Take a permutation and count the number of ways to express it as a product of a fixed number of transpositions — you have calculated a Hurwitz number. By adding a mild constraint on such factorisations, one obtains the notion of a monotone Hurwitz number. We have recently shown that the monotone Hurwitz problem fits into the so-called topological recursion/quantum curve paradigm. This talk will attempt to explain what the previous sentence means.
Simple Hurwitz numbers

Hurwitz numbers count the number of ways to express a permutation as a product of transpositions.

Definition

Let $H_{g,n}(\mu_1, \mu_2, \ldots, \mu_n)$ be $\frac{1}{|\mu|!}$ multiplied by the number of tuples $(\sigma_1, \sigma_2, \ldots, \sigma_m)$ of transpositions in $S_{|\mu|}$ such that

- $m = 2g - 2 + n + |\mu|$;
- $\sigma_1 \sigma_2 \cdots \sigma_m$ has labelled cycles of lengths $\mu_1, \mu_2, \ldots, \mu_n$; and
- $\langle \sigma_1, \sigma_2, \ldots, \sigma_m \rangle$ is transitive.

Fact

Hurwitz numbers equivalently count

- branched covers of $\mathbb{C}P^1$ with respect to ramification over $\infty$;
- edge-labelled embedded graphs with respect to winding number.
Monotone Hurwitz numbers

For monotone Hurwitz numbers, we add a mild constraint.

Definition
Let $\bar{H}_{g,n}(\mu_1, \mu_2, \ldots, \mu_n)$ be $\frac{1}{|\mu|!}$ multiplied by the number of tuples $(\sigma_1, \sigma_2, \ldots, \sigma_m)$ of transpositions in $S_{|\mu|}$ such that

- $m = 2g - 2 + n + |\mu|$;
- $\sigma_1 \sigma_2 \cdots \sigma_m$ has labelled cycles of lengths $\mu_1, \mu_2, \ldots, \mu_n$;
- $\langle \sigma_1, \sigma_2, \ldots, \sigma_m \rangle$ is transitive; and
- if $\sigma_i = (a_i b_i)$ with $a_i < b_i$, then $b_1 \leq b_2 \leq \cdots \leq b_m$.

Why monotone?
Monotone Hurwitz numbers are natural from the viewpoint of

- matrix models (HCIZ integral);
- representation theory (Jucys–Murphy elements); and
- integrability (Toda tau-functions).
Example calculation

Take \((g, n) = (0, 2)\) and \(\mu = (1, 2)\), so \(m = 2g - 2 + n + |\mu| = 3\).

There are 27 products of 3 transpositions in \(S_3\) and 24 are transitive.

\[
\begin{align*}
(1 2) \circ (1 2) & \circ (1 3) & (1 2) \circ (1 3) & \circ (2 3) & (1 3) \circ (1 3) & \circ (2 3) & (2 3) \circ (1 3) & \circ (1 3) \\
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\end{align*}
\]

All 24 products produce cycle type \((1, 2)\), so \(H_{0, 2}(1, 2) = \frac{24}{3!} = 4\).

Only the first 12 products are monotone, so \(\tilde{H}_{0, 2}(1, 2) = \frac{12}{3!} = 2\).
Polynomiality


There are polynomials $P_{g,n}$ and $\vec{P}_{g,n}$ such that

- $H_{g,n}(\mu_1, \ldots, \mu_n) = m! \times \prod \frac{\mu_i}{\mu_i!} \times P_{g,n}(\mu_1, \ldots, \mu_n)$
- $\vec{H}_{g,n}(\mu_1, \ldots, \mu_n) = \prod \binom{2\mu_i}{\mu_i} \times \vec{P}_{g,n}(\mu_1, \ldots, \mu_n)$.

For example,

- $P_{1,2}(\mu_1, \mu_2) = \frac{1}{24}(\mu_1^2 + \mu_2^2 + \mu_1\mu_2 - \mu_1 - \mu_2)$
- $\vec{P}_{1,2}(\mu_1, \mu_2) = \frac{1}{12}(2\mu_1^2 + 2\mu_2^2 + 2\mu_1\mu_2 - \mu_1 - \mu_2 - 1)$.

In fact, $[\mu_1^{a_1} \cdots \mu_n^{a_n}] P_{g,n}(\mu_1, \ldots, \mu_n) = \pm \int_{\overline{M}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} \lambda_{3g-3+n-|a|}.$

**Question**

Do the coefficients of $\vec{P}_{g,n}$ have geometric meaning?
Cut-and-join recursion

(Monotone) Hurwitz numbers of type \((g, n)\) can be calculated from those of types

- \((g, n - 1)\)
- \((g - 1, n + 1)\)
- \((g_1, n_1) \times (g_2, n_2)\) for \(\left\{ \begin{array}{l} g_1 + g_2 = g \\ n_1 + n_2 = n + 1. \end{array} \right. \)

For example,

\[
\mu_1 \tilde{H}_{1,2}(\mu_1, \mu_2) = (\mu_1 + \mu_2) \tilde{H}_{1,1}(\mu_1 + \mu_2) + \sum_{\alpha + \beta = \mu_1} \alpha \beta \tilde{H}_{0,3}(\alpha, \beta, \mu_2) \\
+ \sum_{\alpha + \beta = \mu_1} \alpha \beta \left[ \tilde{H}_{0,1}(\alpha) \tilde{H}_{1,2}(\beta, \mu_2) + \tilde{H}_{1,1}(\alpha) \tilde{H}_{0,2}(\beta, \mu_2) \right].
\]
Topological recursion and quantum curves

- **Topological recursion (Chekhov–Eynard–Orantin):**
  \[
  \omega_{g,n}(z_s) = \sum_{\alpha} \text{Res}_{z=\alpha} K(z_1, z) \left[ \omega_{g-1,n+1}(z, \bar{z}, z_s \setminus \{1\}) + \sum_{g_1 + g_2 = g \atop |I| = S \setminus \{1\}} \omega_{g_1,|I|+1}(z, z_I) \omega_{g_2,|J|+1}(\bar{z}, z_J) \right]
  \]

- **Wave function:**
  \[
  Z(x, \hbar) = \exp \left[ \sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \frac{\hbar^{2g-2+n}}{n!} F_{g,n}(x, \ldots, x) \right]
  \]

- **Polarisation:** \( \hat{x} = x \) and \( \hat{y} = -\hbar \frac{\partial}{\partial x} \), which imply \([\hat{x}, \hat{y}] = \hbar\)
Results

This is joint work with A. Dyer and D. Mathews (arXiv:1408.3992).

- The spectral curve \( A(x, y) = xy^2 + y + 1 = 0 \) yields
  \[
  F_{g,n}(x_1, \ldots, x_n) = \sum_{\mu} \hat{H}_{g,n}(\mu_1, \ldots, \mu_n) x_1^{\mu_1} \cdots x_n^{\mu_n}.
  \]

- The wave function satisfies
  \[
  Z(x, \hbar) = 1 + \sum_{d=1}^{\infty} \sum_{m=0}^{d} \binom{d + m - 1}{d - 1} \frac{x^d \hbar^{m-d}}{d!}.
  \]

- The corresponding quantum curve is \( \hat{A}(\hat{x}, \hat{y}) = \hat{x}\hat{y}^2 + \hat{y} + 1 \), so
  \[
  x\hbar^2 \frac{\partial^2 Z}{\partial x^2} - \hbar \frac{\partial Z}{\partial x} + Z = 0.
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