Take a permutation and count the number of ways to express it as a product of a fixed number of transpositions — you have calculated a Hurwitz number. By adding a mild constraint on such factorisations, one obtains the notion of a monotone Hurwitz number. We have recently shown that the monotone Hurwitz problem fits into the so-called topological recursion/quantum curve paradigm. This talk will attempt to give the flavour of what exactly the previous sentence means.
Hurwitz numbers

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Definition
Let $H_{g,n}(\mu_1, \mu_2, \ldots, \mu_n)$ be $\frac{1}{|\mu|!}$ multiplied by the number of tuples $(\sigma_1, \sigma_2, \ldots, \sigma_m)$ of transpositions in $S_{|\mu|}$ such that
- $m = 2g - 2 + n + |\mu|$;
- $\sigma_1\sigma_2\cdots\sigma_m$ has labelled cycles of lengths $\mu_1, \mu_2, \ldots, \mu_n$; and
- $\langle\sigma_1, \sigma_2, \ldots, \sigma_m\rangle$ is transitive.

Secret
Hurwitz numbers count branched covers of $\mathbb{C}P^1$. 
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- $\langle \sigma_1, \sigma_2, \ldots, \sigma_m \rangle$ is transitive; and
- if $\sigma_i = (a_i \ b_i)$ with $a_i < b_i$, then $b_1 \leq b_2 \leq \cdots \leq b_m$.

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Monotone Hurwitz numbers are natural from the viewpoint of matrix models (HCIZ integral); representation theory (Jucys–Murphy elements); and integrability (Toda tau-functions).
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Example calculation

Take \((g, n) = (0, 2)\) and \(\mu = (1, 2)\), so \(m = 2g - 2 + n + |\mu| = 3\).
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There are 27 products of 3 transpositions in \(S_3\) and 24 are transitive.

\[
\begin{align*}
(1 \, 2) \circ (1 \, 2) \circ (1 \, 3) & \quad (1 \, 2) \circ (1 \, 3) \circ (2 \, 3) & \quad (1 \, 3) \circ (1 \, 3) \circ (2 \, 3) & \quad (2 \, 3) \circ (1 \, 3) \circ (1 \, 3) \\
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\]

All 24 products produce cycle type \((1, 2)\), so \(H_{0,2}(1, 2) = \frac{24}{3!} = 4\).

Only the first 12 products are monotone, so \(\tilde{H}_{0,2}(1, 2) = \frac{12}{3!} = 2\).
Old results

- **Polynomiality.** There are polynomials $P_{g,n}$ and $\tilde{P}_{g,n}$ such that
  
  $H_{g,n}(\mu_1, \ldots, \mu_n) = m! \times \prod \frac{\mu_i^{\mu_i}}{\mu_i!} \times P_{g,n}(\mu_1, \ldots, \mu_n)$

  $\tilde{H}_{g,n}(\mu_1, \ldots, \mu_n) = \prod \left(\frac{2^{\mu_i}}{\mu_i}\right) \times \tilde{P}_{g,n}(\mu_1, \ldots, \mu_n)$.
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  For example, $\tilde{P}_{1,2}(\mu_1, \mu_2) = \frac{1}{12} (2\mu_1^2 + 2\mu_2^2 + 2\mu_1\mu_2 - \mu_1 - \mu_2 - 1)$. 
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- **Polynomiality.** There are polynomials $P_{g,n}$ and $\bar{P}_{g,n}$ such that
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- **Cut-and-join recursion.** (Monotone) Hurwitz numbers of type $(g,n)$ can be calculated from those of types
  - $(g,n-1)$
  - $(g-1,n+1)$
  - $(g_1,n_1) \times (g_2,n_2)$ for $g_1+g_2 = g$ and $n_1 + n_2 = n + 1$. 

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  For example,

  \[
  \mu_1 \tilde{H}_{1,2}(\mu_1, \mu_2) = (\mu_1 + \mu_2) \tilde{H}_{1,1}(\mu_1 + \mu_2) + \sum_{\alpha + \beta = \mu_1} \alpha \beta \tilde{H}_{0,3}(\alpha, \beta, \mu_2)
  \]
  \[
  + 2 \sum_{\alpha + \beta = \mu_1} \alpha \beta \left[ \tilde{H}_{0,1}(\alpha) \tilde{H}_{1,2}(\beta, \mu_2) + \tilde{H}_{1,1}(\alpha) \tilde{H}_{0,2}(\beta, \mu_2) \right].
  \]
Topological recursion and quantum curves

spectral curve $P(x, y) = 0$

differentials $\omega_{g,n}(x_1, \ldots, x_n)$

free energies $F_{g,n}(x_1, \ldots, x_n)$

quantum curve $\hat{P}(\hat{x}, \hat{y})$

Schrödinger eq. $\hat{P}(\hat{x}, \hat{y})Z = 0$

wave function $Z(x, \hbar)$

We use the definitions $Z(x, \hbar) = \exp \sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \frac{g^{-2} + n!}{2} F_{g,n}(x_1, \ldots, x_n)$.
We use the definitions

- $Z(x, \hbar) = \exp \left[ \sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \frac{\hbar^{2g-2+n}}{n!} F_{g,n}(x, \ldots, x) \right]$

- $\hat{x} = x$ and $\hat{y} = -\hbar \frac{\partial}{\partial x}$, which imply $[\hat{x}, \hat{y}] = \hbar$. 
New results

This is joint work with A. Dyer and D. Mathews (arXiv:1408.3992).

- The spectral curve $P(x, y) = xy^2 - y + 1 = 0$ yields

$$F_{g,n}(x_1, \ldots, x_n) = \sum_{\mu} \tilde{H}_{g,n}(\mu_1, \ldots, \mu_n) x_1^{\mu_1} \cdots x_n^{\mu_n}.$$
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- The spectral curve $P(x, y) = xy^2 - y + 1 = 0$ yields
  \[ F_{g,n}(x_1, \ldots, x_n) = \sum_{\mu} \hat{H}_{g,n}(\mu_1, \ldots, \mu_n) x_1^{\mu_1} \cdots x_n^{\mu_n}. \]

- The wave function satisfies
  \[ Z(x, \hbar) = 1 + \sum_{d=1}^{\infty} \sum_{m=0}^{\infty} \left\{ \frac{d + m - 1}{d - 1} \right\} \frac{x^d \hbar^{m-d}}{d!}. \]
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- The corresponding quantum curve is $\hat{P}(\hat{x}, \hat{y}) = \hat{x}\hat{y}^2 - \hat{y} + 1$, so

$$x\hbar^2 \frac{\partial^2 Z}{\partial x^2} + \hbar \frac{\partial Z}{\partial x} + Z = 0.$$