THE GEOMETRY AND COMBINATORICS OF MODULI SPACES

18/01/12 @ University of Queensland

Norman Do

* Moduli spaces and enumerative geometry

Moduli spaces parametrise geometric objects

different points ↔ different objects

nearly points ↔ similar objects

Toy example: Moduli spaces of triangles

\[ M_3 = \left\{ (a, b, c) \in \mathbb{R}_+^3 \mid \frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c} \right\} / S_3 \]

Toy question: How many triangles are isosceles, have a side of length 5, and a side of length 7?

\[ X_{130} \subseteq M_3 \quad X_5 \subseteq M_3 \quad X_7 \subseteq M_3 \]

Equivalently, what is \[ |X_{130} \cap X_5 \cap X_7| = \int_{M_3} X_{130} \cdot X_5 \cdot X_7 \]

Cohomology / Intersection theory: Naively, cohomology is...

spaces \[ \rightarrow \quad \bigwedge \quad \rightarrow \quad \text{rings} \]

elements of \( H^*(X) \) ↔ submanifolds of \( X \)

addition ↔ formal addition

multiplication ↔ intersection

* Moduli spaces of curves

Topology of surfaces: \[ g=0 \quad g=1 \quad g=2 \quad \cdots \]

Geometry of surfaces:

\[ \begin{array}{ccc}
\text{algebraic geometry} & \text{surfaces} & \text{differential geometry} \\
\text{algebraic curves} & \text{Riemann surfaces} & \text{hyperbolic surfaces}
\end{array} \]
Moduli spaces of curves:
\[ M_{g,n} = \{ \text{genus } g \text{ smooth algebraic curves} \} \quad \text{with } n \text{ labelled points} \]
\[ \rightarrow \text{compactify} \]
\[ \overline{M}_{g,n} = \{ \text{genus } g \text{ stable algebraic curves} \} \quad \text{with } n \text{ labelled points} \]

\[ \begin{array}{c}
\text{e.g.} \quad \begin{array}{c}
\circ \quad 1 \quad \circ \quad 2 \quad \circ \quad 3
\end{array}
\end{array} \in M_{2,3} \]
\[ \text{allow degenerations} \]
\[ \begin{array}{c}
\text{e.g.} \quad \begin{array}{c}
\circ \quad 1 \quad \circ \quad 2 \quad \circ \quad 3
\end{array}
\end{array} \in \overline{M}_{2,3}
\]

Here, stable = allow nodes + finitely many automorphisms.

Facts:
- \( \overline{M}_{g,n} \) is an orbifold, so intersection numbers can be rational
- \( \dim \overline{M}_{g,n} = 2(3g - 3 + n) \)
- \( \overline{M}_{g,n} \) is VERY complicated

Witten - Kontsevich theorem:
We have \( \psi_1, \psi_2, \ldots, \psi_n \in H^*(\overline{M}_{g,n}) \) representing natural codimension 2 submanifolds, where
\[
\psi_k = c_1\left(L_k\right)
\]
cotangent line bundle at \( k \)th marked point

If \( a_1 + a_2 + \ldots + a_n = 3g - 3 + n \), then
\[
\int_{\overline{M}_{g,n}} \psi_1^{a_1} \psi_2^{a_2} \ldots \psi_n^{a_n} \in \mathbb{Q}
\]
is an intersection number.

Witten says that these numbers can be stored in a natural generating function which satisfies KdV.

Proofs: Kontsevich, Okounkov - Pandharipande, Mirzakhani, ...

* Tiling surfaces
\[ \text{\# ways to glue edges of a } b_1\text{-gon,} \]
\[ a \text{-gon,}, \ldots, \text{a } b_n\text{-gon to obtain a} \]
genus \( g \) surface
\[ = \text{"lattice points in } M_{g,n}" \]
Strebel's theorem:

surface tilings with lengths assigned to edges \( \rightarrow \) points in \( \mathcal{M}_{g,n} \)

Example: \( N_{0,4}(3,3,3,3) = 8 \)

2 labelings

Now define \( \overline{N}_{g,n}(b_1, b_2, \ldots, b_n) = \# "lattice points in \mathcal{M}_{g,n}" \)

\[ = N_{g,n}(b_1, b_2, \ldots, b_n) \text{ + lower order terms} \]

Example:

\[
\begin{align*}
\overline{M}_{0,5} &= M_{0,5} \text{ U } M_{0,4} \times M_{0,3} \text{ U } M_{0,3} \times M_{0,3} \times M_{0,3} \\
&= 1 \text{ labelling} \text{ U } 10 \text{ labelings} \text{ U } 15 \text{ labelings} \\
\overline{N}_{0,5}(b_1, b_2, b_3, b_4, b_5) &= N_{0,5}(b_1, b_2, b_3, b_4, b_5) + \sum_{10 \text{ labelings}} N_{0,4}(b_i, b_j, b_k, 0) \times N_{0,3}(b_l, b_m, 0) \\
&\quad + \sum_{15 \text{ labelings}} N_{0,3}(b_i, b_j, 0) \times N_{0,3}(b_k, 0, 0) \times N_{0,3}(b_l, b_m, 0)
\end{align*}
\]

Topological recursion: [Do - Norbury, 2011]

\( \overline{N}_{g,n} \) depends on \( \overline{N}_{g,n-1} \)

\( \overline{N}_{g-1,n+1} \)

\( \overline{N}_{g_1,n_1} \times \overline{N}_{g_2,n_2+1} \) for \( g_1 + g_2 = g \)

\( n_1 + n_2 = n - 1 \)

Idea of proof: Remove edges
Theorem: [Dr. Norbury, 2011]

- $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$ is an even symmetric quasi-polynomial of degree $2(3g-3+n)$, depending on parity.

- (Top degree) If $a_1 + a_2 + \cdots + a_n = 3g - 3 + n$, then

$$\left\lfloor \begin{array}{c} b_1^{2a_1} \\ a_1! \\ \vdots \\ b_n^{2a_n} \\ a_n! \end{array} \right\rfloor \overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$$

$$= \frac{1}{2^{3g-3+n}} \int_{\overline{M}_{g,n}} \psi_1^{a_1} \psi_2^{a_2} \cdots \psi_n^{a_n}$$

- (Bottom degree) $\overline{N}_{g,n}(0, 0, \ldots, 0) = \chi(\overline{M}_{g,n})$

Corollary:

- (Old) Witten-Kontsevich theorem

- (New) Recursion for $\chi_{g,n} = \chi(\overline{M}_{g,n})$

$$\chi_{g,n+1} = (2 - 2g - n) \chi_{g,n} + \frac{1}{2} \left[ \chi_{g-1,n+2} + \sum_{h=0}^{3g-3+n} \sum_{k=0}^{n} \binom{n}{k} \chi_{h,k+1} \chi_{g-h,n-k+1} \right]$$

Question: Is there geometric meaning for the intermediate coefficients?

Conjecture: Yes — that’s why they’re positive.

* The remaining puzzle

- moduli spaces of curves

  $\downarrow$

  Gromov-Witten theory of $\mathbb{P}^1$

  $\downarrow$

  Gromov-Witten theory of CY3s

- Eynard-Orantin topological recursion:

  $[\text{relation between } x \text{ and } y] \rightarrow [E-O] \rightarrow \{ W_{g,n}(x_1, x_2, \ldots, x_n) \}$

  e.g. $x = y^2$ $\rightarrow$ Witten-Kontsevich theorem

  $x = y \exp(-y)$ $\rightarrow$ Hurwitz numbers

  complicated $\rightarrow$ (plane) partitions

  $x = \exp(y) + \exp(-y)$ $\rightarrow$ GW theory of $\mathbb{P}^1$

  $x = y \exp(-y^n)$ $\rightarrow$ $r$-spin Hurwitz numbers