Mathellaneous

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Constant curiosity

Not all numbers were created equal. Mathematically minded folk are all aware of the ubiquity of Archimedes' constant π, the importance of Euler's constant e and the beauty of the golden ratio ϕ. These, along with a handful of other numbers, are all a part of popular mathematical culture. They often turn up in the most unexpected places, causing endless wonder and aptly demonstrating the rich tapestry that is mathematics.

However, let us spare a thought for a few of the lesser known mathematical constants — ones which might not permeate the various fields of mathematics but have nevertheless been immortalised in the mathematical literature in one way or another. In this article, we will consider four of these numerical curios and their rise to fame.

A simple formula for primes

The primes are somewhat elusive beasts among the menagerie of natural numbers. Such simple to state problems as the Twin Prime Conjecture and the Goldbach Conjecture have yet to succumb to the might of mathematicians. And trying to find order in the disorder created by the primes has lead to one of the deepest and most difficult challenges in modern mathematics — the Riemann Hypothesis.

Nevertheless, there have been some valiant, and often curious, attempts to harness the primes. For example, the set of positive values taken on by the following bizarre polynomial of degree 25 in 26 variables is precisely the set of primes, where the variables $a, b, c, \ldots, z$ vary over the non-negative integers [4].

$$(k+2)(1-(wz+h+j-q)^2-((gk+2g+k+1)(h+j)+h-z)^2-(2n+p+q+z-e)^2-(16(k+1)^3(k+2)(n+1)^2+1-f^2)^2-(e^3(e+2)(a+1)^2+1-o^2)^2-((a^2-1)y^2+1-x^2)^2-(16r^2y^4(a^2-1)+1-u^2)^2-(((a+u^2(u^2-a))^2-1)(n+4dy)^2+1-(x+cu)^2)^2-(n+l+v-y)^2-((a^2-1)l^2+1-m^2)^2-(ai+k+1-l-i)^2-(p+l(a-n-1)+b(2an+2a-n^2-2n-2)-m)^2-(q+y(a-p-1)+s(2ap+2a-p^2-2p-2)-x)^2-(z+pl(a-p)+t(2ap-p^2-1)-pm)^2)$$

But could there be other prime-generating formulae out there, preferably not quite so unwieldy? If we relax the condition from generating all the primes to merely generating arbitrarily large primes, then we have the following candidate proposed by Mills [5].

> **Theorem:** There exists a positive constant M such that the expression
> $$\left\lfloor M^{3^n} \right\rfloor$$
> yields only primes for all positive integers n.

This prime-producing formula, a formula which contains only five symbols, is astonishing in its sheer simplicity. So why, the keen reader must be wondering, are people expending so much effort in calculating ever larger primes? Why are the prime number records so hard to break when we have such a simple formula at our fingertips? The reason is the fact that Mills gives no explicit value for such a number M in his original paper, nor even an algorithm to calculate one. However, there is much evidence to believe that the smallest

possible value is given by

$$M = 1.30637788386308069046861449926026057129167\ldots$$

and it is this number which has been perpetuated throughout the literature under the name of Mills' constant.

Let us now remove some of the mystery from Mills' mysterious discovery by considering how such a result might be proven. The crux of the proof is a result proven in the 1930's independently by Hoheisel and Ingham which bounds the size of the gap between consecutive primes. In fact, if we believe the Riemann hypothesis to be true, and it seems that many modern mathematicians do, then the Hoheisel-Ingham result can be strengthened to show that there exists a prime between any two consecutive perfect cubes. In particular, this allows us to define the following recursive sequence: $q_1 = 2$ and for each $n \geq 1$, let q_{n+1} be the least prime greater than q_n^3. Thus, the first few terms are $q_1 = 2$, $q_2 = 11$, $q_3 = 1361$ and $q_4 = 2521008887$. Some simple analysis shows that we have the increasing sequence

$$\sqrt[3]{q_1} < \sqrt[9]{q_2} < \sqrt[27]{q_3} < \cdots < \sqrt[3^n]{q_n} < \cdots$$

as well as the decreasing sequence

$$\sqrt[3]{q_1 + 1} > \sqrt[9]{q_2 + 1} > \sqrt[27]{q_3 + 1} > \cdots > \sqrt[3^n]{q_n + 1} > \cdots.$$

It follows that

$$M = \lim_{n \to \infty} \sqrt[3^n]{q_n}$$

exists and satisfies $\lfloor M^{3^n} \rfloor = q_n$ for all positive integers n. However, by definition, the sequence $q_1, q_2, q_3, \ldots$ consists entirely of primes and Mills' result follows.

Using the construction above, we obtain Mills' constant. Of course, Mills removes from this proof the necessity for the Riemann hypothesis, but at the expense of being able to determine a particular value of M. It is unknown whether Mills' constant is rational or not — it seems incredibly doubtful that it would be, or even that it has some other analytical description without reference to the original sequence of primes.

A surprising result about continued fractions

The representation of a real number as a decimal is aesthetically displeasing to the purest of mathematicians. The somewhat arbitrary use of 10 digits is more due to a biological coincidence than any deep-seated mathematical reason. A purer "hands-free" approach is to represent a real number by its continued fraction

$$a = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

where $a_0 = \lfloor a \rfloor$ and $a_1, a_2, a_3, \ldots$ is a sequence of positive integers which may be infinite, finite, or in the case that a is an integer, empty. And to avoid such a cumbersome and unsightly notation, it is common to abbreviate the above expression into the more compact form

$$a = [a_0; a_1, a_2, a_3, \ldots].$$

Note that the sequence terminates precisely when a is rational. Hence, we have a natural bijection between the irrational numbers and sequences of integers $a_0, a_1, a_2, \ldots$ where $a_i > 0$ for all $i > 0$. Many fascinating results have been accumulated on continued fractions — for

example, it is well-known that a continued fraction is periodic if and only if it is a quadratic irrational.

Now let us consider the question: what can be said about the behaviour of the sequence $a_1, a_2, a_3, \ldots$ for a randomly chosen real number a? Well, it seems almost nothing, since any conceivable sequence of positive integers corresponds to a particular real number a. And yet, Khintchine proved the following result, which must surely come as a surprise to any unsuspecting mathematician.

Theorem: For almost all real numbers $a = [a_0; a_1, a_2, a_3, \ldots]$,
$$\lim_{n \to \infty} \sqrt[n]{a_1 a_2 \ldots a_n} = K$$
where K is the constant
$$K = \prod_{k=1}^{\infty} \left(1 + \frac{1}{k(k+2)} \right)^{\frac{\ln(k)}{\ln(2)}} = 2.6854520010\ldots.$$

By the phrase "almost all", we mean for all real numbers x outside of a set with Lebesgue measure zero. This seems at odds with the statement made earlier concerning the bijection between the irrational numbers and sequences. For many positive integer sequences, the limit

$$\lim_{n \to \infty} \sqrt[n]{a_1 a_2 \ldots a_n}$$

might not even exist, and even if it does, the probability that it is equal to Khintchine's constant is zero.

There are obvious exceptions to the theorem, all of which must belong to a subset of the real numbers with Lebesgue measure zero — for example, the rationals, the quadratic irrationals and the number e. It remains to be seen whether or not the numbers $\sqrt[3]{2}$, π and K itself[1] obey Khintchine's law, although it seems likely that they do. Amazingly enough, without resorting to constructing such numbers from their continued fraction representations, no one has managed to find a single number which does obey Khintchine's law!

The Look and Say Sequence

Consider the following sequence

$$1, 11, 21, 1211, 111221, 312211, 13112221, 1113213211, 31131211131221, \ldots$$

Each term of the sequence, except the first, describes the digits appearing in the previous term. For example, to generate the term after 312211, we scan along its digits and note that it is comprised of

"one 3", "one 1", "two 2's", and "two 1's".

And, lo and behold, the next term in the sequence is 13112221. For obvious reasons, it has been coined the Look and Say Sequence. Despite its frivolous definition , John Conway [1] has managed to prove the following amazing result about the sequence.

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[1] It is not even known whether or not K is rational, so its continued fraction may be finite, after all.

Theorem: If C_n denotes the number of digits in the nth term of the Look and Say Sequence, then

$$C = \lim_{n \to \infty} \sqrt[n]{C_n}$$

exists. Amazingly enough, this constant is the unique positive real root of the following irreducible polynomial of degree 71.

$x^{71} - x^{69} - 2x^{68} - x^{67} + 2x^{66} + 2x^{65} + x^{64} - x^{63} - x^{62} - x^{61} - x^{60} - x^{59} + 2x^{58} + 5x^{57} + 3x^{56} - 2x^{55} - 10x^{54} - 3x^{53} - 2x^{52} + 6x^{51} + 6x^{50} + x^{49} + 9x^{48} - 3x^{47} - 7x^{46} - 8x^{45} - 8x^{44} + 10x^{43} + 6x^{42} + 8x^{41} - 5x^{40} - 12x^{39} + 7x^{38} - 7x^{37} + 7x^{36} + x^{35} - 3x^{34} + 10x^{33} + x^{32} - 6x^{31} - 2x^{30} - 10x^{29} - 3x^{28} + 2x^{27} + 9x^{26} - 3x^{25} + 14x^{24} - 8x^{23} - 7x^{21} + 9x^{20} + 3x^{19} - 4x^{18} - 10x^{17} - 7x^{16} + 12x^{15} + 7x^{14} + 2x^{13} - 12x^{12} - 4x^{11} - 2x^{10} + 5x^9 + x^7 - 7x^6 + 7x^5 - 4x^4 + 12x^3 - 6x^2 + 3x - 6.$

Conway's constant cannot be expressed in radicals, but has the following approximate value.

$$C = 1.303577269034296391257099112152551 8907307\ldots$$

This must certainly be one of the most bizarre of the algebraic numbers to appear in the mathematical literature!

One of the main ideas behind Conway's analysis of the sequence is the observation that often, a string of digits can be broken down into substrings which evolve via the Look and Say rule without interfering with each other. For example, the evolution of the string 322 can be considered as the concatenation of the evolutions of the strings 3 and 22, as shown below.

$$3.22 \to 13.22 \to 1113.22 \to 3113.22 \to \ldots$$

In particular, from the eighth term onwards, every term of the Look and Say Sequence is comprised of a combination of 92 substrings which never interfere with each other. Conway names these substrings the atomic elements, giving each an atomic number and its corresponding name from the periodic table.

Atomic Number	Element	String
1	Hydrogen	22
2	Helium	13112221133211322112211213322112
3	Lithium	312211322212221121123222122
4	Beryllium	111312211312113221133211322112211213322112
5	Boron	1321132122211322212221121123222112
$\vdots$	$\vdots$	$\vdots$
92	Uranium	3

Thus, elements give birth to more elements, and the behaviour of the system is governed by what is known as the transfer matrix M. This is the 92×92 matrix where the (i, j) entry is simply the number of atoms of the element with atomic number j formed from the element with atomic number i after one application of the Look and Say rule. Conway's constant is simply the eigenvalue of greatest magnitude of the matrix M.

Random sequences á la Fibonacci

Of course, we have all heard of the notorious Fibonacci sequence and its association with the combinatorics of rabbit breeding and phyllotaxis[2]. Perhaps the most useful result when

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[2]Phyllotaxis refers to the principles governing the arrangement of leaves on the stem of a plant.

studying the Fibonacci sequence is the following closed formula, often named after Binet: the nth term of the Fibonacci sequence is given by

$$F_n = \frac{1}{\sqrt{5}}(\phi_+^n - \phi_-^n)$$

where $\phi_+ = \frac{1+\sqrt{5}}{2}$ is the golden ratio and $\phi_- = \frac{1-\sqrt{5}}{2}$ is its algebraic conjugate.

One particular corollary is the fact that the Fibonacci sequence has exponential behaviour for large n, and it is easy enough to deduce that

$$\lim_{n \to \infty} \sqrt[n]{F_n} = \phi_+.$$

But what happens, one might ask, if we spice up the Fibonacci sequence with an element of randomness? Divakar Viswanath studied random sequences defined by the formula

$$V_1 = 1, V_2 = 1, V_{n+1} = V_n \pm V_{n-1} \text{ for } n > 1$$

where for each n, the sign is chosen independently and with equal probability — for example, by the flip of an unbiased coin.

Of course, we cannot hope to reproduce an analogue of Binet's formula for such random sequences; however, we can still ask whether the set of all such random sequences share the same asymptotic behaviour. At first glance, it would appear otherwise, judging by the following three examples of random Fibonacci sequences. The first has been generated by always taking the negative sign, and shows periodic behaviour. The second has been generated by always taking the positive sign, and is, of course, the beloved Fibonacci sequence. The third was randomly generated by a computer and shows that both the sign and the magnitude of the terms behave erratically.

$$1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, 1, 1, \ldots$$
$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, \ldots$$
$$1, 1, 2, 1, 3, 4, 7, 3, -4, -7, -11, -4, 7, 3, -4, -1, -5, -6, -1, -7, \ldots$$

Nevertheless, Viswanath [6] managed to prove the following.

> **Theorem:** If $V_1, V_2, V_3, \ldots$ is a random Fibonacci sequence, then almost surely (that is, with probability 1)
> $$\lim_{n \to \infty} \sqrt[n]{|V_n|} = 1.3198824\ldots.$$

Despite the randomness involved, this result ensures that there is a certain uniformity to random Fibonacci sequences. Intuition might suggest that Viswanath's constant should be closely related to the golden ratio — however, no such relationship has been found. In fact, it seems that all we can say about Viswanath's constant is that it exists and we can calculate it.

More recently, Mark Embree and Lloyd Trefethen [2] have shown that Viswanath's random Fibonacci sequences fit into a larger picture involving sequences of the form

$$X_1 = 1, X_2 = 1, X_{n+1} = X_n \pm \beta X_{n-1} \text{ for } n > 1.$$

The two have collaborated to show that there exists a critical value $\beta^* \approx 0.70258\ldots$ satisfying the following.

- For $0 < \beta < \beta^*$, then almost surely the sequence will decay exponentially.
- For $\beta > \beta^*$, then almost surely the sequence will grow exponentially.

In fact, for each positive value of β, there exists a positive constant $\sigma(\beta)$ such that for almost all sequences defined by the above formula

$$\lim_{n \to \infty} \sqrt[n]{|X_n|} = \sigma(\beta).$$

This is a generalization of the work of Viswanath, whose constant corresponds to $\sigma(1)$. Perhaps the most amazing part of the work of Embree and Trefethen is their calculation of the σ function. Their computational evidence suggests that the dependence of $\sigma(\beta)$ on β is not only far from smooth, but is in fact fractal in nature!

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