on the moduli space. They consist of $3g-3+n$ length parameters, which are simple to understand, and $3g-3+n$ twist parameters, which are more complicated in nature. It is certainly desirable to have local coordinates for the moduli space, each of which is the length function associated to some simple closed curve. The following theorem due to Wolpert [60] asserts that this is possible and, furthermore, that the Weil–Petersson symplectic form has a reasonably simple description in these coordinates.

Theorem 3.18. *In a genus g hyperbolic surface with n cusps, consider distinct simple closed geodesics $C_1, C_2, \dots, C_{6g-6+2n}$ with lengths $\ell_1, \ell_2, \dots, \ell_{6g-6+2n}$. For a point $p \in C_i \cap C_j$, let θ_p denote the angle between the curves at p, measured anticlockwise from C_i to C_j. Define the $(6g-6+2n) \times (6g-6+2n)$ skew-symmetric matrix X by*

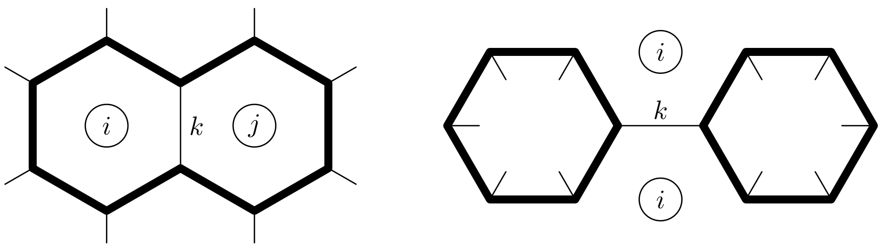
$$X_{ij} = \sum_{p \in C_i \cap C_j} \cos\theta_p, \quad \text{for } i < j.$$

If X is invertible, then $\ell_1, \ell_2, \dots, \ell_{6g-6+2n}$ are local coordinates for the moduli space and the Weil–Petersson symplectic form is given by

$$\omega = -\sum_{i<j} [X^{-1}]_{ij} \, d\ell_i \wedge d\ell_j.$$

On closer inspection of Wolpert's original proof, this theorem extends without amendment to the case of hyperbolic surfaces with geodesic boundaries. By linearity, the theorem also holds if $C_1, C_2, \dots, C_{6g-6+2n}$ are geodesic multicurves or, in other words, if they are finite unions of distinct simple closed geodesics, each with an integer weight. In order to use Theorem 3.18, we require a natural system of multicurves to work with. Begin by labelling the edges of Γ from 1 up to $6g-6+3n$. To the edge labelled k, we associate a multicurve $\widetilde{C}_k$ in Γ as follows.

- Case 1: If the edge labelled k is adjacent to two distinct faces, labelled i and j, then let $\widetilde{C}_k$ be the curve shown in bold in the diagram below left.



- Case 2: If the edge labelled k is adjacent to the face labelled i on both sides, then let $\widetilde{C}_k$ be the union of the two curves shown in bold in the diagram above right.
- Case 3: If the edge labelled k is a loop, then let $\widetilde{C}_k$ be the empty curve.

The ribbon graph Γ occurs as the spine of all hyperbolic surfaces $S(\widetilde{\Gamma})$, for a metric ribbon graph $\widetilde{\Gamma} \in \mathcal{MRG}_\Gamma(\mathbf{x})$. Since Γ is a deformation retract of $S(\widetilde{\Gamma})$, the multicurve $\widetilde{C}_k$ on Γ defines a unique homotopy class of multicurves on $S(\widetilde{\Gamma})$. Let the unique geodesic representative in this homotopy class be C_k and denote its length with respect to the hyperbolic metric by ℓ_k. In this way, we have length functions $\ell_1, \ell_2, \ldots, \ell_{6g-6+3n}$ on $\mathcal{MRG}_\Gamma(\mathbf{x})$ and, in fact, on $\mathcal{MRG}_\Gamma(N\mathbf{x})$ for every positive real number N. However, we will be more interested in the normalised length functions $\widehat{\ell}_k = \frac{\ell_k}{N}$ for $k = 1, 2, \ldots, 6g-6+3n$. Note that on $\mathcal{MRG}_\Gamma(\mathbf{x})$, we also have the edge length functions $e_1, e_2, \ldots, e_{6g-6+3n}$. For a particular value of N, it is difficult to precisely relate these two coordinate systems. However, the picture is much simpler in the $N \to \infty$ limit, where we can use Lemma 3.15 to deduce the following.

- Case 1: If the edge labelled k is adjacent to two distinct faces, labelled i and j, then

$$\lim_{N\to\infty} \widehat{\ell}_k = x_i + x_j - 2e_k.$$

- Case 2: If the edge labelled k is adjacent to the face labelled i on both sides, then

$$\lim_{N\to\infty} \widehat{\ell}_k = x_i - 2e_k.$$

- Case 3: If the edge labelled k is a loop, then $\widehat{\ell}_k = 0$.

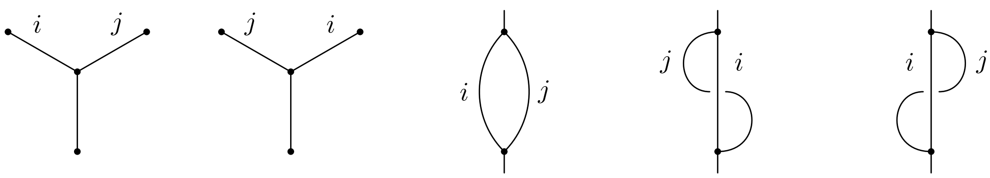
Observe that the function $\widehat{\ell}_k$ can be naturally extended to an open subset U which satisfies $\mathcal{MRG}_\Gamma(\mathbf{x}) \subseteq U \subseteq \mathcal{MRG}_{g,n}(\mathbf{x})$. Therefore, both $\widehat{\ell}_k$ and its derivative converge uniformly on $\mathcal{MRG}_\Gamma(\mathbf{x})$. It follows that one may interchange the operation of limit and derivative. Thus, in the $N \to \infty$ limit, we obtain

$$d\widehat{\ell}_k = -2de_k, \text{ for } k = 1, 2, \ldots, 6g-6+3n.$$

Now we turn our attention to the asymptotic behaviour of the matrix $\widehat{X}$, the $(6g-6+3n) \times (6g-6+3n)$ skew-symmetric matrix defined by

$$\widehat{X}_{ij} = \sum_{p \in C_i \cap C_j} \cos\theta_p, \quad \text{for } i < j.$$

First, we introduce the oriented adjacency $\widehat{B}_{ij}$ of edge i and edge j to be 0 if they are not adjacent or equal, and according to the following convention otherwise.

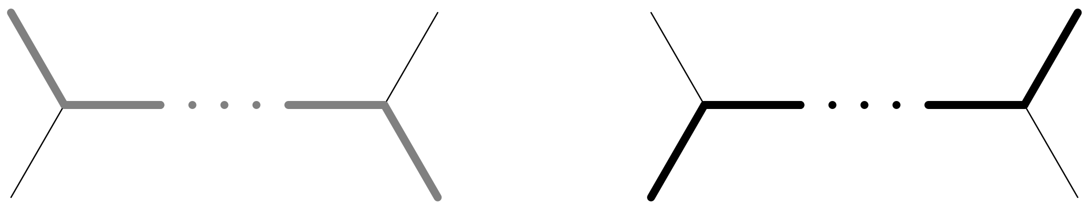


$B_{ij} = -1$ $\quad$ $B_{ij} = +1$ $\quad$ $B_{ij} = 0$ $\quad$ $B_{ij} = -2$ $\quad$ $B_{ij} = +2$

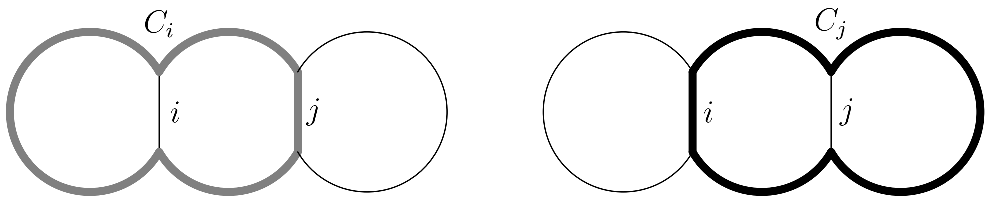
This also defines a $(6g-6+3n) \times (6g-6+3n)$ skew-symmetric matrix. Note that $\widehat{B}$ is an integer matrix by definition while $\widehat{X}$ is an integer matrix as a result of Lemma 3.16. In fact, these two matrices are related by the following lemma.

Lemma 3.19. *In the $N \to \infty$ limit, the matrix $\widehat{X}$ converges to $-2\widehat{B}$.*

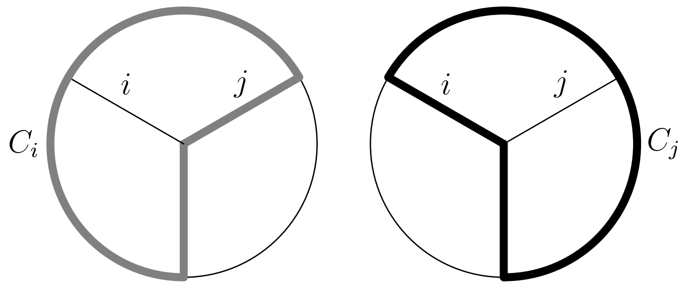
Proof. Suppose that the two curves C_i and C_j traverse a maximal path of consecutive edges. Then in the $N \to \infty$ limit, they will contribute $+1$ to $\widehat{X}_{ij}$ if the diagram is as follows, -1 if the curves are reversed, and 0 otherwise.



It is clear that if C_i and C_j are not adjacent to a common face, then the two curves do not meet at all and $\widehat{X}_{ij} = 0$. Now suppose that edges i and j do share a common face, but are not adjacent. Then the schematic diagram below, combined with our previous observation, shows that C_i and C_j meet precisely twice, but with differing signs, so in the $N \to \infty$ limit we have $\widehat{X}_{ij} = 0$.



Now suppose that the oriented adjacency between edge i and edge j is -1. Then the schematic diagram below, combined with our previous observation, shows that C_i and C_j meet precisely twice with positive orientation so in the $N \to \infty$ limit, we have $\widehat{X}_{ij} = 2$. The same argument can be used to prove that if the oriented adjacency between edge i and edge j is $+1$, then in the $N \to \infty$ limit, we have $\widehat{X}_{ij} = -2$.



There are a few other cases which may arise, for example when the oriented adjacency between edge i and j is ± 2 or when vertices, edges or faces in the diagrams above coincide. However, these may be handled in an entirely analogous manner which is not worthy of reproduction here. In conclusion, we have $\widehat{X}_{ij} = -2\widehat{B}_{ij}$ for all i and j in the $N \to \infty$ limit. □

We state here the fact that the matrix $\widehat{B}$ has rank $6g-6+2n$, although the proof will be delayed until Proposition 3.25. Therefore, we may assume that there exists

$$\mathcal{I} = \{i_1, i_2, \ldots, i_{6g-6+2n}\} \subseteq \{1, 2, \ldots, 6g-6+3n\}$$

such that the matrix B formed from taking the corresponding $6g-6+2n$ rows and $6g-6+2n$ columns of $\widehat{B}$ is invertible. The matrix X is defined analogously from the matrix $\widehat{X}$. The previous results lead to the following result.

Theorem 3.20. *In the $N \to \infty$ limit, $\frac{f^*\omega}{N^2}$ converges pointwise on $\mathcal{MRG}_\Gamma(\mathbf{x})$ to a 2-form Ω.*

Proof. We will assume without loss of generality that $\mathcal{I} = \{1, 2, \ldots, 6g-6+2n\}$, for ease of notation. By Theorem 3.18, Lemma 3.19 and the observation that $d\widehat{\ell}_k = -2de_k$ in the limit, we have

$$\begin{aligned}\Omega = \lim_{N\to\infty} \frac{f^*\omega}{N^2} &= -\lim_{N\to\infty} \sum_{i<j} [X^{-1}]_{ij}\, d\widehat{\ell}_i \wedge d\widehat{\ell}_j \\ &= \frac{1}{2}\sum_{i<j} [B^{-1}]_{ij}\, d\widehat{\ell}_i \wedge d\widehat{\ell}_j = 2\sum_{i<j} [B^{-1}]_{ij}\, de_i \wedge de_j.\end{aligned}$$ □

We note that this 2-form on the combinatorial moduli space $\mathcal{MRG}_\Gamma(\mathbf{x})$ coincides precisely with the 2-form Ω defined by Kontsevich in [26]. It has also been brought to our attention that this result, with an alternative proof due to Mondello, appears in [36]. Among other differences, Mondello uses a system of coordinates dual to ours to compute the Weil–Petersson Poisson structure on Teichmüller space and produces Theorem 3.20 as a byproduct. It seems that our choice of coordinates is more well-suited for the purpose of analysing the asymptotic behaviour of the Weil–Petersson symplectic form. We believe that it also offers a more intuitive, rather than computational, proof of Theorem 3.20.

The proof of Kontsevich's combinatorial formula: Part 2

As discussed in the proof of Proposition 3.12, we have

$$\mathcal{MRG}_\Gamma(\mathbf{x}) = \{\mathbf{e} \in \mathbb{R}_+^{6g-6+3n} \mid A\mathbf{e} = \mathbf{x}\}/\operatorname{Aut}\Gamma,$$

where A is the adjacency matrix between faces and edges. In other words, A_{ij} is the number of times that face i is adjacent to edge j in the cell decomposition of a genus g surface associated to Γ. Therefore, $\mathcal{MRG}_\Gamma(\mathbf{x})$ can be interpreted as the quotient of a polytope by a finite group. Now recall that we are interested in the calculation of the following expression.

$$\lim_{N\to\infty} \frac{V_\Gamma(N\mathbf{x})}{N^{6g-6+2n}} = \frac{1}{(3g-3+n)!} \lim_{N\to\infty} \int_{\mathcal{MRG}_\Gamma(\mathbf{x})} \left(\frac{f^*\omega}{N^2}\right)^{3g-3+n}$$

In Section 1.3, we stated that the Weil–Petersson form extends smoothly to a closed form on $\overline{\mathcal{M}}_{g,n}(\mathbf{L})$. Since $\overline{\mathcal{M}}_{g,n}(\mathbf{L})$ is compact, one consequence is the fact that $V_{g,n}(\mathbf{L})$ is finite. Therefore, we can invoke Lebesgue's dominated convergence theorem to interchange the order of the limit and integration procedures.

$$\begin{aligned}
\lim_{N\to\infty} \frac{V_\Gamma(N\mathbf{x})}{N^{6g-6+2n}} &= \frac{1}{(3g-3+n)!} \int_{\mathcal{MRG}_\Gamma(\mathbf{x})} \left(\lim_{N\to\infty} \frac{f^*\omega}{N^2}\right)^{3g-3+n} \\
&= \int_{\mathcal{MRG}_\Gamma(\mathbf{x})} \frac{\Omega^{3g-3+n}}{(3g-3+n)!} \\
&= 2^{3g-3+n} \operatorname{Pf} B^{-1} \int_{\mathcal{MRG}_\Gamma(\mathbf{x})} d\mathbf{e}_{\mathcal{I}} \\
&= \frac{1}{|\mathrm{Aut}\,\Gamma|} \frac{2^{3g-3+n}}{\sqrt{\det B}} \int_{\substack{A\mathbf{e}=\mathbf{x} \\ \mathbf{e}>\mathbf{0}}} d\mathbf{e}_{\mathcal{I}}
\end{aligned}$$

Here, we have used the notation $d\mathbf{e}_{\mathcal{I}} = de_{i_1} \wedge de_{i_2} \wedge \ldots \wedge de_{i_{6g-6+2n}}$. The second and third equalities follow from Theorem 3.20, while the appearance of the factor $\frac{1}{|\mathrm{Aut}\,\Gamma|}$ in the final line is due to the action of $\mathrm{Aut}\,\Gamma$ on the polytope $\{\mathbf{e} \in \mathbb{R}_+^{6g-6+3n} \mid A\mathbf{e} = \mathbf{x}\}$.

We are now interested in performing the volume calculation

$$\int_{\substack{A\mathbf{e}=\mathbf{x} \\ \mathbf{e}>\mathbf{0}}} d\mathbf{e}_{\mathcal{I}}.$$

As a function of $\mathbf{x}$, this volume is piecewise linear though not so easy to describe, due to the positivity constraints on $\mathbf{e}$. However, its Laplace transform does have a simple description, as demonstrated by the following result.

Theorem 3.21. *Suppose that A is an $n \times (m+n)$ matrix with rank n, non-negative real entries, and positive column sums. Consider sets $\mathcal{I} \sqcup \mathcal{J} = \{1, 2, \ldots, m+n\}$, where $\mathcal{I} = \{i_1, i_2, \ldots, i_m\}$ and $\mathcal{J} = \{j_1, j_2, \ldots, j_n\}$. Let $V(\mathbf{x})$ denote the volume of the polytope $\mathcal{P}(\mathbf{x}) = \{\mathbf{e} \in \mathbb{R}_+^{m+n} \mid A\mathbf{e} = \mathbf{x}\}$ with respect to the volume form $d\mathbf{e}_{\mathcal{I}} = de_{i_1} \wedge de_{i_2} \wedge \ldots \wedge de_{i_m}$. If $A_{\mathcal{J}}$ is the $n \times n$ matrix formed by the columns of A indexed by elements of $\mathcal{J}$ and $\mathbf{s}^t = (s_1, s_2, \ldots, s_n)$, then the Laplace transform of $V(\mathbf{x})$ is given by*

$$\mathcal{L}\{V(\mathbf{x})\} = |\det A_{\mathcal{J}}| \prod_{k=1}^{m+n} \frac{1}{[A^t\mathbf{s}]_k}.$$

Proof. Let $\widehat{\mathcal{P}}(\mathbf{x})$ denote the projection of $\mathcal{P}(\mathbf{x})$ onto the subspace defined by $\{e_j = 0 \mid j \in \mathcal{J}\}$, and note that $V(\mathbf{x})$ is the volume of $\widehat{\mathcal{P}}(\mathbf{x})$ with respect to the volume form $d\mathbf{e}_{\mathcal{I}}$. If the matrix $A_{\mathcal{J}}$ is singular, then $\mathcal{P}(\mathbf{x})$ projects to a set of positive codimension in the subspace $\{e_j = 0 \mid j \in \mathcal{J}\}$. Therefore, both sides of the equation are zero and the theorem is trivially true.

So consider the case $\det A_{\mathcal{J}} \neq 0$. If we write $\mathbf{e}_{\mathcal{I}} = (e_{i_1}, e_{i_2}, \ldots, e_{i_m})$ and $\mathbf{e}_{\mathcal{J}} = (e_{j_1}, e_{j_2}, \ldots, e_{j_n})$, then the following is true.

$$\begin{aligned}\mathbf{e}_{\mathcal{I}} \in \widehat{\mathcal{P}}(\mathbf{x}) &\Leftrightarrow \mathbf{e}_{\mathcal{I}} > \mathbf{0} \text{ and there exists } \mathbf{e}_{\mathcal{J}} > \mathbf{0} \text{ such that } A_{\mathcal{I}}\mathbf{e}_{\mathcal{I}} + A_{\mathcal{J}}\mathbf{e}_{\mathcal{J}} = \mathbf{x} \\ &\Leftrightarrow \mathbf{e}_{\mathcal{I}} > \mathbf{0} \text{ and } A_{\mathcal{J}}^{-1}A_{\mathcal{I}}\mathbf{e}_{\mathcal{I}} < A_{\mathcal{J}}^{-1}\mathbf{x}\end{aligned}$$

It is required to perform two integrations — one for the volume calculation and one for the Laplace transform. Our first step will be to switch the order of integration.

$$\begin{aligned}\mathcal{L}\{V(\mathbf{x})\} = \mathcal{L}\left\{\int_{\substack{A\mathbf{e}=\mathbf{x}\\ \mathbf{e}>\mathbf{0}}} d\mathbf{e}_{\mathcal{I}}\right\} &= \int_{\mathbf{x}>\mathbf{0}} \left(\int_{\substack{A\mathbf{e}=\mathbf{x}\\ \mathbf{e}>\mathbf{0}}} d\mathbf{e}_{\mathcal{I}}\right) \exp(-\langle \mathbf{s}, \mathbf{x}\rangle)\, d\mathbf{x} \\ &= \int_{\mathbf{e}_{\mathcal{I}}>\mathbf{0}} \left(\int_{A_{\mathcal{J}}^{-1}A_{\mathcal{I}}\mathbf{e}_{\mathcal{I}}<A_{\mathcal{J}}^{-1}\mathbf{x}} \exp(-\langle \mathbf{s}, \mathbf{x}\rangle)\, d\mathbf{x}\right) d\mathbf{e}_{\mathcal{I}}\end{aligned}$$

Now use the substitution $\mathbf{y} = A_{\mathcal{J}}^{-1}\mathbf{x}$, which implies that $d\mathbf{x} = |\det A_{\mathcal{J}}|\, d\mathbf{y}$.

$$\begin{aligned}\mathcal{L}\{V(\mathbf{x})\} &= |\det A_{\mathcal{J}}| \int_{\mathbf{e}_{\mathcal{I}}>\mathbf{0}} \left(\int_{A_{\mathcal{J}}^{-1}A_{\mathcal{I}}\mathbf{e}_{\mathcal{I}}<\mathbf{y}} \exp(-\langle \mathbf{s}, A_{\mathcal{J}}\mathbf{y}\rangle)\, d\mathbf{y}\right) d\mathbf{e}_{\mathcal{I}} \\ &= |\det A_{\mathcal{J}}| \int_{\mathbf{e}_{\mathcal{I}}>\mathbf{0}} \left(\int_{A_{\mathcal{J}}^{-1}A_{\mathcal{I}}\mathbf{e}_{\mathcal{I}}<\mathbf{y}} \exp(-\langle A_{\mathcal{J}}^t\mathbf{s}, \mathbf{y}\rangle)\, d\mathbf{y}\right) d\mathbf{e}_{\mathcal{I}}\end{aligned}$$

The inner integral can be directly evaluated to obtain the following.

$$\begin{aligned}\mathcal{L}\{V(\mathbf{x})\} &= |\det A_{\mathcal{J}}| \prod_{j\in\mathcal{J}} \frac{1}{[A_{\mathcal{J}}^t\mathbf{s}]_j} \int_{\mathbf{e}_{\mathcal{I}}>\mathbf{0}} \exp(-\langle A_{\mathcal{I}}^t\mathbf{s}, \mathbf{e}_{\mathcal{I}}\rangle)\, d\mathbf{e}_{\mathcal{I}} \\ &= |\det A_{\mathcal{J}}| \prod_{j\in\mathcal{J}} \frac{1}{[A_{\mathcal{J}}^t\mathbf{s}]_j} \prod_{i\in\mathcal{I}} \int_0^\infty \exp(-[A^t\mathbf{s}]_i e_i)\, de_i \\ &= |\det A_{\mathcal{J}}| \prod_{k=1}^{m+n} \frac{1}{[A^t\mathbf{s}]_k}\end{aligned}$$

□

Now recall that the proof of Kontsevich's combinatorial formula will be complete once we prove equation (3.1).

$$\mathcal{L}\left\{\lim_{N\to\infty} \frac{V_\Gamma(N\mathbf{x})}{N^{6g-6+2n}}\right\} = \frac{2^{2g-2+n}}{|\mathrm{Aut}\,\Gamma|} \prod_{e\in E(\Gamma)} \frac{1}{s_{\ell(e)} + s_{r(e)}}$$

Theorem 3.21 can now be applied to our setting with $m = 6g - 6 + 2n$ and A representing the $n \times (6g - 6 + 3n)$ adjacency matrix between faces and edges of Γ.

$$\begin{aligned}\mathcal{L}\left\{\lim_{N\to\infty}\frac{V_\Gamma(N\mathbf{x})}{N^{6g-6+2n}}\right\} &= \frac{1}{|\mathrm{Aut}\,\Gamma|}\frac{2^{3g-3+n}}{\sqrt{\det B}}\,\mathcal{L}\left\{\int_{\substack{A\mathbf{e}=\mathbf{x}\\ \mathbf{e}>0}} d\mathbf{e}_{\mathcal{I}}\right\}\\ &= \frac{2^{3g-3+n}}{|\mathrm{Aut}\,\Gamma|}\frac{|\det A_{\mathcal{J}}|}{\sqrt{\det B}}\prod_{k=1}^{6g-6+3n}\frac{1}{[A^t\mathbf{s}]_k}\\ &= \frac{2^{3g-3+n}}{|\mathrm{Aut}\,\Gamma|}\frac{|\det A_{\mathcal{J}}|}{\sqrt{\det B}}\prod_{e\in E(\Gamma)}\frac{1}{s_{\ell(e)}+s_{r(e)}}\end{aligned}$$

The first equality we have already deduced from Theorem 3.20, the second equality follows from Theorem 3.21, and the third equality uses the combinatorial structure of the matrix A. So all that remains, in order to deduce Kontsevich's combinatorial formula, is to prove that

$$\frac{2^{3g-3+n}}{|\mathrm{Aut}\,\Gamma|}\frac{|\det A_{\mathcal{J}}|}{\sqrt{\det B}} = \frac{2^{2g-2+n}}{|\mathrm{Aut}\,\Gamma|} \Leftrightarrow \det B = 2^{2g-2}(\det A_{\mathcal{J}})^2. \tag{3.2}$$

3.4 Calculating the combinatorial constant

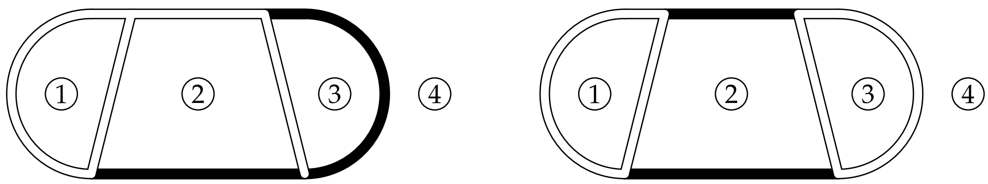
Chain complexes associated to a ribbon graph

The final piece of the puzzle in our proof is to deduce the correct combinatorial constant appearing in Kontsevich's combinatorial formula. From the discussion in the previous section, particularly equation (3.2), this boils down to proving the following theorem concerning determinants associated to a coloured trivalent ribbon graph.

Theorem 3.22. *Let Γ be a trivalent ribbon graph of type (g, n) with n edges coloured white and the remaining $6g - 6 + 2n$ edges coloured black. Let A be the $n \times n$ adjacency matrix formed between the faces and the white edges. Let B be the $(6g - 6 + 2n) \times (6g - 6 + 2n)$ oriented adjacency matrix formed between the black edges. Then*

$$\det B = 2^{2g-2}(\det A)^2.$$

Example 3.23. Consider the following trivalent ribbon graph of type $(0, 4)$, with the two colourings shown.



For the colouring on the left, we have the matrices

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

which satisfy $\det A = -2$ and $\det B = 1$. For the colouring on the right, we have the matrices

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which satisfy $\det A = 0$ and $\det B = 0$. Therefore, Theorem 3.22 is certainly satisfied for the two colourings of this trivalent ribbon graph.

For the remainder of this section, fix a trivalent ribbon graph Γ of type (g, n) with n edges coloured white and the remaining $6g - 6 + 2n$ edges coloured black. Consider Γ to be the 1-skeleton of a cell decomposition of a closed genus g surface. This cell decomposition necessarily consists of $4g - 4 + 2n$ vertices, $6g - 6 + 3n$ edges and n faces. Denote the edges by $e_1, e_2, \ldots, e_{6g-6+3n}$ and the faces by $f_1, f_2, \ldots, f_n$. Consider the sequence $\mathcal{C}$ of free $\mathbb{Z}$-modules

$$0 \longrightarrow C_1 \xrightarrow{d_1} C_2 \xrightarrow{d_2} C_3 \xrightarrow{d_3} C_4 \longrightarrow 0,$$

where $C_1 = C_4 = \langle f_1, f_2, \ldots, f_n \rangle$ and $C_2 = C_3 = \langle e_1, e_2, \ldots, e_{6g-6+3n} \rangle$. We define the $\mathbb{Z}$-module homomorphisms as follows.

- The map d_1 is the adjacency map from faces to edges of the cell decomposition associated to Γ. In other words, for a face f adjacent to the m not necessarily distinct edges $e_1, e_2, \ldots, e_m$ as shown in the diagram below, let

  $$d_1(f) = e_1 + e_2 + \cdots + e_m,$$

  and extend d_1 to a homomorphism of $\mathbb{Z}$-modules.

- The map d_2 is the oriented adjacency map on the edges of the cell decomposition associated to Γ. In other words, for an edge e adjacent to the four not necessarily distinct edges e_1, e_2, e_3, e_4 as shown in the diagram below, let

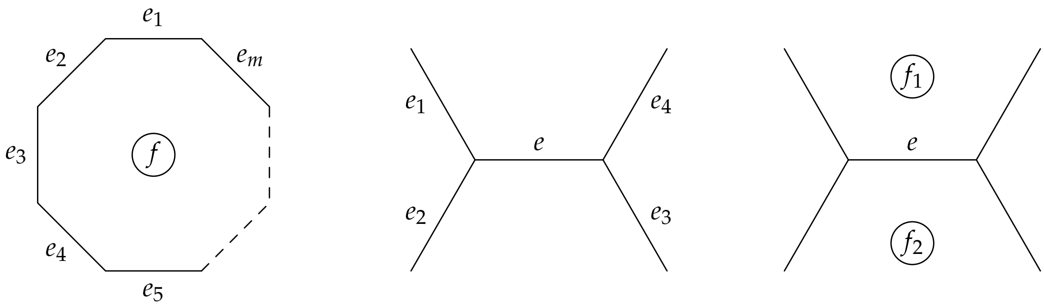
  $$d_2(e) = e_1 - e_2 + e_3 - e_4,$$

and extend d_2 to a homomorphism of $\mathbb{Z}$-modules.

- The map d_3 is the adjacency map from edges to faces of the cell decomposition associated to Γ. In other words, for an edge e adjacent to the two not necessarily distinct faces f_1 and f_2 as shown in the diagram below, let

$$d_3(e) = f_1 + f_2,$$

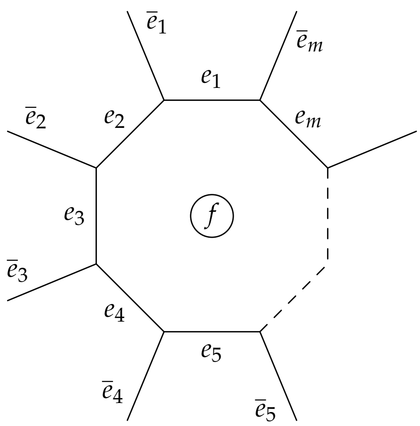
and extend d_3 to a homomorphism of $\mathbb{Z}$-modules.



Proposition 3.24. *The sequence $\mathcal{C}$ is a chain complex of $\mathbb{Z}$-modules.*

Proof. There are two things to check — namely, that $d_2 \circ d_1 = 0$ and $d_3 \circ d_2 = 0$.

Suppose that f is a face which is adjacent to the m not necessarily distinct edges $e_1, e_2, \ldots, e_m$, as shown in the diagram below. Furthermore, suppose that the edges e_k and e_{k+1} are also adjacent to the edge $\overline{e}_k$, where the subscripts are taken modulo m.

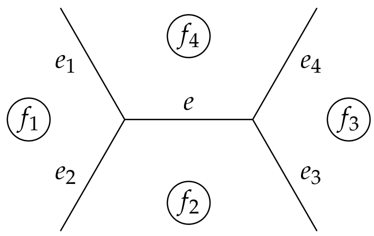


$$\begin{aligned} & d_2 \circ d_1(f) \\ &= d_2(e_1 + e_2 + \cdots + e_m) \\ &= \sum d_2(e_k) \\ &= \sum (e_{k-1} - \overline{e}_{k-1} + \overline{e}_k - e_{k+1}) \\ &= \sum e_{k-1} - \sum \overline{e}_{k-1} + \sum \overline{e}_k - \sum e_{k+1} \\ &= 0 \end{aligned}$$

Since $d_2 \circ d_1 = 0$ on the generators $f_1, f_2, \ldots, f_n$, it holds for all elements of C_1.

Now suppose that e is an edge which is adjacent to the four not necessarily distinct edges e_1, e_2, e_3, e_4, as shown in the diagram below. Furthermore, suppose that the edges e_k and e_{k+1}

are adjacent to the face f_k, where the subscripts are taken modulo 4.



$$\begin{aligned} & d_3 \circ d_2(e) \\ &= d_3(e_1 - e_2 + e_3 - e_4) \\ &= d_3(e_1) - d_3(e_2) + d_3(e_3) - d_3(e_4) \\ &= (f_4 + f_1) - (f_1 + f_2) + (f_2 + f_3) - (f_3 + f_4) \\ &= 0 \end{aligned}$$

Since $d_3 \circ d_2 = 0$ on the generators $e_1, e_2, \ldots, e_{6g-6+3n}$, it holds for all elements of C_2. □

Now let us consider $\mathcal{C} \otimes \mathbb{R}$, the following sequence of vector spaces and linear maps.

$$0 \longrightarrow C_1 \otimes \mathbb{R} \xrightarrow{d_1} C_2 \otimes \mathbb{R} \xrightarrow{d_2} C_3 \otimes \mathbb{R} \xrightarrow{d_3} C_4 \otimes \mathbb{R} \longrightarrow 0$$

Of course, the linear maps are defined on the basis elements in precisely the same way as for the sequence $\mathcal{C}$. In a hopefully excusable abuse of notation, we will use d_1, d_2, d_3 to denote the $\mathbb{Z}$-module homomorphisms of $\mathcal{C}$ as well as the linear maps of $\mathcal{C} \otimes \mathbb{R}$. As a result of Proposition 3.24, the sequence $\mathcal{C} \otimes \mathbb{R}$ is a chain complex of real vector spaces although the following proposition asserts even more.

Proposition 3.25. *The chain complex $\mathcal{C} \otimes \mathbb{R}$ is exact.*

Proof. Since $\mathcal{C} \otimes \mathbb{R}$ is a chain complex, it suffices to prove that

$$\dim(\ker d_1) = 0, \quad \dim(\operatorname{im} d_1) = \dim(\ker d_2), \quad \dim(\operatorname{im} d_2) = \dim(\ker d_3), \quad \dim(\operatorname{im} d_3) = n.$$

The fact that $\dim(\ker d_1) = 0$ follows from the proof of Corollary 3.12. Now observe that d_1 and d_3 are transposes of each other, with respect to the canonical bases. If we invoke the rank-nullity theorem, then all that remains to be proved is that $\dim(\ker d_2) = n$.

As in Proposition 3.24, suppose that f is a face which is adjacent to the m not necessarily distinct edges $e_1, e_2, \ldots, e_m$. Furthermore, suppose that the edges e_k and e_{k+1} are also adjacent to the edge $\overline{e}_k$, where the subscripts are taken modulo m.

If $\sum a_k e_k \in \ker d_2$, then

$$\frac{a_{i-1} + a_i - \overline{a}_{i-1}}{2} = \frac{a_i + a_{i+1} - \overline{a}_i}{2}$$

for all i, where the subscripts are taken modulo m. So to the face f, we can associate the well-defined value

$$b = \frac{a_1 + a_2 - \overline{a}_1}{2} = \frac{a_2 + a_3 - \overline{a}_2}{2} = \cdots = \frac{a_m + a_1 - \overline{a}_m}{2}.$$

Then $d_1(\sum b_k f_k) = \sum a_k e_k$, from which it follows that $\ker d_2 \subseteq \text{im } d_1$. Since we have already established that $\text{im } d_1 \subseteq \ker d_2$, we can now deduce that $\ker d_2 = \text{im } d_1$ and $\dim(\ker d_2) = \dim(\text{im } d_1) = n$. □

There are various implications of the previous result, such as the fact that the matrix $\widehat{B}$ used in the statement of Lemma 3.19 has rank $6g - 6 + 2n$. We also have the following result.

Corollary 3.26. *Consider a trivalent ribbon graph Γ of type (g, n) with n edges coloured white and the remaining $6g - 6 + 2n$ edges coloured black. If one forms the matrices A and B as per the statement of Theorem 3.22, then* $\det A = 0$ *if and only if* $\det B = 0$.

Thus, for the remainder of the section, we may assume that the colouring of the edges of Γ is such that both A and B are invertible.

Torsion of an acyclic complex

Interesting information is captured by the homology of a chain complex. When the complex is acyclic — that is, has zero homology — then one can extract further information by considering its torsion. A simple example arises in the case of a linear map between two real vector spaces of the same dimension. An invariant under change of basis is the rank of the linear map. However, when the rank is maximal and bases for the vector spaces are fixed, then one can also consider the determinant. The notion of torsion is the natural generalisation of this concept to acyclic chain complexes. For more information, see the first chapter of [42].

Recall our definition of $\mathcal{C} \otimes \mathbb{R}$ as a sequence of real vector spaces, each with a canonical basis described in terms of the edges and faces of the cell decomposition associated to Γ. We will calculate the torsion of the acyclic complex $\mathcal{C} \otimes \mathbb{R}$ with respect to these bases in two distinct ways.

- *Method 1: Black and white edges*
  The following is an approach for calculating the torsion of an acyclic complex $\mathcal{C}$ given by

$$0 \longrightarrow C_1 \xrightarrow{d_1} C_2 \xrightarrow{d_2} \cdots \xrightarrow{d_{n-2}} C_{n-1} \xrightarrow{d_{n-1}} C_n \longrightarrow 0.$$

  For each k, choose finite ordered collections $B_k \subseteq C_k$ with $B_n = \emptyset$ such that the restriction of d_k to B_k is one-to-one and $d_k B_k \cup B_{k+1}$ is a basis for C_{k+1}. The fact that this is possible relies on the acyclicity of the complex. For two bases U and V of a vector space, let V/U denote the matrix describing the base change $U \mapsto V$. Also, let X_k be a basis for C_k for all k and note that $X = \bigoplus X_k$ is a basis for $\mathcal{C} = \bigoplus C_k$. Then the torsion of $\mathcal{C}$ with respect to the basis X is given by the equation

$$\mathcal{T}(\mathcal{C}, X) = \prod_{k=0}^{n-1} \det[d_k B_k \cup B_{k+1} / X_{k+1}]^{(-1)^{n-k}}.$$

Let us apply this to the acyclic complex $\mathcal{C} \otimes \mathbb{R}$ associated to the trivalent ribbon graph Γ. Suppose that the white edges of Γ are $e_{i_1}, e_{i_2}, \ldots, e_{i_{6g-6+2n}}$. Let $X_1 = X_4 = \{f_1, f_2, \ldots, f_n\}$ and $X_2 = X_3 = \{e_1, e_2, \ldots, e_{6g-6+3n}\}$ be the canonical bases for $\mathcal{C} \otimes \mathbb{R}$. Note that, by assumption, the sets

$$\begin{aligned} B_0 &= \emptyset & B_3 &= \{e_{i_1}, e_{i_2}, \ldots, e_{i_{6g-6+2n}}\} \\ B_1 &= \{f_1, f_2, \ldots, f_n\} & B_4 &= \emptyset \\ B_2 &= \{e_{i_1}, e_{i_2}, \ldots, e_{i_{6g-6+2n}}\} \end{aligned}$$

satisfy the conditions required, so we have the following equation for the torsion of $\mathcal{C} \otimes \mathbb{R}$.

$$\begin{aligned} \mathcal{T}(\mathcal{C}, X) &= \det[B_1/X_1] \times \frac{1}{\det[d_1 B_1 \cup B_2 / X_2]} \times \det[d_2 B_2 \cup B_3 / X_3] \times \frac{1}{\det[d_3 B_3 \cup B_4 / X_4]} \\ &= 1 \times \frac{1}{\det A} \times \det B \times \frac{1}{\det A} \\ &= \frac{\det B}{(\det A)^2} \end{aligned}$$

- *Method 2: Smith normal form*
  Interpret the chain complex of real vector spaces associated to Γ as a linear map $d : \mathcal{C} \to \mathcal{C}$ where $\mathcal{C} = \bigoplus C_k$. With respect to the canonical basis X, the map d is defined over the integers, so diagonal coordinates are provided by the Smith–Jordan normal form. In other words, we can write

$$\begin{aligned} C_1 &= \langle A_1, A_2, \ldots, A_n \rangle \\ C_2 &= \langle B_1, B_2, \ldots, B_n, P_1, P_2, \ldots P_{6g-6+2n} \rangle \\ C_3 &= \langle Q_1, Q_2, \ldots, Q_{6g-6+2n}, Y_1, Y_2, \ldots, Y_n \rangle \\ C_4 &= \langle Z_1, Z_2, \ldots, Z_n \rangle \end{aligned}$$

where the boundary maps are all diagonal in the following sense.

$$\begin{aligned} &d_1 A_k = a_k B_k \quad \text{for } k = 1, 2, \ldots, n \\ &d_2 P_k = b_k Q_k \quad \text{for } k = 1, 2, \ldots, 6g-6+2n \qquad d_2 B_k = 0 \quad \text{for } k = 1, 2, \ldots, n \\ &d_3 Y_k = c_k Z_k \quad \text{for } k = 1, 2, \ldots, n \qquad d_3 Q_k = 0 \quad \text{for } k = 1, 2, \ldots, 6g-6+2n \end{aligned}$$

Note that a_k, b_k and c_k are integers which, by a simple change of basis, we may assume to be non-negative. Furthermore, they must be non-zero in order for $\mathcal{C} \otimes \mathbb{R}$ to be acyclic.

Once again, we can express the torsion as an alternating product of determinants. However, the diagonal coordinates allow us to express these determinants as products of positive integers.

$$\mathcal{T}(\mathcal{C}, X) = \frac{(\prod b_k)}{(\prod a_k)\,(\prod c_k)}$$

However, the construction of the diagonal coordinates and the numbers a_k, b_k, c_k leads directly to the following relations.

$$\prod_{k=1}^{n} a_k = |H^2(\mathcal{C})| \qquad \prod_{k=1}^{6g-6+2n} b_k = |H^3(\mathcal{C})| \qquad \prod_{k=1}^{n} c_k = |H^4(\mathcal{C})|$$

Equating these two torsion calculations yields the equation

$$\frac{\det B}{(\det A)^2} = \frac{|H^3(\mathcal{C})|}{|H^2(\mathcal{C})| \times |H^4(\mathcal{C})|}. \tag{3.3}$$

Proposition 3.27. *The $\mathbb{Z}$-modules $H^2(\mathcal{C})$, $H^3(\mathcal{C})$ and $H^4(\mathcal{C})$ are vector spaces over $\mathbb{F}_2$.*

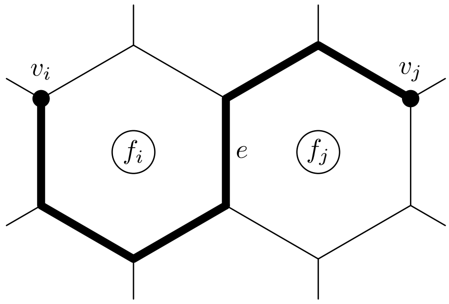
Proof. The proposition follows from the following three facts.

- *If $d_2 y = 0$ for $y \in C_2$, then there exists $x \in C_1$ such that $d_1 x = 2y$.*
  Suppose that $y = \sum a_k e_k$, where $a_1, a_2, \ldots, a_{6g-6+3n}$ are integers. Then by Proposition 3.25, there exist real numbers $b_1, b_2, \ldots, b_n$ such that $d_1(\sum b_k f_k) = \sum a_k e_k$. Consider an edge e_1 and a vertex v adjacent to e_1. Assume without loss of generality that the three not necessarily distinct edges e_1, e_2, e_3 and the three not necessarily distinct faces f_1, f_2, f_3 are adjacent to v. Then one can deduce the following.

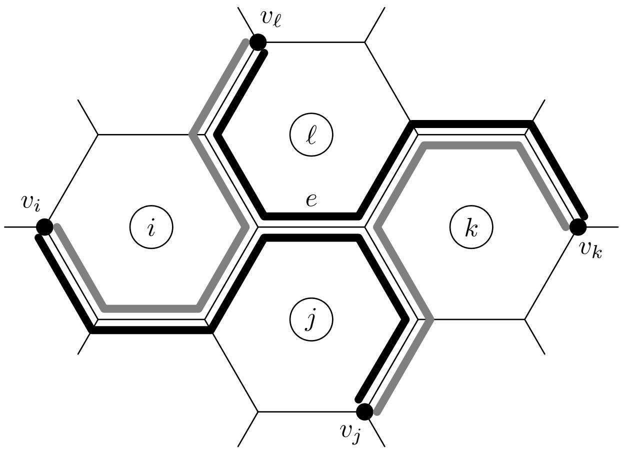
$$\begin{array}{rcl} a_1 &=& b_2 + b_3 \\ a_2 &=& b_3 + b_1 \\ a_3 &=& b_1 + b_2 \end{array} \quad \Rightarrow \quad \begin{array}{rcl} 2b_1 &=& a_2 + a_3 - a_1 \\ 2b_2 &=& a_3 + a_1 - a_2 \\ 2b_3 &=& a_1 + a_2 - a_3 \end{array}$$

  Therefore, b_1, b_2, b_3 are half-integers and the integral linear combination of faces defined by $x = \sum 2b_k f_k \in C_1$ satisfies $d_1 x = 2y$, as required.

- *If $d_3 y = 0$ for $y \in C_3$, then there exists $x \in C_2$ such that $d_2 x = 2y$.*
  Fix vertices $v_1, v_2, \ldots, v_n$, not necessarily distinct, with the condition that the vertex v_k lies on the face f_k. For an edge e adjacent to the faces f_i and f_j, consider the path from v_i to v_j which travels anticlockwise around f_i, clockwise around f_j and passes through the edge e precisely once.



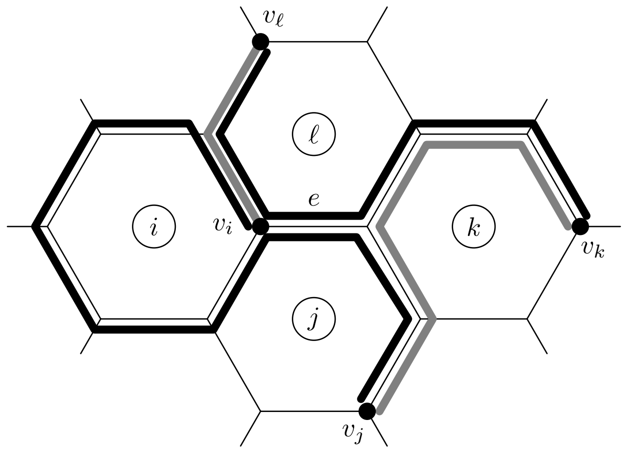
For example, the schematic diagram above indicates the path for a particular edge e. Let $F(e)$ denote the formal sum of the edges traversed by this path and extend the map $F : C_3 \to C_2$ to a homomorphism of $\mathbb{Z}$-modules. Now consider an edge e whose end vertices lie on the faces f_i, f_j, f_k, f_ℓ, as shown in the diagram below. If e is not adjacent to any of v_i, v_j, v_k, v_ℓ, then $F \circ d_2(e) = F(e_1) - F(e_2) + F(e_3) - F(e_4) = -2e$. The paths corresponding to $F(e_1)$ and $F(e_3)$ are shown in grey, while the paths corresponding to $F(e_2)$ and $F(e_4)$ are shown in black. Note the cancellation of black and grey edges everywhere apart from along the edge e itself.



If e is adjacent to any of v_i, v_j, v_k, v_ℓ, then the picture is slightly different, and we have the equation

$$F \circ d_2(e) = F(e_1) - F(e_2) + F(e_3) - F(e_4) = -2e - m_i d_1 f_i + m_j d_1 f_j - m_k d_1 f_k + m_\ell d_1 f_\ell.$$

Here, $m_i = 1$ or 0 depending on whether v_i is adjacent to e or not, respectively, and m_j, m_k, m_ℓ are similarly defined. The following diagram shows an example where e is adjacent to v_i, where the paths corresponding to $F(e_1)$ and $F(e_3)$ are shown in grey, while the paths corresponding to $F(e_2)$ and $F(e_4)$ are shown in black.



Now suppose that $y = \sum a_k e_k$, where $a_1, a_2, \ldots, a_{6g-6+3n}$ are integers. In the particular case when $d_3 y = 0$, we have the particularly nice result that $F \circ d_2(y) = -2y$. It follows that $d_2 \circ F^t(y) = 2y$, where we have used the fact that the matrix representing the map d_2 is skew-symmetric. Since F^t is defined over the integers, it now suffices to take $x = F^t(y)$.

- *For all $y \in C_4$, there exists $x \in C_3$ such that $d_3 x = 2y$.*
  Consider a face f_1 and a vertex v on that face. Assume without loss of generality that the three not necessarily distinct faces f_1, f_2, f_3 and the three not necessarily distinct edges e_1, e_2, e_3 are adjacent to v. Then we can deduce the following.

$$\begin{aligned} d_3(e_2 + e_3 - e_1) &= d_3 e_2 + d_3 e_3 - d_3 e_1 \\ &= (f_3 + f_1) + (f_1 + f_2) - (f_2 + f_3) \\ &= 2f_1 \end{aligned}$$

Since the statement is true for the generators of C_4, it holds for all $y \in C_4$. $\square$

The proof of Kontsevich's combinatorial formula: Part 3

Recall that Kontsevich's combinatorial formula follows immediately once we have proven Theorem 3.22. To do so, we invoke the universal coefficient formula.

$$H^k(\mathcal{C}; \mathbb{F}_2) = \left(H^k(\mathcal{C}) \otimes \mathbb{F}_2\right) \oplus \operatorname{Tor}\left(H^{k+1}(\mathcal{C}); \mathbb{F}_2\right)$$

Proposition 3.27 implies that $H^k(\mathcal{C}) \otimes \mathbb{F}_2 = H^k(\mathcal{C})$ and $\operatorname{Tor}\left(H^{k+1}(\mathcal{C}); \mathbb{F}_2\right) = H^{k+1}(\mathcal{C})$, so we have the relation

$$H^k(\mathcal{C}; \mathbb{F}_2) = H^k(\mathcal{C}) \oplus H^{k+1}(\mathcal{C}).$$

The cell decomposition associated to Γ gives rise to a chain complex of free $\mathbb{Z}$-modules $\mathcal{X}$.

$$0 \longrightarrow X_2 \xrightarrow{\partial_2} X_1 \xrightarrow{\partial_1} X_0 \longrightarrow 0$$

Here, X_2 is generated by the faces of Γ, X_1 is generated by the edges of Γ, and X_0 is generated by the vertices of Γ. There is also the dual cell decomposition Γ^* which gives rise to the associated chain complex of free $\mathbb{Z}$-modules $\mathcal{X}^*$.

$$0 \longrightarrow X_2^* \xrightarrow{\partial_2^*} X_1^* \xrightarrow{\partial_1^*} X_0^* \longrightarrow 0$$

Here, X_2^* is generated by the faces of Γ^*, X_1^* is generated by the edges of Γ^*, and X_0^* is generated by the vertices of Γ^*. Since X_0 is canonically isomorphic to X_2^*, these two chain complexes can be glued together to obtain the following sequence of homomorphisms $\overline{\mathcal{X}}$.

$$0 \longrightarrow X_2 \xrightarrow{\partial_2} X_1 \xrightarrow{\partial_1} X_0 \cong X_2^* \xrightarrow{\partial_2^*} X_1^* \xrightarrow{\partial_1^*} X_0^* \longrightarrow 0$$

Tensoring with $\mathbb{F}_2$, we obtain a chain complex $\overline{\mathcal{X}} \otimes \mathbb{F}_2$ which coincides precisely with $\mathcal{C} \otimes \mathbb{F}_2$.

$$0 \longrightarrow X_2 \otimes \mathbb{F}_2 \xrightarrow{\partial_2} X_1 \otimes \mathbb{F}_2 \xrightarrow{\partial_2^* \circ \partial_1} X_1^* \otimes \mathbb{F}_2 \xrightarrow{\partial_1^*} X_0^* \otimes \mathbb{F}_2 \longrightarrow 0$$

Note that $H^1(\overline{\mathcal{X}} \otimes \mathbb{F}_2) = H^4(\overline{\mathcal{X}} \otimes \mathbb{F}_2) = \mathbb{F}_2$, since they are both equal to the corresponding homology groups of the underlying genus g surface. Now consider the equation

$$\dim(\ker \partial_2^* \circ \partial_1) = \dim(\ker \partial_1) + \dim(\operatorname{im} \partial_1 \cap \ker \partial_2^*).$$

In order to put this to use, we note that ker ∂_2^* is generated by the fundamental class, which is represented by the sum of the dual faces. Since this generator lies in im ∂_1 — it is equal to ∂_1 applied to the sum of all edges — it follows that $\dim(\ker \partial_2^* \circ \partial_1) = \dim(\ker \partial_1) + 1$. So $H^2(\overline{\mathcal{X}} \otimes \mathbb{F}_2) = \mathbb{F}_2^{2g+1}$ and, in summary, the following equations hold true.

$$\begin{array}{lllllll} H^1(\overline{\mathcal{X}} \otimes \mathbb{F}_2) & = & H^1(\mathcal{C} \otimes \mathbb{F}_2) & = & H^1(\mathcal{C}) \oplus H^2(\mathcal{C}) & = & \mathbb{F}_2 \\ H^2(\overline{\mathcal{X}} \otimes \mathbb{F}_2) & = & H^2(\mathcal{C} \otimes \mathbb{F}_2) & = & H^2(\mathcal{C}) \oplus H^3(\mathcal{C}) & = & \mathbb{F}_2^{2g+1} \\ H^3(\overline{\mathcal{X}} \otimes \mathbb{F}_2) & = & H^3(\mathcal{C} \otimes \mathbb{F}_2) & = & H^3(\mathcal{C}) \oplus H^4(\mathcal{C}) & & \\ H^4(\overline{\mathcal{X}} \otimes \mathbb{F}_2) & = & H^4(\mathcal{C} \otimes \mathbb{F}_2) & = & H^4(\mathcal{C}) & = & \mathbb{F}_2 \end{array}$$

The matrix representing d_1 has full rank — see Theorem 3.12 — so we have $H^1(\mathcal{C}) = 0$ and it follows that $H^2(\mathcal{C}) = \mathbb{F}_2$, $H^3(\mathcal{C}) = \mathbb{F}_2^{2g}$, and $H^4(\mathcal{C}) = \mathbb{F}_2$. Finally, combining these homology calculations with equation (3.3) completes the proof of Theorem 3.22.

$$\frac{\det B}{(\det A)^2} = \frac{|H^3(\mathcal{C})|}{|H^2(\mathcal{C})| \times |H^4(\mathcal{C})|} = 2^{2g-2}$$

Chapter 4

Concluding remarks

In this thesis, we have explored the fascinating world of intersection theory on moduli spaces of curves. The recent work of Mirzakhani allows one to adopt a hyperbolic geometric approach to obtain not only new results, but also new insights in this area. For example, we earlier introduced the generalised string and dilaton equations, which relate Weil–Petersson volumes. These, in turn, can be interpreted as relations between intersection numbers on moduli spaces of curves. However, one of the most interesting aspects of this work is the connection revealed between the intersection theory on $\overline{\mathcal{M}}_{g,n}$ and the geometry of hyperbolic cone surfaces.

In another direction, this thesis includes a proof of Kontsevich's combinatorial formula which, of course, is not in itself a new result. What is novel in this work is the crucial use of hyperbolic geometry and the close relationship uncovered between the work of Kontsevich and Mirzakhani. One of the strengths of our proof is the fact that it is both intuitive in nature and devoid of the technical difficulties inherent in the original. The unifying concept in this thesis is the philosophy that any meaningful statement about the volume $V_{g,n}(\mathbf{L})$ gives a meaningful statement about the intersection theory on $\overline{\mathcal{M}}_{g,n}$, and vice versa. It seems likely that one can extend this idea far beyond the content of this thesis.

Our proof of Kontsevich's combinatorial formula capitalised on the fact that the psi-class intersection numbers are stored in the top degree part — in other words, the asymptotics — of the Weil–Petersson volumes, which we now denote by

$$\widetilde{V}_{g,n}(\mathbf{L}) = \lim_{N\to\infty} \frac{V_{g,n}(N\mathbf{L})}{N^{6g-6+2n}}.$$

The crux of the proof was the direct calculation of $\widetilde{V}_{g,n}(\mathbf{L})$ using results from hyperbolic geometry and some combinatorics. As part of joint work with Safnuk [10], we have recently extended some of the ideas in this thesis to prove the following recursive formula for these polynomials.

Theorem 4.1. *The asymptotics of the Weil–Petersson volumes satisfy the following formula.*

$$\begin{aligned} L_1 \widetilde{V}_{g,n}(\mathbf{L}) &= \iint_{0<x+y<L_1} xy\ (L_1 - x - y)\ \widetilde{V}_{g-1,n+1}(x, y, \widehat{\mathbf{L}})\ dx\ dy \\ &+ \sum_{\substack{g_1+g_2=g \\ \mathcal{I}_1 \sqcup \mathcal{I}_2 = [2,n]}} \iint_{0<x+y<L_1} xy\ (L_1 - x - y)\ \widetilde{V}_{g_1,|\mathcal{I}_1|+1}(x, \mathbf{L}_{\mathcal{I}_1})\ \widetilde{V}_{g_2,|\mathcal{I}_2|+1}(y, \mathbf{L}_{\mathcal{I}_2})\ dx\ dy \\ &+ \sum_{k=2}^{n} \int_0^{L_1-L_k} x(L_1 - x)\ \widetilde{V}_{g,n-1}(x, \widehat{\mathbf{L}}_k)\ dx + \int_{L_1-L_k}^{L_1+L_k} \frac{1}{2} x(L_1 + L_k - x)\ \widetilde{V}_{g,n-1}(x, \widehat{\mathbf{L}}_k)\ dx \end{aligned}$$

We have used $\widehat{\mathbf{L}} = (L_2, L_3, \ldots, L_n)$, $\widehat{\mathbf{L}}_k = (L_2, \ldots, \widehat{L_k}, \ldots, L_n)$ *and* $\mathbf{L}_{\mathcal{I}} = (L_{i_1}, L_{i_2}, \ldots, L_{i_m})$ *for* $\mathcal{I} = \{i_1, i_2, \ldots, i_m\}$.

The resemblance of this equation to Mirzakhani's recursion is more than cosmetic, since it is proved in an entirely analogous manner. In particular, the main idea is to unfold the desired volume integral, although the mechanics of the proof are much simpler in this case. For example, standing in the place of the generalised McShane identity in the proof of this statement is the trivial fact that the sum of the edge lengths around a boundary component of a ribbon graph is equal to the length of the boundary. The result itself is also simpler than Mirzakhani's recursion in the sense that all integrands are polynomials. We note here that the differential version of Safnuk's recursion precisely captures the Virasoro constraint condition of Witten's conjecture.

The viewpoint adopted in this thesis might also be used to obtain results concerning integration over Witten cycles and Kontsevich cycles. These are subcomplexes of the moduli space of metric ribbon graphs which are combinatorially defined by constraints on the degree sequence of the metric ribbon graph. Witten and Kontsevich conjectured that these correspond to cycles which are Poincaré dual to polynomials in the tautological classes, a fact now known due to work by Mondello [35] and Igusa [23]. The techniques used in this thesis are well-suited for an investigation along these lines, although this largely remains work in progress. It appears that this thesis, like the vast majority of mathematical research, poses many more questions than it answers.

Bibliography

[1] ABIKOFF, W. *The real analytic theory of Teichmüller space*, vol. 820 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[2] ARBARELLO, E., AND CORNALBA, M. Calculating cohomology groups of moduli spaces of curves via algebraic geometry. *Inst. Hautes Études Sci. Publ. Math. 88* (1998), 97–127.

[3] BERS, L. Spaces of degenerating Riemann surfaces. In *Discontinuous groups and Riemann surfaces*. Princeton Univ. Press, Princeton, N.J., 1974, pp. 43–55.

[4] BOGGI, M., AND PIKAART, M. Galois covers of moduli of curves. *Compositio Math. 120*, 2 (2000), 171–191.

[5] BOWDITCH, B. H., AND EPSTEIN, D. B. A. Natural triangulations associated to a surface. *Topology 27*, 1 (1988), 91–117.

[6] BUSER, P. *Geometry and spectra of compact Riemann surfaces*, vol. 106 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1992.

[7] CANNAS DA SILVA, A. *Lectures on symplectic geometry*, vol. 1764 of *Lecture Notes in Mathematics*. Springer–Verlag, Berlin, 2001.

[8] DELIGNE, P., AND MUMFORD, D. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*, 36 (1969), 75–109.

[9] DO, N., AND NORBURY, P. Weil-Petersson volumes and cone surfaces. Available at `arXiv:math/0603406v1 [math.AG]`.

[10] DO, N., AND SAFNUK, B. Hyperbolic rescaling and symplectic structures on moduli spaces of curves. Preprint.

[11] EKEDAHL, T., LANDO, S., SHAPIRO, M., AND VAINSHTEIN, A. Hurwitz numbers and intersections on moduli spaces of curves. *Invent. Math. 146*, 2 (2001), 297–327.

[12] FABER, C. A conjectural description of the tautological ring of the moduli space of curves. In *Moduli of curves and abelian varieties*, Aspects of Math. E33. Vieweg, Braunschweig, 1999, pp. 109–129.

[13] FABER, C., AND PANDHARIPANDE, R. Relative maps and tautological classes. *J. Eur. Math. Soc. (JEMS) 7*, 1 (2005), 13–49.

[14] FANTECHI, B. Stacks for everybody. In *European Congress of Mathematics, Vol. I (Barcelona, 2000)*, vol. 201 of *Progr. Math.* Birkhäuser, Basel, 2001, pp. 349–359.

[15] FULTON, W. *Intersection theory*, Second ed., vol. 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete 3.* Springer–Verlag, Berlin, 1998.

[16] GOLDMAN, W. M. The symplectic nature of fundamental groups of surfaces. *Adv. in Math. 54*, 2 (1984), 200–225.

[17] GOLDMAN, W. M. Ergodic theory on moduli spaces. *Ann. of Math. (2) 146*, 3 (1997), 475–507.

[18] GRABER, T., AND VAKIL, R. Relative virtual localization and vanishing of tautological classes on moduli spaces of curves. *Duke Math. J. 130*, 1 (2005), 1–37.

[19] GUILLEMIN, V. *Moment maps and combinatorial invariants of Hamiltonian T^n-spaces*, vol. 122 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1994.

[20] HAIN, R., AND LOOIJENGA, E. Mapping class groups and moduli spaces of curves. In *Algebraic geometry — Santa Cruz 1995*, vol. 62 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI, 1997, pp. 97–142.

[21] HARER, J., AND ZAGIER, D. The Euler characteristic of the moduli space of curves. *Invent. Math. 85*, 3 (1986), 457–485.

[22] HARRIS, J., AND MORRISON, I. *Moduli of curves*, vol. 187 of *Graduate Texts in Mathematics*. Springer–Verlag, New York, 1998.

[23] IGUSA, K. Combinatorial Miller-Morita-Mumford classes and Witten cycles. *Algebr. Geom. Topol. 4* (2004), 473–520.

[24] KAZARIAN, M. E., AND LANDO, S. K. An algebro-geometric proof of Witten's conjecture. *J. Amer. Math. Soc. 20*, 4 (2007), 1079–1089.

[25] KIM, Y.-S., AND LIU, K. A simple proof of Witten conjecture through localization. Available at `arXiv:math/0508384v2 [math.AG]`.

[26] KONTSEVICH, M. Intersection theory on the moduli space of curves and the matrix Airy function. *Comm. Math. Phys. 147*, 1 (1992), 1–23.

[27] KRONHEIMER, P. B., AND MROWKA, T. S. Embedded surfaces and the structure of Donaldson's polynomial invariants. *J. Differential Geom. 41*, 3 (1995), 573–734.

[28] LOOIJENGA, E. Smooth Deligne-Mumford compactifications by means of Prym level structures. *J. Algebraic Geom. 3*, 2 (1994), 283–293.

[29] LOOIJENGA, E. Cellular decompositions of compactified moduli spaces of pointed curves. In *The moduli space of curves (Texel Island, 1994)*, vol. 129 of *Progr. Math.* Birkhäuser Boston, Boston, MA, 1995, pp. 369–400.

[30] MCSHANE, G. Simple geodesics and a series constant over Teichmuller space. *Invent. Math. 132*, 3 (1998), 607–632.

[31] MCSHANE, G. Simple geodesics on surfaces of genus 2. *Ann. Acad. Sci. Fenn. Math. 31*, 1 (2006), 31–38.

[32] MILLER, E. Y. The homology of the mapping class group. *J. Differential Geom. 24*, 1 (1986), 1–14.

[33] MIRZAKHANI, M. Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces. *Invent. Math. 167*, 1 (2007), 179–222.

[34] MIRZAKHANI, M. Weil-Petersson volumes and intersection theory on the moduli space of curves. *J. Amer. Math. Soc. 20*, 1 (2007), 1–23.

[35] MONDELLO, G. Combinatorial classes on $\overline{\mathcal{M}}_{g,n}$ are tautological. *Int. Math. Res. Not.*, 44 (2004), 2329–2390.

[36] MONDELLO, G. Triangulated riemann surfaces with boundary and the Weil–Petersson Poisson structure. Available at `arXiv:math/0610698v3 [math.DG]`.

[37] MORITA, S. Characteristic classes of surface bundles. *Invent. Math. 90*, 3 (1987), 551–577.

[38] MULASE, M., AND SAFNUK, B. Mirzakhani's recursion relations, Virasoro constraints and the KdV hierarchy. *Indian J. Math. 50*, 1 (2008), 189–218.

[39] MUMFORD, D. Towards an enumerative geometry of the moduli space of curves. In *Arithmetic and geometry, Vol. II*, vol. 36 of *Progr. Math.* Birkhäuser Boston, Boston, MA, 1983, pp. 271–328.

[40] NÄÄTÄNEN, M., AND NAKANISHI, T. Weil-Petersson areas of the moduli spaces of tori. *Results Math. 33*, 1-2 (1998), 120–133.

[41] NAKANISHI, T., AND NÄÄTÄNEN, M. Areas of two-dimensional moduli spaces. *Proc. Amer. Math. Soc. 129*, 11 (2001), 3241–3252.

[42] NICOLAESCU, L. I. *The Reidemeister torsion of 3-manifolds*, vol. 30 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 2003.

[43] OKOUNKOV, A. Random trees and moduli of curves. In *Asymptotic combinatorics with applications to mathematical physics (St. Petersburg, 2001)*, vol. 1815 of *Lecture Notes in Math.* Springer, Berlin, 2003, pp. 89–126.

[44] OKOUNKOV, A., AND PANDHARIPANDE, R. Gromov-Witten theory, Hurwitz numbers, and matrix models, I. Available at `arXiv:math/0101147v2 [math.AG]`.

[45] OKOUNKOV, A., AND PANDHARIPANDE, R. The equivariant Gromov-Witten theory of $\mathbf{P}^1$. *Ann. of Math. (2) 163*, 2 (2006), 561–605.

[46] OKOUNKOV, A., AND PANDHARIPANDE, R. Gromov-Witten theory, Hurwitz theory, and completed cycles. *Ann. of Math. (2) 163*, 2 (2006), 517–560.

[47] OKOUNKOV, A., AND PANDHARIPANDE, R. Virasoro constraints for target curves. *Invent. Math. 163*, 1 (2006), 47–108.

[48] PENNER, R. C. The moduli space of a punctured surface and perturbative series. *Bull. Amer. Math. Soc. (N.S.) 15*, 1 (1986), 73–77.

[49] PENNER, R. C. Perturbative series and the moduli space of Riemann surfaces. *J. Differential Geom. 27*, 1 (1988), 35–53.

[50] PENNER, R. C. Weil-Petersson volumes. *J. Differential Geom. 35*, 3 (1992), 559–608.

[51] RAUSSEN, M., AND SKAU, C. Interview with Michael Atiyah and Isadore Singer. *European Mathematical Society Newsletter*, 53 (September 2004).

[52] STREBEL, K. *Quadratic differentials*, vol. 5 of *Ergebnisse der Mathematik und ihrer Grenzgebiete 3*. Springer–Verlag, Berlin, 1984.

[53] TAN, S. P., WONG, Y. L., AND ZHANG, Y. Generalizations of McShane's identity to hyperbolic cone-surfaces. *J. Differential Geom. 72*, 1 (2006), 73–112.

[54] THURSTON, W. P. *Three-dimensional geometry and topology. Vol. 1*, vol. 35 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1997. Edited by Silvio Levy.

[55] VAKIL, R. The moduli space of curves and its tautological ring. *Notices Amer. Math. Soc. 50*, 6 (2003), 647–658.

[56] VAKIL, R. The moduli space of curves and Gromov–Witten theory. Available at `arXiv:math/0602347v2 [math.AG]`.

[57] WITTEN, E. Two-dimensional gravity and intersection theory on moduli space. In *Surveys in differential geometry (Cambridge, MA, 1990)*. Lehigh Univ., Bethlehem, PA, 1991, pp. 243–310.

[58] WOLPERT, S. On the homology of the moduli space of stable curves. *Ann. of Math. (2) 118*, 3 (1983), 491–523.

[59] WOLPERT, S. On the Kähler form of the moduli space of once punctured tori. *Comment. Math. Helv. 58*, 2 (1983), 246–256.

[60] WOLPERT, S. On the symplectic geometry of deformations of a hyperbolic surface. *Ann. of Math. (2) 117*, 2 (1983), 207–234.

[61] WOLPERT, S. On the Weil-Petersson geometry of the moduli space of curves. *Amer. J. Math. 107*, 4 (1985), 969–997.

[62] ZOGRAF, P. The Weil-Petersson volume of the moduli space of punctured spheres. In *Mapping class groups and moduli spaces of Riemann surfaces (Göttingen/Seattle, WA, 1991)*, vol. 150 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 1993, pp. 367–372.

[63] ZOGRAF, P. G. Liouville action on moduli spaces and uniformization of degenerate Riemann surfaces. *Algebra i Analiz 1*, 4 (1989), 136–160.

[64] ZVONKINE, D. Strebel differentials on stable curves and Kontsevich's proof of Witten's conjecture. Available at `arXiv:math/0209071v2 [math.AG]`.

Appendix A

Weil–Petersson volumes

A.1 Table of Weil–Petersson volumes

The following table shows various Weil–Petersson volumes $V_{g,n}(L_1, L_2, \ldots, L_n)$. As previously mentioned in Section 1.4, there may be a discrepancy of a power of two between these results and others in the literature. These are caused by a differing normalisation of the Weil–Petersson volume form and by the particular orbifold nature of the spaces $\mathcal{M}_{1,1}(L_1)$ and $\mathcal{M}_{2,0}$.

g	n	$V_{g,n}(L_1, L_2, \ldots, L_n)$
0	3	1
	4	$\frac{1}{2}\sum L_i^2 + 2\pi^2$
	5	$\frac{1}{8}\sum L_i^4 + \frac{1}{2}\sum L_i^2 L_j^2 + 3\pi^2 \sum L_i^2 + 10\pi^4$
	6	$\frac{1}{48}\sum L_i^6 + \frac{3}{16}\sum L_i^4 L_j^2 + \frac{3}{4}\sum L_i^2 L_j^2 L_k^2 + \frac{3\pi^2}{2}\sum L_i^4 + 6\pi^2 \sum L_i^2 L_j^2 + 26\pi^4 \sum L_i^2 + \frac{244\pi^6}{3}$
	7	$\frac{1}{384}\sum L_i^8 + \frac{1}{24}\sum L_i^6 L_j^2 + \frac{3}{32}\sum L_i^4 L_j^4 + \frac{3}{8}\sum L_i^4 L_j^2 L_k^2 + \frac{3}{2}\sum L_i^2 L_j^2 L_k^2 L_m^2 + \frac{5\pi^2}{12}\sum L_i^6$
		$+\frac{15\pi^2}{4}\sum L_i^4 L_j^2 + 15\pi^2 \sum L_i^2 L_j^2 L_k^2 + 20\pi^4 \sum L_i^4 + 80\pi^4 \sum L_i^2 L_j^2 + \frac{910\pi^6}{3}\sum L_i^2 + \frac{2758\pi^8}{3}$
1	1	$\frac{1}{48}L_1^2 + \frac{\pi^2}{12}$
	2	$\frac{1}{192}\sum L_i^4 + \frac{1}{96}L_1^2 L_2^2 + \frac{\pi^2}{12}\sum L_i^2 + \frac{\pi^4}{4}$
	3	$\frac{1}{1152}\sum L_1^6 + \frac{1}{192}\sum L_i^4 L_j^2 + \frac{1}{96}L_1^2 L_2^2 L_3^2 + \frac{\pi^2}{24}\sum L_i^4 + \frac{\pi^2}{8}\sum L_i^2 L_j^2 + \frac{13\pi^4}{24}\sum L_i^2 + \frac{14\pi^6}{9}$
	4	$\frac{1}{9216}\sum L_i^8 + \frac{1}{768}\sum L_i^6 L_j^2 + \frac{1}{384}\sum L_i^4 L_j^4 + \frac{1}{128}\sum L_i^4 L_j^2 L_k^2 + \frac{1}{64}L_1^2 L_2^2 L_3^2 L_4^2 + \frac{7\pi^2}{576}\sum L_i^6$
		$+\frac{\pi^2}{12}\sum L_i^4 L_j^2 + \frac{\pi^2}{4}\sum L_i^2 L_j^2 L_k^2 + \frac{41\pi^4}{96}\sum L_i^4 + \frac{17\pi^4}{12}\sum L_i^2 L_j^2 + \frac{187\pi^6}{36}\sum L_i^2 + \frac{529\pi^8}{36}$
	5	$\frac{1}{92160}\sum L_i^{10} + \frac{1}{4608}\sum L_i^8 L_j^2 + \frac{7}{9216}\sum L_i^6 L_j^4 + \frac{1}{384}\sum L_i^6 L_j^2 L_k^2 + \frac{1}{192}\sum L_i^4 L_j^4 L_k^2$
		$+\frac{1}{64}\sum L_i^4 L_j^2 L_k^2 L_m^2 + \frac{1}{32}L_1^2 L_2^2 L_3^2 L_4^2 L_5^2 + \frac{11\pi^2}{4608}\sum L_i^8 + \frac{35\pi^2}{1152}\sum L_i^6 L_j^2 + \frac{pi^2}{16}\sum L_i^4 L_j^4$
		$+\frac{5\pi^2}{24}\sum L_i^4 L_j^2 L_k^2 + \frac{5\pi^2}{8}\sum L_i^2 L_j^2 L_k^2 L_m^2 + \frac{13\pi^4}{62}\sum L_i^6 + \frac{253\pi^4}{192}\sum L_i^4 L_j^2 + \frac{35\pi^4}{8}\sum L_i^2 L_j^2 L_k^2$
		$+\frac{809\pi^6}{144}\sum L_i^4 + \frac{703\pi^6}{36}\sum L_i^2 L_j^2 + \frac{4771\pi^8}{72}\sum L_i^2 + \frac{16751\pi^{10}}{90}$

g	n	$V_{g,n}(L_1, L_2, \ldots, L_n)$
2	0	$\frac{43\pi^6}{2160}$
	1	$\frac{1}{442368}L_1^8 + \frac{29\pi^2}{138240}L_1^6 + \frac{139\pi^4}{23040}L_1^4 + \frac{169\pi^6}{2880}L_1^2 + \frac{29\pi^8}{192}$
	2	$\frac{1}{4423680}\sum L_i^{10} + \frac{1}{294912}\sum L_i^8 L_j^2 + \frac{29}{2211840}\sum L_i^6 L_j^4 + \frac{11\pi^2}{276480}\sum L_i^8 + \frac{29\pi^2}{69120}\sum L_i^6 L_j^2 + \frac{7\pi^2}{7680}L_1^4 L_2^4 + \frac{19\pi^4}{7680}\sum L_i^6 + \frac{181\pi^4}{11520}\sum L_i^4 L_j^2 + \frac{551\pi^6}{8640}\sum L_i^4 + \frac{7\pi^6}{36}L_1^2 L_2^2 + \frac{1085\pi^8}{1728}\sum L_i^2 + \frac{787\pi^{10}}{480}$
	3	$\frac{1}{53084160}\sum L_i^{12} + \frac{1}{2211840}\sum L_i^{10} L_j^2 + \frac{11}{4423680}\sum L_i^8 L_j^4 + \frac{1}{147456}\sum L_i^8 L_j^2 L_k^2 + \frac{29}{6635520}\sum L_i^6 L_j^6 + \frac{29}{1105920}\sum L_i^6 L_j^4 L_k^2 + \frac{7}{122880}L_1^4 L_2^4 L_3^4 + \frac{\pi^2}{172800}\sum L_i^{10} + \frac{11\pi^2}{110592}\sum L_i^8 L_j^2 + \frac{5\pi^2}{13824}\sum L_i^6 L_j^4 + \frac{29\pi^2}{27648}\sum L_i^6 L_j^2 L_k^2 + \frac{7\pi^2}{3072}\sum L_i^4 L_j^4 L_k^2 + \frac{41\pi^4}{61440}\sum L_i^8 + \frac{211\pi^4}{27648}\sum L_i^6 L_j^2 + \frac{37\pi^4}{2304}\sum L_i^4 L_j^4 + \frac{223\pi^4}{4608}\sum L_i^4 L_j^2 L_k^2 + \frac{77\pi^6}{2160}\sum L_i^6 + \frac{827\pi^6}{3456}\sum L_i^4 L_j^2 + \frac{419\pi^6}{576}L_1^2 L_2^2 L_3^2 + \frac{30403\pi^8}{34560}\sum L_i^4 + \frac{611\pi^8}{216}\sum L_i^2 L_j^2 + \frac{75767\pi^{10}}{8640}\sum L_i^2 + \frac{1498069\pi^{12}}{64800}$
3	0	$\frac{176557\pi^{12}}{1209600}$
	1	$\frac{1}{53508833280}L_1^{14} + \frac{77\pi^2}{9555148800}L_1^{12} + \frac{3781\pi^4}{2786918400}L_1^{10} + \frac{47209\pi^6}{418037760}L_1^8 + \frac{127189\pi^8}{26127360}L_1^6 + \frac{8983379\pi^{10}}{87091200}L_1^4 + \frac{8497697\pi^{12}}{9331200}L_1^2 + \frac{9292841\pi^{14}}{4082400}$
	2	$\frac{1}{856141332480}\sum L_i^{16} + \frac{1}{21403533312}\sum L_i^{14} L_j^2 + \frac{77}{152882380800}\sum L_i^{12} L_j^4 + \frac{503}{267544166400}\sum L_i^{10} L_j^6 + \frac{607}{214035333120}L_1^8 L_2^8 + \frac{17\pi^2}{22295347200}\sum L_i^{14} + \frac{77\pi^2}{3185049600}\sum L_i^{12} L_j^2 + \frac{17\pi^2}{88473600}\sum L_i^{10} L_j^4 + \frac{1121\pi^2}{2229534720}\sum L_i^8 L_j^6 + \frac{1499\pi^4}{7431782400}\sum L_i^{12} + \frac{899\pi^4}{185794560}\sum L_i^{10} L_j^2 + \frac{10009\pi^4}{371589120}\sum L_i^8 L_j^4 + \frac{191\pi^4}{4128768}L_1^6 L_2^6 + \frac{3859\pi^6}{139345920}\sum L_i^{10} + \frac{33053\pi^6}{69672960}\sum L_i^8 L_j^2 + \frac{120191\pi^6}{69672960}\sum L_i^6 L_j^4 + \frac{195697\pi^8}{92897280}\sum L_i^8 + \frac{110903\pi^8}{4644864}\sum L_i^6 L_j^2 \frac{6977\pi^8}{138240}L_1^4 L_2^4 + \frac{37817\pi^{10}}{430080}\sum L_i^6 + \frac{2428117\pi^{10}}{4147200}\sum L_i^4 L_j^2 + \frac{5803333\pi^{12}}{3110400}\sum L_i^4 + \frac{18444319\pi^{12}}{3110400}L_1^2 L_2^2 + \frac{20444023\pi^{14}}{1209600}\sum L_i^2 + \frac{2800144027\pi^{16}}{65318400}$
4	0	$\frac{1959225867017\pi^{18}}{493807104000}$
	1	$\frac{1}{29588244450508800}L_1^{20} + \frac{149\pi^2}{3698530556313600}L_1^{18} + \frac{48689\pi^4}{2397195730944000}L_1 16 + \frac{50713\pi^6}{8989483991040}L_1^{14} + \frac{30279589\pi^8}{32105299968000}L_1^{12} + \frac{43440449\pi^{10}}{445906944000}L_1^{10} + \frac{274101371\pi^{12}}{44590694400}L_1^8 + \frac{66210015481\pi^{14}}{292626432000}L_1^6 + \frac{221508280867\pi^{16}}{50164531200}L_1^4 + \frac{74706907467169\pi^{18}}{1975228416000}L_1^2 + \frac{92480712720869\pi^{20}}{987614208000}$
5	0	$\frac{84374265930915479\pi^{24}}{355541114880000}$
	1	$\frac{1}{487424903779901767680000}L_1^{26} + \frac{7\pi^2}{1339079405988741120000}L_1^{24} + \frac{1823\pi^4}{3106765667303424000000}L_1^{22} + \frac{296531\pi^6}{7766914168258560000}L_1^{20} + \frac{68114707\pi^8}{427180279254220800000}L_1^{18} + \frac{2123300941\pi^{10}}{4746447547269120000}L_1^{16} + \frac{42408901133\pi^{12}}{49442161950720000}L_1^{14} + \frac{19817320001\pi^{14}}{17657914982400000}L_1^{12} + \frac{11171220559409\pi^{16}}{1135151677440000}L_1^{10} + \frac{62028372646367\pi^{18}}{111244864389120}L_1^8 + \frac{202087901261599\pi^{20}}{10534551552000}L_1^6 + \frac{6266936808901001 21\pi^{22}}{1738201006080000}L_1^4 + \frac{881728936440038779\pi^{24}}{289700167680000}L_1^2 + \frac{21185241498983729441\pi^{26}}{2824576634880000}$

A.2 Program for genus 0 Weil–Petersson volumes

The following Maple routine, due to Norbury [9], uses the generalised string equation in order to calculate $V_{0,n+1}(\mathbf{L}, L_{n+1})$ from $V_{0,n}(\mathbf{L})$. A similar algorithm may be used to calculate $V_{1,n+1}(\mathbf{L}, L_{n+1})$ from $V_{1,n}(\mathbf{L})$. These programs have the advantage of being much faster than implementing Mirzakhani's recursion.

```
> # input:  symmetric polynomial f in n variables L1,...,Ln
# output:  symmetric polynomial S in n+1 variables L1,...,L(n+1)
# satisfying S(L(n+1)=0)=f
sym:=proc(f) local i,j,k,m,S,T,T1,prod,sum,epsilon:
S:=f:
epsilon:=array[1..100]:
for i from 1 to 100 do epsilon[i]:=0:  od:
while epsilon[n+1]<1 do
T:=subs(seq(L||j=(1-epsilon[j])*L||j,j=1..n),f):
T1:=0:
for i from 1 to n do
prod:=1:
for j from i+1 to n+1 do
prod:=prod*(1-epsilon[j])
od:
T1:=T1+prod*subs(L||i=L||(n+1),T):
od:
sum:=0:  for k from 1 to n do sum:=sum+epsilon[k] od:
S:=S+(-1)^sum*T1:
for k from 1 to 100 do
if epsilon[k]=1 then epsilon[k]:=0
else epsilon[k]:=1:  k:=100 end if:
od:
od:
S:=simplify(S):
end:
```

```
> # calculate the genus zero volumes recursively from evaluation
# of V_(0,n+1) at L(n+1)=2*Pi*I
for n from 3 to 12 do
P:=0:
for j from 1 to n do
P:=P+int(L||j*V[n],L||j)
od:
Q0:=P:
C0:=simplify(coeff(Q0,Pi,0)):
sim:=sym(C0):
V[n+1]:=sim:
for k from 1 to n-2 do
P||k:=sim-C||(k-1):
Q||k:=subs(L||(n+1)=2*Pi*I,Q||(k-1)-P||k*Pi^(2*k-2)):
C||k:=simplify(coeff(Q||k,Pi,2*k)):
sim:=sym(C||k):
V[n+1]:=V[n+1]+sim*Pi^(2*k):
od:
od:
```