Mathellaneous

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Unsolved problems for young and old

1 The lure of an unsolved problem

In his superb article entitled "*The two cultures of mathematics*" [4], Fields medallist Tim Gowers writes on the ever more prevalent dichotomy between mathematicians who are primarily problem-solvers and those who are primarily theory-builders. In particular, he defends the currently less fashionable problem-solving areas, such as combinatorics, and proposes that they should be just as highly regarded as theoretical ones. Despite this growing trend towards theory-building, unsolved problems remain the driving force behind mathematical progress. For example, it seems highly doubtful that the state of the Taniyama-Shimura Conjecture would now be known had Fermat the margin space to pen his infamous, and most likely mythical, proof. For those unaware, the Taniyama-Shimura Conjecture was finally proven in 1999 through the collaboration of Breuil, Conrad, Diamond and Taylor, forty-four years after Taniyama first speculated the deep connection between rational elliptic curves and modular forms. Much of the groundwork for the proof was provided by Andrew Wiles after an incredible eight years of continuous toil spent searching for the Holy Grail of mathematics, a proof of Fermat's Last Theorem. But who amongst us has not felt the lure of an unsolved problem?

This article contains concise expositions on four beautiful, though elementary, unsolved problems which have lured me in the past. Of course, the word "elementary" refers only to the lack of technical prerequisites needed to understand and approach the problems. Indeed, the fact that they have all withstood attack from mathematicians the world over is testament to their difficulty. Old mathematicians will hopefully be reminded that learning a wealth of mathematical machinery is not necessary to attack interesting unsolved problems. Young mathematicians can use such problems as these to cut their teeth on with a minimum of knowledge before progressing on to higher mathematical pursuits. Since an article such as this could easily contain hundreds of fascinating elementary unsolved problems, I have adopted the following three criteria in problem selection, along with my own personal taste and discretion.

- *The problem can be understood by any person with a working knowledge of mathematics.*
  The reader will not be reading about the much publicized Clay Mathematics Institute Millennium Prize Problems here, all of which require mathematical mastery of their respective areas.
- *The problem is not very well-known.*
  Many of the well-known unsolved problems are considered intractable by the mathematical community, such as as Goldbach's Conjecture or the Twin Prime Conjecture. Anyway, what is the point of telling a joke that everyone already knows?
- *The problem is interesting, tantalizing and surprising.*
  My hope is that readers might also feel the lure of these unsolved problems. Perhaps they will be bandied around classrooms and offices, written on the backs of envelopes,

and passed on by word of mouth until some very clever person can lay claim to a solution. Remember that all maths problems were, at some stage, unsolved!

2 The Angel Problem

The Angel and the Devil are playing a game on an infinite grid of unit squares. The Devil, on his turn, may remove any unoccupied square from the playing field. The Angel is a chess piece on the board and, on her turn, can move to any remaining square up to 1000 king moves away. Since she has wings, the Angel can fly over any intervening squares that have been removed. In other words, she can move to any other available square within the 2001×2001 square centred at her current location. The Devil wins if he can trap the Angel while the Angel wins if she can continue flying forever. Can the Devil beat the Angel?

There are few people who would suspect that the Devil has a chance of winning such a biased game. For it seems that the Angel can fly far quicker than the Devil can remove squares from the board. So it should come as a surprise that this problem, initially proposed by John Conway over twenty years ago, remains unsolved to this day.

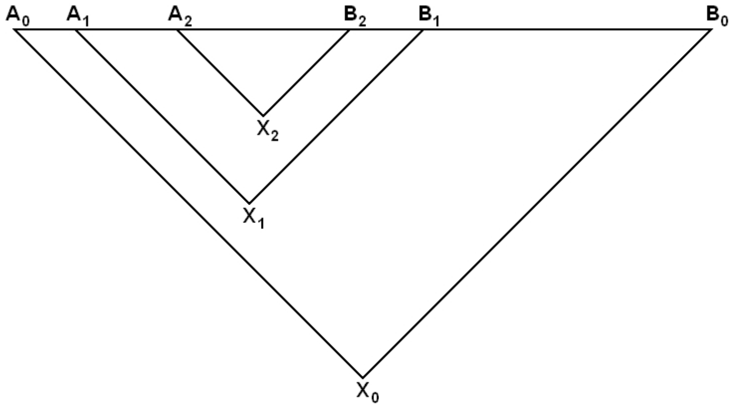
The Angel that we have described here is said to have power 1000, since she can move up to 1000 king moves in one graceful bound. Of course, the problem admits generalization to any positive integer power. It turns out that the Devil has the upper hand in a battle against an Angel of power 1, otherwise known as a king in chess terminology. Surprisingly, that is about the extent of our knowledge, since the winner of the game is unknown even when we consider an Angel with any power greater than or equal to 2. The main difficulty in the problem lies in the fact that the Devil can never make a wrong move — no matter how the Devil moves, he is always better off than he was before.

In an excellent survey of the problem [2], Conway shows that such battles between the Angel and the Devil may not be as one-sided as at first seems. For example, suppose that the Angel adopts the naive strategy of flying as quickly as possible in one direction, gracefully weaving around any squares that the Devil may have removed. The following result ensures that this is bound to end in doom for the Angel.

Theorem 1 *A* Fool *is an Angel who must increase her y-coordinate with every move. The Devil can catch a Fool.*

Proof. We will show that the Devil can start building a wall far to the North which, by the time the Fool arrives, will be an impenetrable blockade of width 1000. First observe that a Fool who begins at the point X_0 is subsequently contained within the upward cone emanating from X_0 with a gradient of $\pm\frac{1}{1000}$. Let us truncate this cone with the horizontal line A_0B_0 at some yet to be chosen height H far far above X_0. The Devil can now eat one out of every M squares along A_0B_0, where M is chosen so that the Devil has completed this task by the time the Fool has reached a distance of $\frac{H}{2}$ away from A_0B_0.

Now the Fool is at the point X_1 and is subsequently contained within the upward cone formed by X_1 and the segment A_1B_1, which is exactly half the length of A_0B_0. The Devil can now eat the second of every M squares along A_1B_1, and should be able to complete this task by the time the Fool has reached a distance of $\frac{H}{4}$ away from A_0B_0.



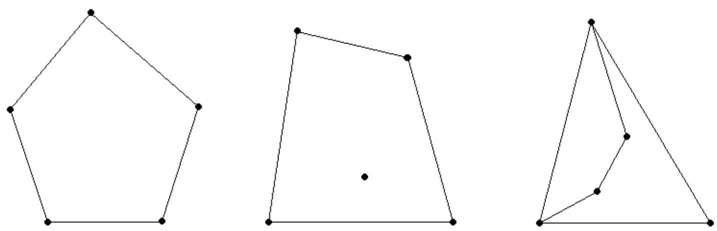
The Devil can merrily continue building his blockade, taking the third of every M squares, then the fourth of every M squares, and so on. So there will be a point in time when the Devil will have eaten all of the squares along A_0B_0 which the Fool has a chance of reaching. The Fool should now be a distance of $\frac{H}{2^M}$ away from A_0B_0. The Devil can repeat the very same trick along the row just below A_0B_0. In this manner, he should easily be able to remove all of the squares along or just below A_0B_0 which the Fool can subsequently reach by the time the Fool has reached a distance of $\frac{H}{2^{2M}}$ away from A_0B_0. Now if we take H to be 1000×2^N, where $N > 1000M$, then continuing this strategy ensures that the Devil can build his wall until it is 1000 squares in thickness before the Fool has reached it. $\square$

Conway believes that this problem has been alive far too long and offers \$100 for a proof that a sufficiently high-powered Angel can win, and \$1000 for a proof that the Devil can trap an Angel of any finite power.

Problem: An *Out-and-Out Fool* is an Angel who promises always to increase her distance from the origin. Prove that the Devil can catch an Out-and-Out Fool.

3 The Happy End Problem

It is a somewhat trivial fact that three non-collinear points in the plane will always be the vertices of a triangle. More interesting is the fact that among any five points in the plane, with no three collinear, there are always four which form the vertices of a convex quadrilateral. This is certainly not true for four points and is due to the fact that five such points must be in one of the following three configurations.



One is now inclined to ask how many points in the plane are required to guarantee a convex pentagon, or more generally, the following unsolved problem.

Let $f(n)$ denote the number of points in the plane, with no three collinear, that are required to guarantee a convex n-gon. What is $f(n)$?

This problem was first examined in the 1930's by a group of talented Hungarian mathematicians, including Paul Erdös, George Szekeres and Esther Klein, who may be familiar to many readers. It was Erdös who named it the Happy End Problem after the latter two became engaged and eventually married and the name has since been propagated throughout the literature. Their early work on the problem produced the results $f(3) = 3$, $f(4) = 5$ and $f(5) = 9$ which suggests that the correct expression for $f(n)$ may be $2^{n-2} + 1$. In fact, it did not take long before they found a construction to show that this conjecture was indeed a lower bound for $f(n)$. And not long after they also discovered the following upper bound.

Theorem 2 *Let $f(n)$ denote the smallest number of points in the plane, with no three collinear, that are required to guarantee a convex n-gon. Then $f(n) \leq \binom{2n-4}{n-2} + 1$.*

Proof. Given n points in the plane, we can find a Cartesian coordinate system in which it is possible to list the points in strictly increasing order of x-coordinate. For two points A and B in the plane, let $g(A, B)$ denote the gradient of the line which passes through them. Now define an m-cap to be a set of points $Q_1, Q_2, \ldots, Q_m$ listed in strictly increasing order of x-coordinate such that

$$g(Q_1, Q_2) \geq g(Q_2, Q_3) \geq \ldots \geq g(Q_{m-1}, Q_m).$$

Similarly, we can define an n-cup to be a subsequence $Q_1, Q_2, \ldots, Q_n$ listed in strictly increasing order of x-coordinate such that

$$g(Q_1, Q_2) \leq g(Q_2, Q_3) \leq \ldots \leq g(Q_{n-1}, Q_n).$$

Let $F(m, n)$ be the smallest number of points in the plane, with no three collinear, that are required to guarantee an m-cap or an n-cup. Then this function must satisfy the inequality $F(m, n) \leq F(m-1, n) + F(m, n-1) - 1$. Now using the simple fact that $F(3, n) = F(n, 3) = n$ and applying the inequality recursively yields $F(m, n) \leq \binom{m+n-4}{m-2} + 1$. Since $f(n) \leq F(n, n)$, it follows that $f(n) \leq \binom{2n-4}{n-2} + 1$. □

Unfortunately, this upper bound happens to be exponentially far away from the lower bound of $2^{n-2} + 1$, which is the conjectured value for $f(n)$. However, it did pave the way for further progress on the problem. It was not until 1998 that a second married couple, consisting of Ronald Graham and Fan Chung, managed to cleverly tweak the original proof to knock the upper bound down by one to $\binom{2n-4}{n-2}$. Such a slight improvement may not seem like much of a dent on the original problem, but did mark the first significant progress made in over sixty years. Not only that, their result was the start of a cascade of increasingly strong upper bounds on the problem. In the very same volume of the journal Discrete and Computational Geometry, Kleitman and Pachter managed to bring the upper bound down to $\binom{2n-4}{n-2} + 7 - 2n$ while Tóth and Valtr did better still with an upper bound of $\binom{2n-5}{n-3} + 2$

n	Lower Bound for $f(n)$	Upper Bound for $f(n)$
3	3	3
4	5	5
5	9	9
6	17	37
7	33	128
8	65	464
9	129	1718
$\vdots$	$\vdots$	$\vdots$
n	$2^{n-2}+1$	$\binom{2n-5}{n-3}+2$

The table above gives the current upper and lower bounds for $f(n)$ and it is amazing that we still do not know the value of $f(6)$. In fact, we cannot even close the upper and lower bounds to within a factor of 2. The interested reader may like to consult the excellent survey of this problem by Morris and Soltan [6].

Problem: Prove that $F(m,n) \leq F(m-1,n) + F(m,n-1) - 1$.

4 The Burnt Pancake Problem

The following problem was introduced by W. H. Gates and C. H. Papadimitriou [3] in the former author's only published paper before founding a certain well-known software corporation.

> A chef decides to make n pancakes but, due to incredible sloppiness, the pancakes all have different sizes and each is burnt on one side. A waiter wishes to sort them in increasing order of size from top to bottom as well as ensure that all pancakes have their burnt side down. With only one free hand, the waiter must achieve this by picking up some number of pancakes from the top and flipping them over. How many flips are required to sort any given stack of n burnt pancakes?

Let us consider the stack of pancakes as a permutation of the numbers $1, 2, \ldots, n$ with some elements negated, commonly known as a *signed permutation.* Here, the number $+k$ denotes the kth smallest pancake with burnt side down while $-k$ denotes the kth smallest pancake with burnt side up. In this notation, the flips of the waiter now take the following form.

$$(a_1, a_2, \ldots, a_k, a_{k+1}, \ldots, a_n) \rightarrow (-a_k, \ldots, -a_2, -a_1, a_{k+1}, \ldots, a_n)$$

Thus, a large pancake with burnt side up sitting on a small pancake with burnt side up sitting on a medium pancake with burnt side down is designated by the signed permutation $(-3, -1, 2)$. And a clever waiter can sort this configuration in three flips as follows.

$$(-3, -1, 2) \rightarrow (-2, 1, 3) \rightarrow (-1, 2, 3) \rightarrow (1, 2, 3)$$

The status of the Burnt Pancake Problem is found in [5] and can be summarized in the following theorem.

Theorem 3 *Let $B(n)$ denote the number of flips required to sort any signed permutation of the numbers $(1, 2, \ldots, n)$. Also, let $R(n)$ denote the number of flips required to sort the signed permutation $(-1, -2, \ldots, -n)$. Then $B(n) \geq \frac{3n}{2}$, $B(n) \leq 2n - 2$ for $n \geq 10$, and $R(n) \leq \frac{3n+3}{2}$ for $n \geq 23$.*

Some known values for $B(n)$ and $R(n)$ are shown in the table below. Of course, there seems to be striking evidence for the conjecture that $B(n) = R(n)$ for all n. In other words, it is conjectured that the worst case scenario occurs when the initial stack of pancakes is in increasing order of size from top to bottom, but all pancakes have their burnt side up. Proving this open conjecture would significantly help to sandwich the upper and lower bounds for $B(n)$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$B(n)$	1	4	6	8	10	12	14	15	17	18	??	??	??	??	??	??	??	??
$R(n)$	1	4	6	8	10	12	14	15	17	18	19	21	22	23	24	26	28	29

Problem: The Unburnt Pancake Problem, as the name suggests, is the analogous problem where the waiter is trying to sort a stack of n *unburnt* pancakes of varying sizes. Show how to sort a stack of n unburnt pancakes with at most $2n - 3$ flips. How much better can you do?

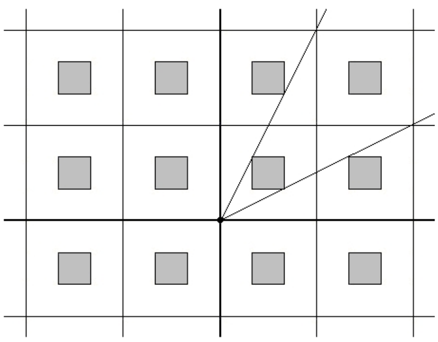
5 The Lonely Runner Conjecture

Suppose that n people are running at distinct constant speeds around a circular track of unit length. They all start at the same time and place and never stop. A runner is said to be *lonely* whenever every other runner is at least $\frac{1}{n}$ away. Is it true that every runner must get lonely at some time?

Despite appearances, this beautiful conjecture is number theoretic in nature, since it has been shown that the speeds can be assumed to be integral. Of course, it is sufficient to concentrate on one particular runner and show that he or she must get lonely at some time. Since we are only concerned with the relative speeds and distances between the runners, it is also safe to assume that this particular runner has zero speed. Thus, the problem can be restated more mathematically, though far less poetically, thus.

Let $v_1, v_2, \ldots, v_{n-1}$ be distinct integers. Then there exists a positive real number t such that $\frac{1}{n} \leq \{v_k t\} \leq \frac{n-1}{n}$ for all k. (Here, $\{x\}$ denotes the fractional part of x.)

Note that the Lonely Runner Conjecture can be stated as a visibility problem. For example, the $n = 3$ case is equivalent to the fact that a ray from the origin can avoid all points in the plane whose coordinates both have fractional parts in the interval $\left[\frac{1}{3}, \frac{2}{3}\right]$ only if the ray travels along one of the axes. A proof of this is obvious upon consideration of the following diagram.



The furthest progress to date on the conjecture is the following theorem which was first published in 2001 by the team of Bohman, Holzman and Kleitman [1].

Theorem 4 *The lonely runner conjecture holds for up to six runners.*

The proof of this statement was simplified by Renault [7] in 2004, using involved arguments in numerous cases based on congruence classes of the runners' velocities. It remains to be seen whether the proof can be simplified further, whether there exists a proof which applies more generally, or even whether the conjecture is true for all runners.

Problem: Prove the lonely runner conjecture for four runners.

References

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