have you ever tried to count how many panels there are on a soccer ball? have you ever wondered what the hairy ball theorem is and whether it applies to you? have you ever thought about how curvaceous a person can be? if you’ve read this much of the abstract, then you should definitely come along to the seminar and learn about the amazing euler characteristic!
WHAT IS A POLYHEDRON?

Naively speaking, a **polyhedron** is a shape consisting of vertices, edges, and faces.

**Nice polyhedra**

- tetrahedron
- cube
- soccer ball

**A scary polyhedron**
### PROPERTIES OF POLYHEDRA

Let’s count the number of vertices ($V$), edges ($E$), and faces ($F$).

<table>
<thead>
<tr>
<th>polyhedron</th>
<th>$V$</th>
<th>$E$</th>
<th>$F$</th>
<th>$V - E + F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>tetrahedron</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>cube</td>
<td>8</td>
<td>12</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>soccer ball</td>
<td>60</td>
<td>90</td>
<td>32</td>
<td>2</td>
</tr>
<tr>
<td>dinosaur</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>2</td>
</tr>
</tbody>
</table>

#### Euler’s formula

If a polyhedron has $V$ vertices, $E$ edges, and $F$ faces, then

$$V - E + F = 2.$$
PLANAR GRAPHS

A graph consists of vertices and edges that join vertices in pairs. A graph is called planar if you can draw it without edges crossing.

Euler’s formula
If a planar graph has \( V \) vertices, \( E \) edges, and divides the plane into \( F \) faces including the outside one, then

\[ V - E + F = 2. \]

Fact
The vertices and edges of a polyhedron form a planar graph.
WHY YOU MIGHT BELIEVE EULER

\[ V - E + F = ? \]
WHY YOU MIGHT BELIEVE EULER

\[ V - E + F = ? \]
WHY YOU MIGHT BELIEVE EULER

\[ V - E + F = ? \]
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\[ V - E + F = 2 \]
WHY YOU MIGHT BELIEVE EULER

\[ V - E + F = 2 \]
AMAZING FACTS ABOUT PLANAR GRAPHS

Fact
The graphs $K_5$ and $K_{3,3}$ are not planar.

\[ K_5 \quad K_{3,3} \]

Kuratowski’s theorem
A graph is planar if and only if it does not contain “$K_5$” or “$K_{3,3}$”.

Fáry’s theorem
Every planar graph can be drawn in the plane without edges crossing such that every edge is a straight line segment.
EULER’S FORMULA IN ACTION

- **How many faces does a soccer ball have?**
  
  Suppose that there are $P$ pentagons and $H$ hexagons. Every pentagon is surrounded by five hexagons while every hexagon is surrounded by three pentagons and three hexagons.

  
  \[
  3V = 2E \\
  5P + 6H = 2E \\
  5P = 3H \\
  V - E + (P + H) = 2
  \]

  
  \[
  \Rightarrow P = 12 \\
  \Rightarrow H = 20
  \]

- **Why is the graph $K_5$ not planar?**
  
  Suppose that $K_5$ is planar and can be drawn in the plane using $V = 5$ vertices, $E = 10$ edges, and $F$ faces. Then

  \[
  V - E + F = 2 \Rightarrow F = 7 \
  \text{and} \quad 3F \leq 2E \Rightarrow 21 \leq 20.
  \]

  This is a blatant contradiction!
Here, $V - E + F = 8 - 8 + 3 = 3$. Each face must be a “2D blob” without holes.

Here, $V - E + F = 16 - 28 + 12 = 0$. The polyhedron must enclose a “3D blob” without holes.

Euler’s formula generalised
A polyhedron with genus $g$ — in other words, $g$ holes — satisfies

$$V - E + F = 2 - 2g.$$  

The Euler characteristic of a surface is the magic number $2 - 2g$.  

COUNTING VORTICES

- Is it possible to comb a hairy ball without creating a cowlick?
- Is it possible for the wind on Earth to not have a vortex?
- Is it possible for a vector field on the sphere to not have a zero?

Hairy ball theorem [Brouwer, 1912]
It is impossible for a vector field on the sphere to not have a zero.

It is possible to comb a hairy doughnut without creating a cowlick.
Poincaré–Hopf theorem [Poincaré, 1881 and Hopf, 1926]

There is a special way to write a number at every vortex so that the sum of the numbers is equal to the Euler characteristic of the planet.

How to count vortices
Walk around the vortex in a small anticlockwise loop, always facing the wind. The index of the vortex is the number of anticlockwise turns that you make.

index = +1  
index = +1  
index = −1  
index = −2
WHY YOU MIGHT BELIEVE POINCARÉ AND HOPF

- The gravy flow on the genus two surface has six vortices.
- The ones at the top and bottom have index $+1$, while the remaining four have index $-1$.
- These numbers sum up to $-2$, which is the Euler characteristic.
Naively speaking, curvature measures how “bendy” or “curvy” a surface is at a particular point.

To find the curvature at a point, take the smallest and largest circles of best fit and multiply the inverse of their radii.

For example, a sphere of radius $R$ has curvature $\frac{1}{R^2}$.

Below are shapes with positive curvature (sphere), zero curvature (cylinder), and negative curvature (hyperboloid).
THE GAUSS-BONNET THEOREM

Gauss–Bonnet theorem [Bonnet, 1848]

If you integrate the curvature $K$ over a surface $S$ with respect to the area $dA$, then you will find that

$$\int_S K \, dA = 2\pi \chi(S).$$

It seems like it could be true — when you deform a surface, you’re only spreading out the curvature, never creating or destroying it.
The Gauss map takes a point $P$ on your surface and returns a point $G(P)$ on the unit sphere.

- Consider the vector that points directly out of the surface at $P$.
- Translate to the origin and shrink/expand until it has length one.
- Then $G(P)$ is the endpoint of this vector.
WHY YOU MIGHT BELIEVE GAUSS AND BONNET

Calculating curvature

Draw a tiny triangle $\Delta$ around $P$. The curvature at $P$ is the ratio

$$\frac{\text{Area } G(\Delta)}{\text{Area } \Delta}.$$ 

A sketch proof

Divide your surface $S$ into many tiny triangles $\Delta_1, \Delta_2, \Delta_3, \ldots$.

$$\int_S K \, dA = \sum K(\Delta) \times \text{Area } \Delta = \sum \frac{\text{Area } G(\Delta)}{\text{Area } \Delta} \times \text{Area } \Delta$$

$$= \sum \text{Area } G(\Delta) = \deg G \times \text{Area sphere}$$

$$= 4\pi \deg G = 4\pi (1 - g) = 2\pi \chi(S)$$

Intuitively, $\deg G$ is the number of times that the surface $S$ is wrapped around the sphere by the map $G$. 
GENERALISING THE EULER CHARACTERISTIC

- **Higher dimensions**
  The higher dimensional analogue of a surface is a manifold. The analogues of vertices, edges, and faces are simplices. If \( a_k \) is the number of \( k \)-dimensional simplices in \( M \), then
  \[
  \chi(M) = a_0 - a_1 + a_2 - a_3 + \cdots.
  \]

- **Bundles**
  Vector fields choose a vector in each tangent plane of a surface. Instead of tangent planes, you can use other collections of planes to create vector bundles. The analogue of the Euler characteristic is called the Euler class.

- **Cardinality**
  The Euler characteristic is like cardinality — it “counts” objects. Now sets can have a negative number of objects. There is also a way to have a fractional number of objects. What does it all mean?!
THANKS

If you would like more information, you can

- find the slides at http://users.monash.edu.au/~normd
- email me at norm.do@monash.edu
- speak to me at the front of the lecture theatre