How to Win at Tic-Tac-Toe

Norm Do

Undoubtably, one of the most popular pencil and paper games in the world is tic-tac-toe, also commonly known as noughts and crosses. In this talk, you will learn how to beat your friends (at tic-tac-toe), discover why snaky is so shaky, and see the amazing tic-tac-toe playing chicken!

March 2007

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- Tic-Tac-Toe is boring: People who are mildly clever should never lose.

How not to lose at tic-tac-toe

Perform as many of the following actions as possible on your turn — listed in order of priority — without sacrificing higher priorities.

1. Complete three in a row.
2. Block your opponent from completing three in a row.
3. Threaten a win with two possible completions in two rows.
4. Avoid a configuration in which your opponent can force the win.
5. Threaten a win with a possible completion (two in a row).
6. If you are the second player and the center space is not already taken, then take it.
7. Prevent your opponent from getting two in a row.


SHOT TAC TOE
Three In A Row & Down They Go!
Includes Board And 9 Game Piece Shotglasses

Fundamental Theorem of Game Theory

In a game which has two players, no luck, and perfect information, exactly one of the following things is true:

- the first player can force a win;
- the second player can force a win; or
- both players can force a draw.

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Examples

The Fundamental Theorem of Game Theory applies to the following games

- tic-tac-toe;
- Connect Four;
- chess; and
- all of the games that we will consider today.

Count Foxy Words

- Two players take turns to select one of the following words.

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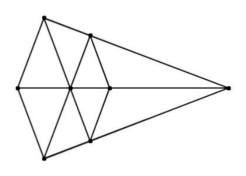
- If all of the words have been selected without one of the players winning, then the game is declared a draw.

Jam

- The diagram below shows a map of towns and roads, represented by points and line segments, respectively.



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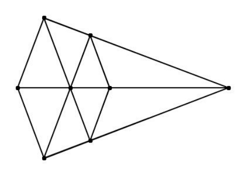

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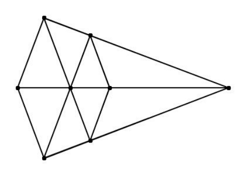

- Two players take turns to select a road and a road may not be chosen if it has already been used.
- The first player to take all of the roads passing through a town wins. If all roads have been selected without one of the players winning, then the game is declared a draw.

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- Two players take turns to select an integer from 1 to 9.
- An integer may not be chosen if it has already been used and a player wins once they have chosen three distinct numbers which add to 15.
- If all of the nine numbers have been selected without one of the players winning, then the game is declared a draw.

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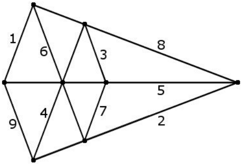

8	1	6
3	5	7
4	9	2

COUNT	FOXY	WORDS
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N-in-a-row

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Intuition

How can the second player possibly have the advantage after beginning the game one move behind?

Theorem

The second player does not have a winning strategy for N-in-a-row.

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Proof.

Let us suppose that the second player has a winning strategy. But now the first player can win by making his or her first move at random and thereafter adopting the second player's winning strategy. If this calls for the first player to play in an already occupied square, he or she just makes another random move. Since having an extra square on the board cannot possibly hurt the first player, this gives the contradiction that both players can force a win. So we must conclude that the second player cannot have a winning strategy, as desired. □

Theorem

The second player can force a draw in 9-in-a-row.

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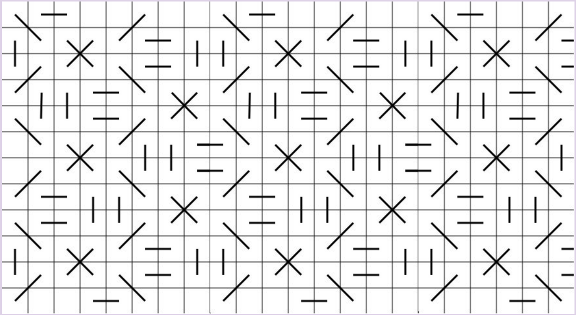
Proof.

Pairing strategy



□

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Proof.

Proven in 1993 using new computer algorithms, sheer brute force, and hundreds of hours of CPU time! □

Theorem

The second player can force a draw in 8-in-a-row.

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Proof.

Well tricky! □

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Well tricky! □

Unsolved problem

Can the first player force a win for 6-in-a-row or 7-in-a-row?

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Strategy stealing □

Change the rules!

To even up the game, let us say that the first player wins if they can create the animal, and the second player wins if they can prevent the first player from doing so.

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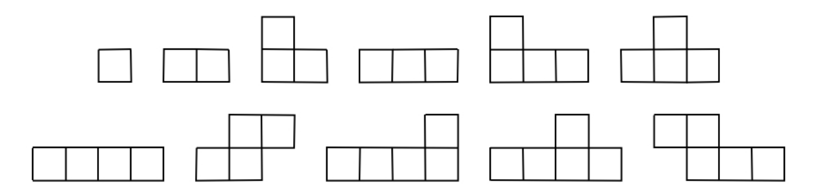
Definition

Animals are *winners* or *losers* depending on whether the first player wins or loses, respectively.

Theorem

The first player can win in animal tic-tac-toe if the animal used is one of the following twelve.



Small animals

- The only animal of size 1 is a winner.

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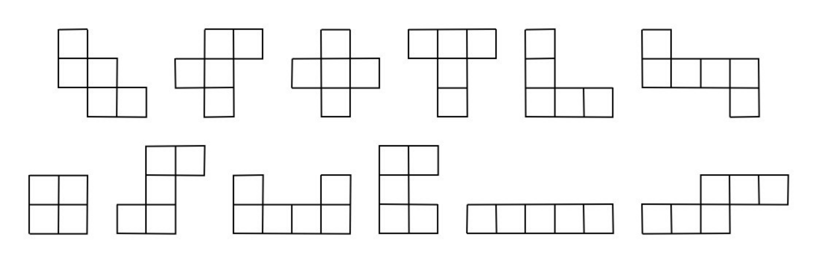
- . . . except for "fatty", the 2×2 square.
- Let us call fatty a "basic loser", since it is a loser which does not contain a smaller one.

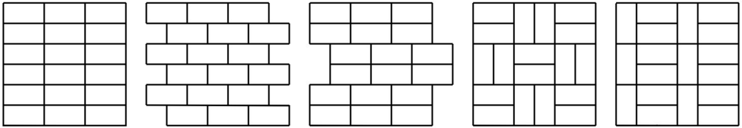
Theorem

The twelve animals in the figure below are all basic losers.



Proof.

Pairing strategies



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Large animals

- Of the twelve animals of size five:
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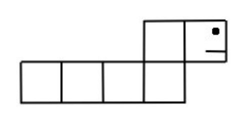
  - every single one contains a basic loser and hence, are losers themselves.
- It follows that every animal of size greater than seven is also a loser since they all contain an animal of size seven.

Snaky

- And what about the one animal of size six which has been left unaccounted for? Let us now meet this exotic animal which, in the literature, goes by the name of Snaky.



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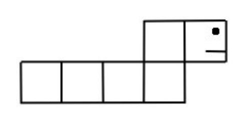

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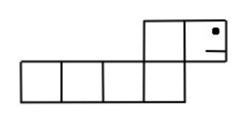

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Unsolved problem

Is Snaky a winner or a loser?

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- If hypercube tic-tac-toe is played on a k-dimensional hypercube of side length n, then we call the game n^k tic-tac-toe. The original game of tic-tac-toe is simply 3^2 tic-tac-toe.

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- Strategy stealing arguments can be used to prove that the second player cannot force a win.

Theorem

Hypercube tic-tac-toe on the 3^3 board is a win for the first player. In fact, it is impossible for a game of 3^3 tic-tac-toe to result in a draw.

Theorem

Hypercube tic-tac-toe on the 5^2 *board is a theoretical draw.*

Theorem

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Proof.

Use the pairing strategy indicated by the diagram below.

V	I	A	A	F
J	B	H	U	B
C	I		G	C
D	U	H	D	F
J	E	E	G	V

□

Theorem

Hypercube tic-tac-toe on the 4^3 board is a win for the first player.

Theorem

Hypercube tic-tac-toe on the 4^3 board is a win for the first player.

Proof.

Proven in 1980 using sheer brute force, symmetry considerations, some clever programming and 1500 hours of CPU time! $\square$

The following table gives results and conjectures for hypercube tic-tac-toe for small values of n and k. An entry labelled "W" denotes a win for the first player while "D" denotes a theoretical draw and an entry in red indicates that the result is merely conjectured but not actually proven.

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$n=1$	W	W	W	W	W	W
$n=2$	D	W	W	W	W	W
$n=3$	D	D	W	W	W	W
$n=4$	D	D	W	W	W	W
$n=5$	D	D	D	W	W	W
$n=6$	D	D	D	W	W	W
$n=7$	D	D	D	W	W	W
$n=8$	D	D	D	D	W	W
$n=9$	D	D	D	D	W	W
$n=10$	D	D	D	D	D	W

Unsolved problem

- If the n^k game is a draw, then the n^{k-1} game is a draw.
- If the n^k game is a draw, then the $(n+1)^k$ game is a draw.

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- We have seen that pairing strategies can be used to prove that the second player can force a draw.
- When do pairing strategies exist?
- Pairing strategies can only exist if the number of cells on the board is at least twice the number of winning paths.
- So how many winning paths are there in n^k tic-tac-toe?

Theorem

The number of winning paths on the n^k hypercube is

$$\frac{(n+2)^k - n^k}{2}.$$

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Proof.

- Embed your n^k hypercube inside an $(n+2)^k$ hypercube.
- Every winning path can be extended to give two cells in the outer shell.
- Every cell in the outer shell corresponds to a unique winning path.
- So the number of winning paths is half the number of cells in the outer shell — that is, $\frac{(n+2)^k - n^k}{2}$.

$\square$

Theorem

If a pairing strategy exists for the second player in n^k tic-tac-toe, then

$$n \geq \frac{2}{\sqrt[k]{2} - 1}.$$

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Proof.

For a pairing strategy to exist, the number of cells on the board must be at least twice the number of winning paths.Therefore

$$\begin{aligned} \#\text{ cells on the board} &\geq 2 \times \#\text{ winning paths} \\ n^k &\geq (n+2)^k - n^k. \end{aligned}$$

This rearranges to give the desired inequality. □

k	$\frac{2}{\sqrt[k]{2}-1}$	$\frac{2k}{\log_e 2}$	difference
1	2.000	2.885	0.885
2	4.828	5.771	0.942
3	7.695	8.656	0.962
4	10.570	11.542	0.971
5	13.450	14.427	0.977
6	16.332	17.312	0.981
7	19.214	20.198	0.983
8	22.098	23.083	0.986
100	287.540	288.539	0.999
1000	2884.390	2885.390	1.000

Conjecture

For every positive integer k, the following equation holds.

$$\left\lceil \frac{2}{\sqrt[k]{2}-1} \right\rceil = \left\lfloor \frac{2k}{\log_e 2} \right\rfloor$$

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Disproof

Consider $k = 6,847,196,937$ or $k = 27,637,329,632$ or $\ldots$?!

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  `http://www.austms.org.au/Publ/Gazette/`

- E-mail me at

  `norm@ms.unimelb.edu.au`

- See me at

  `the front of the Russell Love Theatre`