THE GEOMETRY AND COMBINATORICS OF MODULI SPACES

18/01/12 @ University of Queensland

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* Moduli spaces and enumerative geometry

Moduli spaces parametrise geometric objects

different points $\longleftrightarrow$ different objects

nearby points $\longleftrightarrow$ similar objects

Toy example: Moduli spaces of triangles

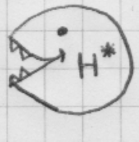
$$\mathcal{M}_\Delta = \left\{ (a,b,c) \in \mathbb{R}_+^3 \;\middle|\; \begin{matrix} b+c>a \\ c+a>b \\ a+b>c \end{matrix} \right\} \Big/ S_3$$

Toy question: How many triangles are

$$\underbrace{\text{isosceles,}}_{X_{iso} \subseteq \mathcal{M}_\Delta} \quad \underbrace{\text{have a side of length 5,}}_{X_5 \subseteq \mathcal{M}_\Delta} \quad \underbrace{\text{and a side of length 7?}}_{X_7 \subseteq \mathcal{M}_\Delta}$$

Equivalently, what is $|X_{iso} \cap X_5 \cap X_7| = \int_{\mathcal{M}_\Delta} X_{iso} \cdot X_5 \cdot X_7$?

Cohomology / Intersection theory: Naively, cohomology is . . .

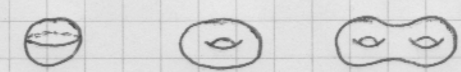
spaces $\longrightarrow$  $\longrightarrow$ rings

elements of $H^*(X) \longleftrightarrow$ submanifolds of X

addition $\longleftrightarrow$ formal addition

multiplication $\longleftrightarrow$ intersection

* Moduli spaces of curves

Topology of surfaces:  . . .

$g=0$ $\quad$ $g=1$ $\quad$ $g=2$

Geometry of surfaces:

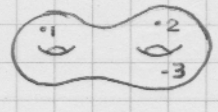
surfaces

algebraic geometry | complex analysis | differential geometry

algebraic curves $\xleftrightarrow{1:1}$ Riemann surfaces $\xleftrightarrow{1:1}$ hyperbolic surfaces

Moduli spaces of curves:

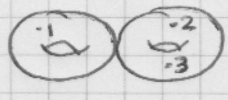
$$\mathcal{M}_{g,n} = \left\{ \begin{array}{l} \text{genus } g \text{ smooth algebraic curves} \\ \text{with } n \text{ labelled points} \end{array} \right\}$$

e.g.  $\in \mathcal{M}_{2,3}$

$\downarrow$ compactify

$\downarrow$ allow degenerations

$$\overline{\mathcal{M}}_{g,n} = \left\{ \begin{array}{l} \text{genus } g \text{ stable algebraic curves} \\ \text{with } n \text{ labelled points} \end{array} \right\}$$

e.g.  $\in \overline{\mathcal{M}}_{2,3}$

Here, stable = allow nodes + finitely many automorphisms.

Facts:

- $\overline{\mathcal{M}}_{g,n}$ is an orbifold, so intersection numbers can be rational
- $\dim \overline{\mathcal{M}}_{g,n} = 2(3g-3+n)$
- $\overline{\mathcal{M}}_{g,n}$ is VERY complicated

Witten - Kontsevich theorem:

We have $\psi_1, \psi_2, \ldots, \psi_n \in H^*(\overline{\mathcal{M}}_{g,n})$ representing natural codimension 2 submanifolds, where

$$\psi_k = c_1(\mathcal{L}_k).$$

$\mathcal{L}_k$: cotangent line bundle at k^{th} marked point

If $a_1 + a_2 + \cdots + a_n = 3g-3+n$, then

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \cdot \psi_2^{a_2} \cdots \psi_n^{a_n} \in \mathbb{Q}$$

is an intersection number.

Witten says that these numbers can be stored in a natural generating function which satisfies KdV.

Proofs: Kontsevich, Okounkov - Pandharipande, Mirzakhani, ...

* Tiling surfaces

$N_{g,n}(b_1, b_2, \ldots, b_n)$ = # ways to glue edges of a b_1-gon, a b_2-gon, ..., a b_n-gon to obtain a genus g surface

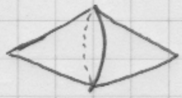
= # "lattice points in $\mathcal{M}_{g,n}$"

Strebel's theorem:

surface tilings with lengths assigned to edges $\longrightarrow$ points in $\mathcal{M}_{g,n}$

Example: $N_{0,4}(3,3,3,3) = 8$

2 labellings



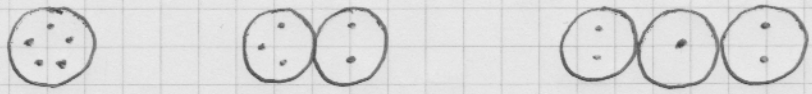
6 labellings

Now define $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n) = \#$ "lattice points in $\overline{\mathcal{M}}_{g,n}$"

$= N_{g,n}(b_1, b_2, \ldots, b_n)$ + lower order terms

Example:

$$\overline{\mathcal{M}}_{0,5} = \underbrace{\mathcal{M}_{0,5}}_{\text{1 labelling}} \cup \underbrace{\mathcal{M}_{0,4} \times \mathcal{M}_{0,3}}_{\text{10 labellings}} \cup \underbrace{\mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \times \mathcal{M}_{0,3}}_{\text{15 labellings}}$$



$$\overline{N}_{0,5}(b_1, b_2, b_3, b_4, b_5) = N_{0,5}(b_1, b_2, b_3, b_4, b_5) + \sum_{\text{10 labellings}} N_{0,4}(b_i, b_j, b_k, 0) \times N_{0,3}(b_l, b_m, 0)$$

$$+ \sum_{\text{15 labellings}} N_{0,3}(b_i, b_j, 0) \times N_{0,3}(b_k, 0, 0) \times N_{0,3}(b_l, b_m, 0)$$

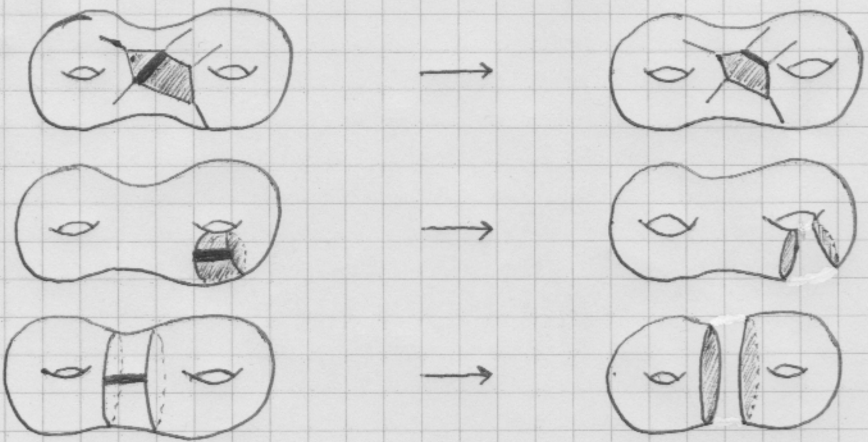
Topological recursion: [Do - Norbury, 2011]

$\overline{N}_{g,n}$ depends on $\overline{N}_{g,n-1}$

$\overline{N}_{g-1,n+1}$

$\overline{N}_{g_1,n_1+1} \times \overline{N}_{g_2,n_2+1}$ for $g_1 + g_2 = g$, $n_1 + n_2 = n-1$

Idea of proof: Remove edges



Theorem: [Do - Norbury, 2011]

- $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$ is an even symmetric quasi-polynomial of degree $2(3g-3+n)$, depending on parity
- (Top degree) If $a_1 + a_2 + \cdots + a_n = 3g-3+n$, then

$$\left[\frac{b_1^{2a_1}}{a_1!} \cdots \frac{b_n^{2a_n}}{a_n!} \right] \overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$$
$$= \frac{1}{2^{5g-6+2n}} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \psi_2^{a_2} \cdots \psi_n^{a_n}$$

- (Bottom degree) $\overline{N}_{g,n}(0, 0, \ldots, 0) = \chi(\overline{\mathcal{M}}_{g,n})$

Corollary:

- (Old) Witten - Kontsevich theorem
- (New) Recursion for $\chi_{g,n} = \chi(\overline{\mathcal{M}}_{g,n})$

$$\chi_{g,n+1} = (2-2g-n)\,\chi_{g,n} + \tfrac{1}{2}\left[\chi_{g-1,n+2} + \sum_{h=0}^{g} \sum_{k=0}^{n} \binom{n}{k} \chi_{h,k+1}\, \chi_{g-h,n-k+1} \right]$$

Question: Is there geometric meaning for the intermediate coefficients?

Conjecture: Yes — that's why they're positive.

* The remaining puzzle

- moduli spaces of curves

$$\downarrow$$

Gromov - Witten theory of $\mathbb{P}^1$

$$\downarrow$$

Gromov - Witten theory of CY3s

- Eynard - Orantin topological recursion:

[relation between x and y] $\longrightarrow$ $\boxed{\text{E - O}}$ $\longrightarrow$ $\{ W_{g,n}(x_1, x_2, \ldots, x_n) \}$

e.g.	$x = y^2$	$\longrightarrow$	Witten - Kontsevich theorem
	$x = y \exp(-y)$	$\longrightarrow$	Hurwitz numbers
	complicated	$\longrightarrow$	(plane) partitions
	$x = \exp(y) + \exp(-y)$	$\xrightarrow{?}$	GW theory of $\mathbb{P}^1$
	$x = y \exp(-y^r)$	$\xrightarrow{??}$	r - spin Hurwitz numbers