How to calculate in the infinite wedge space.

INFINITE WEDGE SPACE

Vector space: $V$ has basis $\{ s \mid s \in \mathbb{Z} + \frac{1}{2} \}$

Infinite wedge space: $\bigwedge^\infty V$ has basis

$$\{ s_1 \wedge s_2 \wedge s_3 \wedge \cdots \mid s_1 > s_2 > s_3 > \cdots \text{ eventually saturates} \}$$

That is, $s_{n+1} = s_n - 1$ for $n$ sufficiently large. As usual, $a \wedge b = -b \wedge a$.

If $S = \{ s_1 > s_2 > s_3 > \cdots \}$, then we write $v_S = s_1 \wedge s_2 \wedge s_3 \wedge \cdots$

Inner product: Declare the basis $\{ v_S \}$ to be orthonormal.

Diagrammatic representation: $s \in S \longleftrightarrow$ down step at $s$

$s \notin S \longleftrightarrow$ up step at $s$

$$v_S = \frac{7}{2} \wedge \frac{1}{2} \wedge -\frac{1}{2} \wedge -\frac{5}{2} \wedge -\frac{7}{2} \wedge -\frac{13}{2} \wedge \cdots$$

So a basis vector $v_S$ corresponds to a pair $(\lambda, c)$ where $\lambda$ is a Young diagram and $c \in \mathbb{Z}$ is the charge. Sometimes, we prefer to work in the charge 0 subspace $\bigwedge^\infty_0 V$.

OPERATORS

Fermionic operators: $\Psi_k v = k \wedge v$ for $k \in \mathbb{Z} + \frac{1}{2}$

$\Psi^*_k$ is the adjoint
In other words,

\[ \psi_k u_s = \begin{cases} u_{s+k} & \text{if } k \in S \\ 0 & \text{if } k \notin S \end{cases} \]

\[ \psi_k^* u_s = \begin{cases} u_{s+k} & \text{if } k \in S \\ 0 & \text{if } k \notin S \end{cases} \]

\[ \text{sign} = (-1)^{\# \{ s \in S | s > k \} } \]

Anti-commutation relations:

\[ [\psi_i, \psi_j]^*] = \delta_{ij} \]

\[ [\psi_i, \psi_j]^*] = 0 \]

\[ [\psi_i^*, \psi_j^*]^*] = 0 \]

Exercise: Prove these.

Bosonic operators:

\[ \alpha_n = \sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi_k \psi_k^* \text{ for } n \in \mathbb{Z} \setminus \{0\} \]

\[ \alpha_n^* = (\sum_{k \in \mathbb{Z}} \psi_k \psi_k^*)^* = \sum_{k \in \mathbb{Z}} \psi_k^* \psi_k^* = \sum_{k \in \mathbb{Z}} \psi_k \psi_k = \alpha_n \]

Commutation relations:

\[ [\alpha_m, \alpha_n] = m \delta_{m+n} \]

Exercise: Prove this.

Diagrammatic representation: \( \alpha_n \) tends to change DU into UD, when possible.

\[ \text{So } \alpha_n \text{ "adds a box" in all possible ways.} \]

\[ \alpha_{n+1} U_\lambda = \sum_{\mu = \lambda + \alpha} U_\mu \]

It follows that

\[ (\alpha_n)^d U_\rho = \sum_{\lambda \vdash d} (\dim \lambda) U_\lambda. \]

Calculation: Consider the inner product \( ((\alpha_n)^d U_\rho, (\alpha_n)^d U_\rho) \).

1. From the fact above,

\[ ((\alpha_n)^d U_\rho, (\alpha_n)^d U_\rho) = (\sum_{\lambda \vdash d} (\dim \lambda) U_\lambda, \sum_{\lambda \vdash d} (\dim \lambda) U_\lambda) = \sum_{\lambda \vdash d} (\dim \lambda)^2 \]

2. \[ ((\alpha_n)^d U_\rho, (\alpha_n)^d U_\rho) = ((\alpha_n)^d (\alpha_n)^d U_\rho, U_\rho) \]

Note that \( \alpha_n \) "removes a box" in all possible ways.
\[ \alpha_i U_{\lambda} = \sum_{\kappa = \lambda - 0} U_{\mu} \]

In particular, \( \alpha_i U_{\lambda} = 0 \). So use \([\alpha_i, \alpha_i^{-1}] = 1\) to commute \( \alpha_i \), s to the right. The fact that \( \alpha_i \alpha_i^{-1} = \alpha_i \alpha_i + 1 \) means that we obtain a contribution to the inner product only when each \( \alpha_i \) "cancels" with an \( \alpha_i \). There are \( d! \) ways this can happen, so

\[ ((\alpha_i)^d U_\varphi, (\alpha_i)^d U_\psi) = (\alpha_i)^d (\alpha_i)^d U_\varphi, U_\psi) = d! \]

3. So we conclude that \( \sum_{\lambda \vdash d} (\dim \lambda)^k = d! \).

**Exercise:** Expand \( (\alpha_i)^d (\alpha_i)^d \) using \( \alpha_i \alpha_i^{-1} = \alpha_i \alpha_i + 1 \) for \( d = 1, 2, 3 \).

**Expectations:** To obtain \( \langle f(\lambda) \rangle_{\text{Plancherel}} = \sum_{\lambda \vdash d} \frac{(\dim \lambda)^k}{d!} f(\lambda) \), consider the operator \( F \) on \( \Lambda^d V \) satisfying \( FU_\lambda = f(\lambda) U_\lambda \).

\[
\frac{1}{d!} ((\alpha_i)^d F (\alpha_i)^d U_\varphi, U_\psi) = \frac{1}{d!} (F (\alpha_i)^d U_\varphi, (\alpha_i)^d U_\psi)
\]

\[
= \frac{1}{d!} (F \sum_{\lambda \vdash d} (\dim \lambda) U_\lambda, \sum_{\lambda \vdash d} (\dim \lambda) U_\lambda) = \frac{1}{d!} (\sum_{\lambda \vdash d} (\dim \lambda) f(\lambda) U_\lambda, \sum_{\lambda \vdash d} (\dim \lambda) U_\lambda)
\]

\[
= \sum_{\lambda \vdash d} \frac{(\dim \lambda)^k}{d!} f(\lambda) = \langle f(\lambda) \rangle_{\text{Plancherel}}
\]

Then you would try to calculate the inner product by commuting \( (\alpha_i)^d \) past \( F \) and then past \( (\alpha_i)^d \).

**Gromov–Witten Theory**

Birational formula: \( H^0_{\mathbb{P}^1}((k_i + 1), (k_2 + 1), \ldots, (k_n + 1)) = \sum_{\lambda \vdash d} \frac{(\dim \lambda)^2}{d!} \prod_{i=1}^{\infty} f_{k_i + 1}(\lambda) \)

GW/H correspondence: \( \langle \tau_{k_1}(\omega) \tau_{k_2}(\omega) \cdots \tau_{k_n}(\omega) \rangle_{d} = \sum_{\lambda \vdash d} \frac{(\dim \lambda)^2}{d!} \prod_{i=1}^{\infty} \mathbb{P}_{k_i + 1}(\lambda) \)

These count non-singular / possibly singular degree \( d \) covers of \( \mathbb{P}^1 \) with a \((k_i + 1)\)-fold branch point over \( p_i \in \mathbb{P}^1 \).

Here, \( f_k \) and \( p_k \) are shifted symmetric functions defined by

\[
f_k(\lambda) = \left( \lambda_{\lambda_1} \right) \frac{\lambda_{(k_1, \ldots, 1)}}{(\dim \lambda)} \quad \text{and} \quad p_k(\lambda) = k! [z^k] \sum_{\infty} \frac{e((\lambda_i - i + \frac{1}{2}) e(\lambda, z) \)
1-point function: \[ F_1^0 (Z) = \sum_{\lambda} \sum_{d=1}^{\text{dim } \lambda} \frac{1}{(\text{dim } \lambda)^2} \frac{\mathcal{P}_{\lambda}(\lambda)}{(k+1)!} Z^{k+1} \]

\[ = \sum_{\lambda} \sum_{d=1}^{\text{dim } \lambda} \frac{1}{d!} e(\lambda, Z) \]

\[ = \frac{1}{d!} \langle e(\lambda, Z) \rangle_{\text{Plancherel}} \]

So let \( E_0(Z) U_\lambda = e(\lambda, Z) U_\lambda \) be an operator on \( \mathcal{D} \). Then

\[ F_1^0 (Z) = \frac{1}{(d!)^2} \left( (\alpha_1)^d E_0(Z) (\alpha_{-1})^d U_\phi, U_\phi \right). \]

**Question:** How do you commute \( E_0(Z) \) with \( \alpha_1 \)?

**Exercise:** Define \( E_m(Z) = \sum_{k \in \mathbb{Z}+1} e^{2 \left( k - \frac{m}{2} \right)} \Psi_{k-m} \Psi_k^* \) for \( m \in \mathbb{Z} \setminus \{0\} \). Then

\[ [\alpha_1, E_m(Z)] = s(nZ) E_{m+n}(Z) \text{ where } s(Z) = e^{2Z} - e^{-Z}. \]

**Case** \( d = 1 \): \( \alpha_1 E_0(Z) \alpha_{-1} = (E_0(Z) \alpha_1 + s(Z) E_1(Z)) \alpha_{-1} \)

\[ = \underbrace{E_0(Z) \alpha_1 \alpha_{-1}}_{\text{disconnected contribution}} + \underbrace{s(Z) \alpha_1 E_1(Z) + s(Z)^2 E_0(Z)}_{\text{connected contribution}} \]

**General case:**

\[ F_1^0 (Z) = \frac{1}{(d!)^2} \left( (\alpha_1)^d E_0(Z) (\alpha_{-1})^d U_\phi, U_\phi \right) \]

\[ = \frac{1}{(d!)^2} \left( (s(Z)^{2d} E_0(Z) + \text{disconnected contribution}) U_\phi, U_\phi \right) \]

\[ \Rightarrow F_1^0 (Z) = \frac{1}{(d!)^2} s(Z)^{2d} E_0(Z) U_\phi, U_\phi \]

\[ = \frac{1}{(d!)^2} s(Z)^{2d} e(\emptyset, Z) = \frac{1}{(d!)^2} s(Z)^{2d-1} \]

**Other Calculations**

- n-point function for GW invariants of \( \mathbb{P}^1 \) and \( E \)
- MacMahon formula for plane partitions and other DT generating functions
- many quantities pertaining to Plancherel/Schur and uniform measure on partitions (determinantal processes, theta functions)
- generating functions for Hurwitz numbers (integrability)
change of basis coefficients between $\{f_\mu\}$ and $\{p_\mu\}$ for the
algebra of shifted symmetric functions

REFERENCES

- Okounkov - Infinite wedge and random partitions
- Okounkov and Pandharipande - Gromov-Witten theory, Hurwitz theory,
  and completed cycles