How to calculate in the infinite wedge space

INFINITE WEDGE SPACE

Vector space: V has basis $\{\underline{s} \mid s \in \mathbb{Z} + \frac{1}{2}\}$

Infinite wedge space: $\Lambda^{\frac{\infty}{2}} V$ has basis

$$\{\underline{s_1} \wedge \underline{s_2} \wedge \underline{s_3} \wedge \cdots \mid s_1 > s_2 > s_3 > \cdots \text{ eventually saturates}\}$$

That is, $s_{n+1} = s_n - 1$ for n sufficiently large. As usual, $\underline{a} \wedge \underline{b} = -\underline{b} \wedge \underline{a}$.

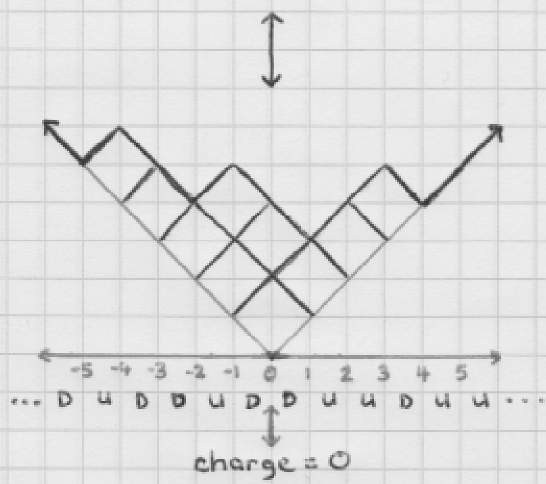
If $S = \{s_1 > s_2 > s_3 > \cdots\}$, then we write $v_S = \underline{s_1} \wedge \underline{s_2} \wedge \underline{s_3} \wedge \cdots$.

Inner product: Declare the basis $\{v_S\}$ to be orthonormal.

Diagrammatic representation: $s \in S \longleftrightarrow$ down step at s

$s \notin S \longleftrightarrow$ up step at s

$$v_S = \underline{\tfrac{7}{2}} \wedge \underline{\tfrac{1}{2}} \wedge \underline{-\tfrac{1}{2}} \wedge \underline{-\tfrac{5}{2}} \wedge \underline{-\tfrac{7}{2}} \wedge \underline{-\tfrac{11}{2}} \wedge \underline{-\tfrac{13}{2}} \wedge \cdots$$



So a basis vector v_S corresponds to a pair (λ, c) where λ is a Young diagram and $c \in \mathbb{Z}$ is the charge. Sometimes, we prefer to work in the charge 0 subspace $\Lambda_0^{\frac{\infty}{2}} V$.

OPERATORS

Fermionic operators: $\psi_k v = \underline{k} \wedge v$ for $k \in \mathbb{Z} + \frac{1}{2}$

ψ_k^* is the adjoint

In other words,

$$\psi_k v_S = \pm v_{S\cup\{k\}} \quad \text{if } k \notin S \qquad\qquad \psi_k^* v_S = \pm v_{S\setminus\{k\}} \quad \text{if } k \in S$$

$$= 0 \quad \text{if } k \in S \qquad\qquad = 0 \quad \text{if } k \notin S.$$

$$\text{sign} = (-1)^{\#\{s\in S \mid s>k\}} \qquad\qquad \text{sign} = (-1)^{\#\{s\in S \mid s>k\}}$$

Anti-commutation relations:

$$[\psi_i, \psi_j^*]_+ = \delta_{ij} \qquad [\psi_i, \psi_j]_+ = 0 \qquad [\psi_i^*, \psi_j^*]_+ = 0$$

Exercise: Prove these.

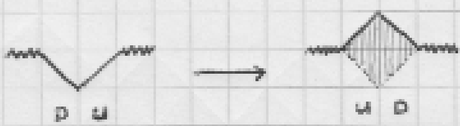
Bosonic operators: $\alpha_n = \sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi_{k-n}\psi_k^*$ for $n \in \mathbb{Z}\setminus\{0\}$

$$\alpha_n^* = (\Sigma\, \psi_{k-n}\psi_k^*)^* = \Sigma\, \psi_k \psi_{k-n}^* = \Sigma\, \psi_{k+n}\psi_k^* = \alpha_{-n}$$

Commutation relations: $[\alpha_m, \alpha_n] = m\,\delta_{m+n}$

Exercise: Prove this.

Diagrammatic representation: α_{-1} tries to change DU into UD, when possible



So α_{-1} "adds a box" in all possible ways.

$$\alpha_{-1} v_\lambda = \sum_{\mu=\lambda+\square} v_\mu$$

It follows that $(\alpha_{-1})^d v_\emptyset = \sum_{|\lambda|=d} (\dim\lambda)\, v_\lambda$.

Calculation: Consider the inner product $((\alpha_{-1})^d v_\emptyset, (\alpha_{-1})^d v_\emptyset)$.

① From the fact above,

$$((\alpha_{-1})^d v_\emptyset, (\alpha_{-1})^d v_\emptyset) = \Big(\sum_{|\lambda|=d} (\dim\lambda)\, v_\lambda, \sum_{|\lambda|=d} (\dim\lambda)\, v_\lambda\Big) = \sum_{|\lambda|=d} (\dim\lambda)^2$$

② $((\alpha_{-1})^d v_\emptyset, (\alpha_{-1})^d v_\emptyset) = ((\alpha_1)^d (\alpha_{-1})^d v_\emptyset, v_\emptyset)$

Note that α_1 "removes a box" in all possible ways.

$$\alpha_1 v_\lambda = \sum_{\mu = \lambda - \square} v_\mu$$

In particular, $\alpha_1 v_\emptyset = 0$. So use $[\alpha_1, \alpha_{-1}] = 1$ to commute α_1s to the right. The fact that $\alpha_1 \alpha_{-1} = \underbrace{\alpha_{-1}\alpha_1}_{\text{commute}} + \underbrace{1}_{\text{cancel}}$ means that we obtain a contribution to the inner product only when each α_1 "cancels" with an α_{-1}. There are $d!$ ways this can happen, so

$$((\alpha_{-1})^d v_\emptyset, (\alpha_{-1})^d v_\emptyset) = ((\alpha_1)^d (\alpha_{-1})^d v_\emptyset, v_\emptyset) = d!.$$

③ So we conclude that $\sum_{|\lambda| = d} (\dim \lambda)^2 = d!$.

Exercise: Expand $(\alpha_1)^d (\alpha_{-1})^d$ using $\alpha_1\alpha_{-1} = \alpha_{-1}\alpha_1 + 1$ for $d = 1, 2, 3$.

Expectations: To obtain $\langle f(\lambda)\rangle_{\text{Plancherel}} = \sum_{|\lambda|=d} \frac{(\dim\lambda)^2}{d!} f(\lambda)$, consider the operator F on $\Lambda_0^{\frac{\infty}{2}} V$ satisfying $F v_\lambda = f(\lambda) v_\lambda$.

$$\frac{1}{d!}((\alpha_1)^d F (\alpha_{-1})^d v_\emptyset, v_\emptyset) = \frac{1}{d!}(F(\alpha_{-1})^d v_\emptyset, (\alpha_{-1})^d v_\emptyset)$$

$$= \frac{1}{d!}\Big(F \sum_{|\lambda|=d} (\dim\lambda) v_\lambda, \sum_{|\lambda|=d} (\dim\lambda) v_\lambda\Big) = \frac{1}{d!}\Big(\sum_{|\lambda|=d} (\dim\lambda) f(\lambda) v_\lambda, \sum_{|\lambda|=d} (\dim\lambda) v_\lambda\Big)$$

$$= \sum_{|\lambda|=d} \frac{(\dim\lambda)^2}{d!} f(\lambda) = \langle f(\lambda)\rangle_{\text{Plancherel}}$$

Then you would try to calculate the inner product by commuting $(\alpha_1)^d$ past F and then past $(\alpha_{-1})^d$.

GROMOV – WITTEN THEORY

Burnside formula: $H_d^{\mathbb{P}^1}((k_1+1), (k_2+1), \ldots, (k_n+1)) = \sum_{|\lambda|=d} \left(\frac{\dim\lambda}{d!}\right)^2 \prod_{i=1}^n f_{k_i+1}(\lambda)$

GW/H correspondence: $\langle \tau_{k_1}(\omega)\tau_{k_2}(\omega)\cdots\tau_{k_n}(\omega)\rangle_d^{\bullet\mathbb{P}^1} = \sum_{|\lambda|=d} \left(\frac{\dim\lambda}{d!}\right)^2 \prod_{i=1}^n \frac{p_{k_i+1}(\lambda)}{(k_i+1)!}$

These count non-singular / possibly singular degree d covers of $\mathbb{P}^1$ with a (k_i+1)-fold branch point over $p_i \in \mathbb{P}^1$.

Here, f_k and p_k are shifted symmetric functions defined by

$$f_k(\lambda) = \binom{|\lambda|}{k} |C_k| \frac{\chi^\lambda_{(k,1,\ldots,1)}}{\dim\lambda} \quad \text{and} \quad p_k(\lambda) = k!\,[z^k] \underbrace{\sum_{i=0}^\infty e^{z(\lambda_i - i + \frac{1}{2})}}_{e(\lambda, z)}$$

1-point function: $F^{\circ}_{d}(z) = \sum_{k} \langle \tau_k(\omega) \rangle^{\circ \mathbb{P}^1}_{d} z^{k+1}$

$$= \sum_{k} \sum_{|\lambda|=d} \left(\frac{\dim \lambda}{d!}\right)^2 \frac{p_{k+1}(\lambda)}{(k+1)!} z^{k+1}$$

$$= \sum_{|\lambda|=d} \left(\frac{\dim \lambda}{d!}\right)^2 e(\lambda, z)$$

$$= \frac{1}{d!} \langle e(\lambda, z) \rangle_{\text{Plancherel}}$$

So let $\mathcal{E}_0(z) v_\lambda = e(\lambda, z) v_\lambda$ be an operator on $\Lambda^{\frac{\infty}{2}}_0 V$. Then

$$F^{\circ}_{d}(z) = \frac{1}{(d!)^2} \left((\alpha_1)^d \, \mathcal{E}_0(z) \, (\alpha_{-1})^d v_\emptyset, v_\emptyset \right).$$

Question: How do you commute $\mathcal{E}_0(z)$ with $\alpha_{\pm 1}$?

Exercise: Define $\mathcal{E}_m(z) = \sum_{k \in \mathbb{Z}+\frac{1}{2}} e^{z(k - \frac{m}{2})} \psi_{k-m} \psi_k^*$ for $m \in \mathbb{Z} \setminus \{0\}$. Then $[\alpha_n, \mathcal{E}_m(z)] = s(nz) \, \mathcal{E}_{m+n}(z)$ where $s(z) = e^{z/2} - e^{-z/2}$.

Case $d = 1$: $\alpha_1 \mathcal{E}_0(z) \alpha_{-1} = (\mathcal{E}_0(z) \alpha_1 + s(z) \, \mathcal{E}_1(z)) \alpha_{-1}$

$$= \underbrace{\mathcal{E}_0(z) \alpha_1 \alpha_{-1} + s(z) \alpha_{-1} \mathcal{E}_1(z)}_{\text{disconnected contribution}} + \underbrace{s(z)^2 \, \mathcal{E}_0(z)}_{\text{connected contribution}}$$

General case:

$$F^{\circ}_{d}(z) = \frac{1}{(d!)^2} \left((\alpha_1)^d \, \mathcal{E}_0(z) (\alpha_{-1})^d v_\emptyset, v_\emptyset \right)$$

$$= \frac{1}{(d!)^2} \left((s(z)^{2d} \, \mathcal{E}_0(z) + \text{disconnected contribution}) v_\emptyset, v_\emptyset \right)$$

$$\Rightarrow \quad F^{\circ}_{d}(z) = \frac{1}{(d!)^2} s(z)^{2d} \left(\mathcal{E}_0(z) v_\emptyset, v_\emptyset \right)$$

$$= \frac{1}{(d!)^2} s(z)^{2d} \, e(\emptyset, z) = \frac{1}{(d!)^2} \, s(z)^{2d-1}$$

OTHER CALCULATIONS

- n-point function for GW invariants of $\mathbb{P}^1$ and E
- MacMahon formula for plane partitions and other DT generating functions
- many quantities pertaining to Plancherel / Schur and uniform measures on partitions (determinantal processes, theta functions)
- generating functions for Hurwitz numbers (integrability)

- change of basis coefficients between $\{f_\mu\}$ and $\{p_\mu\}$ for the algebra of shifted symmetric functions

REFERENCES

- Okounkov - Infinite wedge and random partitions
- Okounkov and Pandharipande - Gromov-Witten theory, Hurwitz theory, and completed cycles