THE MATHEMATICAL ART OF TILING

LunchMaths seminar

17 September 2014

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Monash University

Tiles have been used in art and architecture since the dawn of civilisation. Toddlers grapple with tiling problems when they pack away their wooden blocks and home renovators encounter similar conundrums in the bathroom. However, rather than being a frivolous pastime, mathematicians have found the art of tiling to be brimming with amazing results. In this seminar, we will discover the colourful world of tiles, learn about faulty tilings, unlock the secrets of the Aztec diamond, and discuss a sequence of numbers which (I bet) grows faster than any you have ever imagined!

Dominoes on a checkerboard

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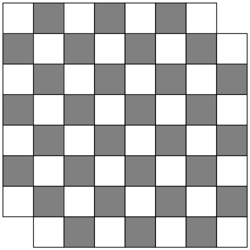

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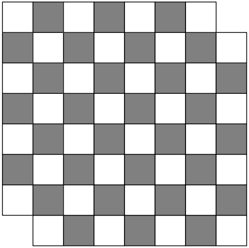
- Can you tile an 8×8 checkerboard with dominoes?
- What if one corner corner is removed?
- What if opposite corner squares are removed?
- Can you always tile the checkerboard if two squares of opposite colours are removed?



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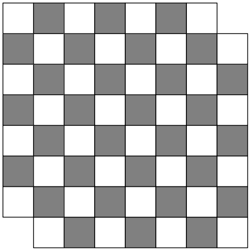

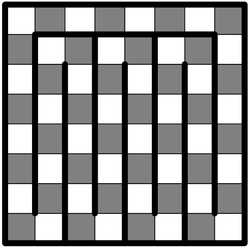

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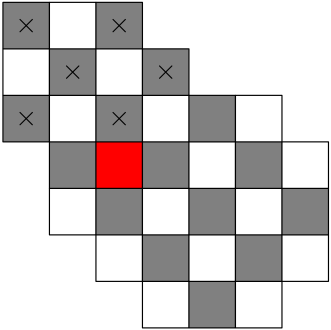
When can you tile a mutilated checkerboard with dominoes?



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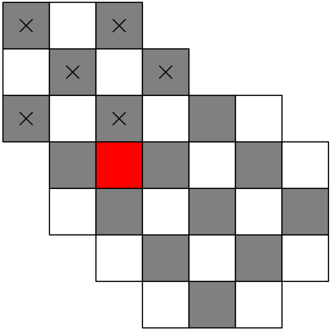

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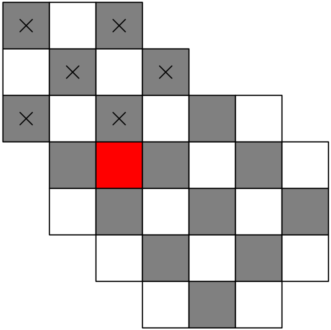

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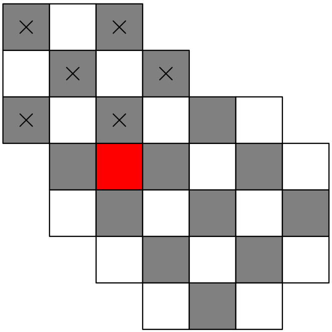

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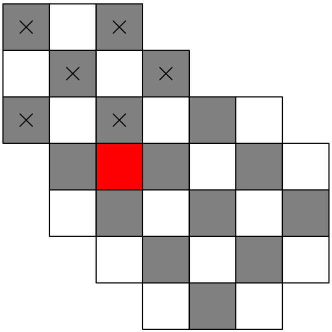

- Black squares are men and white squares are women.
- We want to marry everyone off to their neighbour.
- We need to have gender balance. . . but we also need every group of men to have enough women to marry.

Answer

If every group of men has enough women to marry and vice versa, then you can tile the mutilated checkerboard with dominoes.

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How many ways can you tile an 8×8 checkerboard with dominoes?

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Theorem (Fisher–Temperley and Kasteleyn, 1961)

The number of tilings of a $2m \times 2n$ checkerboard with dominoes is

$$\prod_{j=1}^{m} \prod_{k=1}^{n} \left(4\cos^2 \frac{j\pi}{2m+1} + 4\cos^2 \frac{k\pi}{2n+1} \right).$$

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Answer

$$\begin{matrix}
\left(4\cos^2 \frac{\pi}{9} + 4\cos^2 \frac{\pi}{9}\right) & \left(4\cos^2 \frac{\pi}{9} + 4\cos^2 \frac{2\pi}{9}\right) & \left(4\cos^2 \frac{\pi}{9} + 4\cos^2 \frac{3\pi}{9}\right) & \left(4\cos^2 \frac{\pi}{9} + 4\cos^2 \frac{4\pi}{9}\right) \\
\left(4\cos^2 \frac{2\pi}{9} + 4\cos^2 \frac{\pi}{9}\right) & \left(4\cos^2 \frac{2\pi}{9} + 4\cos^2 \frac{2\pi}{9}\right) & \left(4\cos^2 \frac{2\pi}{9} + 4\cos^2 \frac{3\pi}{9}\right) & \left(4\cos^2 \frac{2\pi}{9} + 4\cos^2 \frac{4\pi}{9}\right) \\
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\end{matrix}$$

$$= 7.064\ldots \times 5.879\ldots \times 4.532\ldots \times \cdots = 12,988,816 = 3604^2$$

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When can you tile an $m \times n$ checkerboard with $a \times b$ rectangles?

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Some observations

- Can you tile a 12×15 checkerboard with 4×7 rectangles?
  NO... the area of one tile does not divide the area of the board.

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- Can you tile a 17×28 checkerboard with 4×7 rectangles?
  NO. . . you can't even tile the first column.

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  NO... the area of one tile does not divide the area of the board.
- Can you tile a 17×28 checkerboard with 4×7 rectangles?
  NO... you can't even tile the first column.
- Can you tile a 14×18 checkerboard with 4×7 rectangles?
  NO... you can't even tile it with 4×1 rectangles.

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- Can you tile a 14×18 checkerboard with 4×7 rectangles?
  NO. . . you can't even tile it with 4×1 rectangles.

We will prove this with COLOURS!

Rectangles on a checkerboard

Every 4×1 rectangle covers one square of each colour. On a 14×18 checkerboard, there are fewer squares of colour 1 than of colour 2.

1	2	3	4	1	2	3	4	1	2	3	4	⋯
2	3	4	1	2	3	4	1	2	3	4	1	⋯
3	4	1	2	3	4	1	2	3	4	1	2	⋯
4	1	2	3	4	1	2	3	4	1	2	3	⋯
1	2	3	4	1	2	3	4	1	2	3	4	⋯
2	3	4	1	2	3	4	1	2	3	4	1	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

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3	4	1	2	3	4	1	2	3	4	1	2	⋯
4	1	2	3	4	1	2	3	4	1	2	3	⋯
1	2	3	4	1	2	3	4	1	2	3	4	⋯
2	3	4	1	2	3	4	1	2	3	4	1	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Answer

Let a and b be relatively prime positive integers. A tiling of an $m \times n$ rectangle with $a \times b$ rectangles exists if and only if

- both m and n can be written as $xa + yb$, where x and y are non-negative integers; and
- either m or n is divisible by a, and either m or n is divisible by b.

Faulty tilings

All Soviet Union Mathematical Olympiad 1963

A 6×6 checkerboard is tiled with dominoes. Prove that you can cut the board with a line that does not pass through any domino.

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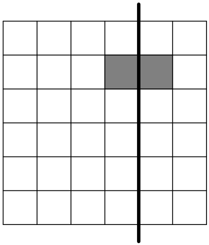
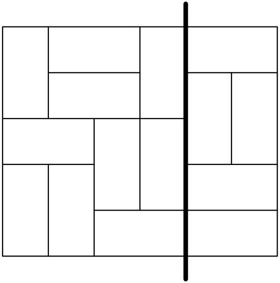
Proof.

- To obtain a contradiction, we suppose otherwise.
- There are 10 lines — 5 horizontal and 5 vertical — each of which must be crossed by at least 1 domino.
- Each of these lines must actually be crossed by at least 2 dominoes.
- So there must be at least 20 dominoes, which is a contradiction! □

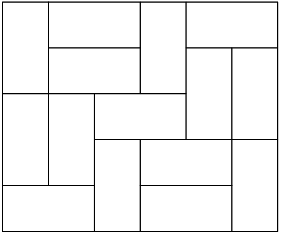


Faulty tilings

If you can cut a checkerboard tiling with a line that does not pass through any tile, then the tiling is faulty — otherwise, it is faultless.



faulty



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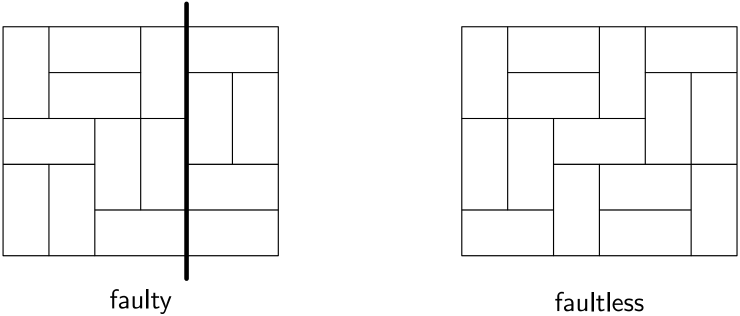

We know that every domino tiling of a 6×6 checkerboard is faulty.

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When can you find a faultless tiling of an $m \times n$ checkerboard with $a \times b$ rectangles?

Faulty tilings

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When can you find a faultless tiling of an $m \times n$ checkerboard with $a \times b$ rectangles?

Answer

Assume that a and b are relatively prime. You can find a faultless tiling of an $m \times n$ checkerboard with $a \times b$ rectangles if and only if

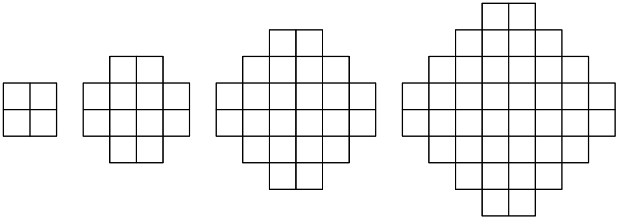
- either m or n is divisible by a, and either m or n is divisible by b;
- both m and n can be expressed as $xa + yb$ in at least two ways, where x and y are positive integers; and
- if the tiles are dominoes, the checkerboard is not 6×6.

Aztec diamonds

Question

How many ways are there to tile an Aztec diamond with dominoes?

Here are the Aztec diamonds $AZ(1), AZ(2), AZ(3)$ and $AZ(4)$.



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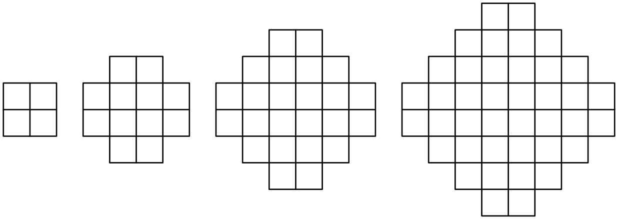
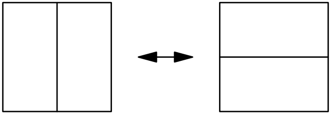


Answer

The number of ways to tile $AZ(n)$ is $2^{n(n+1)/2}$.

Aztec diamonds

A domino flip rotates two adjacent dominoes by 90°.



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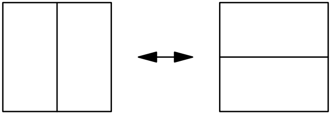
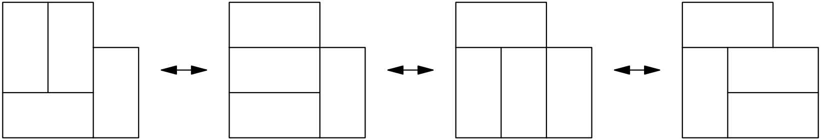


Domino flipping theorem

Domino tilings of a shape without holes are related by domino flips.



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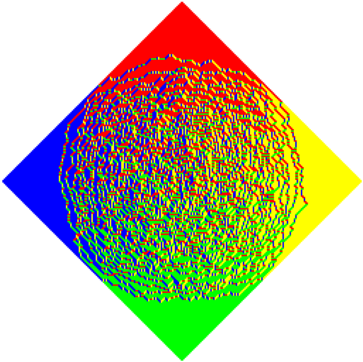

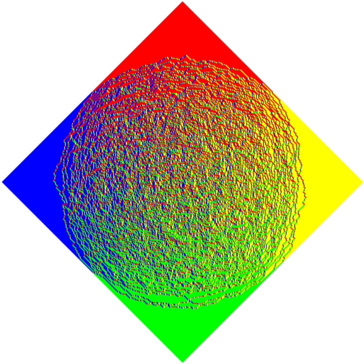

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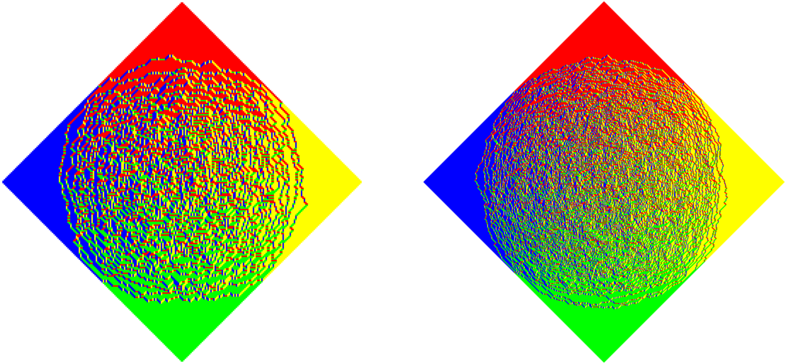

Arctic circle theorem

As n approaches infinity, the disorded region of a random domino tiling of the Aztec diamond $AZ(n)$ will approach a circle.

Tiling a square with similar rectangles

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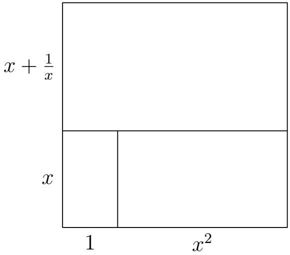
- All rational numbers are happy... but are all happy numbers rational?
- No, some irrational numbers can be happy too!

$$x^2 + 1 = 2x + \frac{1}{x}$$

$$x^3 - 2x^2 + x - 1 = 0$$

$$x = 1.75488 \text{ or } 0.122561 \pm 0.744862i$$



Tiling a square with similar rectangles

Polynomials and algebraic numbers

- A number is called algebraic if it is the root of a non-zero polynomial with integer coefficients.
  For example, $\sqrt{2}$ and $1 + i$ are algebraic but π and e are not.

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- Every algebraic number is the root of infinitely many non-zero polynomials with integer coefficients.
  For example, $\sqrt{2}$ is the root of x^2-2 and $-7x^2+14$ and $(x^2-2)(x^3+3x+1)$ and more.

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- The most efficient one is called the minimal polynomial.
  For example, the minimal polynomial of $\sqrt{2}$ is x^2-2.
- If two algebraic numbers are roots of the same minimal polynomial, then we call them friends.
  For example $\sqrt{2}$ and $-\sqrt{2}$ are friends.

Tiling a square with similar rectangles

Answer

The number r is happy if and only if it is a positive real algebraic number whose friends all have positive real part.

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- The number $\sqrt{2}$ is not happy, because the polynomial $x^2 - 2$ also has the root $-\sqrt{2}$.
- In fact, $\frac{a}{b} + \sqrt{2}$ is happy if and only if $\frac{a}{b} > \sqrt{2}$. This is because the minimal polynomial of $\frac{a}{b} + \sqrt{2}$ is

$$b^2x^2 - 2abx + a^2 - 2b^2,$$

  whose other root is $\frac{a}{b} - \sqrt{2}$.

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Challenge

Tile a square with rectangles similar to a $\left(\frac{3}{2} + \sqrt{2}\right) \times 1$ rectangle.

Tiling a square with infinitely many rectangles

Consider the equation

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots=1.$$

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Question

Can you tile a unit square with one rectangle of size $1 \times \frac{1}{2}$, one rectangle of size $\frac{1}{2} \times \frac{1}{3}$, one rectangle of size $\frac{1}{3} \times \frac{1}{4}$, and so on?

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- Amazingly, no one knows the answer to this question!
- Someone has squeezed these rectangles into a square of side length 1.000000001!

Polyominoes in the plane

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The who-can-tile-lots-but-not-all game

- I give you n unit squares and you construct a set of polyominoes.
- If it is possible to tile the whole plane with tiles of these shapes, then you lose.
- If it is not possible, then you win L dollars, where L is the side length of the largest square you can cover.

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Let $L(n)$ be the largest number of dollars you can win.

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Theorem

The sequence $L(1), L(2), L(3), \ldots$ grows quicker than any sequence that can be output by a computer program.

Polyominoes in the plane

Consequences

- A computer program can output the sequence

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- It is impossible to write a computer program to calculate $L(n)$.
- There probably exists a set of polyominoes constructed from 100 squares of side length one centimetre such that it is impossible to tile the whole plane with them, but it is possible to tile a region that covers Australia.