RESOLUTION OF A QUESTION ON DETERMINANTS

J. HADAMARD

1. Given a determinant

$$(1) \qquad \Delta = \begin{vmatrix} a_1 & b_1 & \dots & l_1 \\ a_2 & b_2 & \dots & l_2 \\ \vdots & \vdots & \vdots & \vdots \\ a_n & b_n & \dots & l_n \end{vmatrix}$$

with the assumption that the elements are less in absolute value than some known quantity A, we wish to determine a limit that the modulus of Δ does not exceed.

One sees immediately that $|\Delta|$ is less than $n!A^n$. But it is clear that this limit is too high as it cannot be attained unless all of the terms in the determinant have the same sign, this is manifestly impossible.

As a consequence, I propose to investigate the maximum of the determinant Δ under the indicated conditions.

2. Without any assumptions on the moduli of the elements $a_1, \dots, b_1 \dots,$ denote by $a_1^0, \dots, b_1^0, \dots,$ their conjugates in the determinant Δ_0, which is the conjugate of Δ. Given, in the determinant Δ, p distinct rows from which to form a rectangular tableau (T), let (T_0) be the corresponding tableau from the determinant Δ_0; we consider the product

$$P_p = (T)(T_0).$$

If $p = n$, this product gives $\Delta\Delta_0$, that is to say the square of the modulus of Δ; for $p < n$, it provides similarly the sum of the squares of the moduli of the different determinants that can be derived from the tableau (T). In every case, the quantity obtained is necessarily real and positive.[1]

[1] We have not considered the case that $P_p = 0$, in which case the determinant Δ is zero and, in any case, is not of interest to the question under consideration.

To form the product P_p, as per the rules for the multiplication of determinants, it is necessary to multiply every row of (T) by every column of (T_0). If we choose a corresponding row and column, for example both of rank h, the resulting s_h gives the sum of the squares of the moduli of the elements $a_h, b_h, \ldots, l_h$. If on the contrary we choose a row and column of different ranks h and h', we obtain the expression

$$s_{h,h'} = a_h a^0_{h'} + b_h b^0_{h'} + \ldots l_h l^0_{h'}. \tag{2}$$

We remark that $s_{h,h'}$ is conjugate to $s_{h',h}$.

The quantities s_h and $s_{h,h'}$ are the elements of the determinant P_p. If, for example the lines comprising the tableau (T) are the first p, we have

$$P_p = \begin{vmatrix} s_1 & s_{1,2} & \ldots & s_{1,p} \\ s_{2,1} & s_2 & \ldots & s_{2,p} \\ \vdots & \vdots & \vdots & \vdots \\ s_{p,1} & s_{p,2} & \ldots & s_p \end{vmatrix}$$

and, if we isolate the part which contains as a factor a given principal element, the last for example, we can write

$$P_p = s_p P_{p-1} + Q_p \tag{3}$$

where Q_p is the determinant

$$\begin{vmatrix} s_1 & s_{1,2} & \ldots & s_{1,p-1} & s_{1,p} \\ s_{2,1} & s_2 & \ldots & s_{2,p-1} & s_{2,p} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{p-1,1} & s_{p-1,2} & \ldots & s_{p-1} & s_{p-1,p} \\ s_{p,1} & s_{p,2} & \ldots & s_{p,p-1} & 0 \end{vmatrix}.$$

The minor of this determinant relative to the h^{th} row and to the h'^{th} column, being designated $\frac{\partial Q_p}{\partial(h,h')}$, we note that the principal minors are the expressions Q_{p-1}, except the last, which is the expression P_{p-1}. Regarding the other minors, we remark only that $\frac{\partial Q_p}{\partial(h,h')}$ is conjugate to $\frac{\partial Q_p}{\partial(h',h)}$.

3. Being posed, it is easy to show that the determinant Q_p is negative or zero, the second case cannot occur unless all of the elements in the last column are zero.

It suffices for this (h being one of the numbers $1, 2, \ldots, p-1$) to consider the four minors $\frac{\partial Q_p}{\partial(h,h)}, \frac{\partial Q_p}{\partial(p,p)}, \frac{\partial Q_p}{\partial(h,p)}, \frac{\partial Q_p}{\partial(p,h)}$. A well known identity that exists between these minors gives us a relation of the form

$$Q_p P_{p-2} = Q_{p-1} P_{p-1} - \left[\frac{\partial Q_p}{\partial(h,p)} \right]^2 .$$

The P's being positive, we see that Q_p is necessarily negative or zero since this conclusion has been demonstrated already for Q_{p-1}. We can thus assume this result for every value of p, it being evident for Q_1, which is equal to zero, and for Q_2, which is equal to $-|s_{1,2}|^2$.

Furthermore, Q_p cannot be 0 unless Q_{p-1} is, which is to say (assuming always that our conclusion is established for Q_{p-1}) if all of the elements in the final column are zero with the exception of the h^{th}. But if h is a variable and p is at least 3, we cannot have an element in the last column different from zero.

4. Returning to the determinant P_p: we can now establish that this determinant is less than or equal to its principal term $s_1 s_2 \ldots s_p$, with equality only if all the non-principal elements are zero.

In effect, we can assume that this fact is true for P_{p-1}, and as a result of equation (3), demonstrate only the inequality $P_p < s_1 s_2 \ldots s_p$, since Q_p is negative.

Furthermore, P_p cannot be equal to $s_1 s_2 \ldots s_p$ unless, for its part, P_{p-1} is equal to $s_1 s_2 \ldots s_{p-1}$ and so, on the other hand, we have that $Q_p = 0$. As we have seen earlier, this double condition implies that all of the non-principal entries are zero.

In particular, for $p = n$, we have that

$$|\Delta|^2 \leq s_1 s_2 \ldots s_n .$$

Assuming the moduli of the elements to be at most 1, the s_h

take maximum value n, and thus $|\Delta|$ is at most $n^{\frac{n}{2}}$.

We see that the maximal value of a determinant of the n^{th} order grows less rapidly than $n!$. From the formula for the approximation of the Γ function, it grows slightly more rapidly than the square root of this product.

5. For Δ to obtain its maximum, it is necessary in the first place, that all of the elements have modulus 1; also that all of the $s_{h,h'}$ be zero $(h \neq h')$.

In writing the equation $s_{h,h'} = 0$ for all values of h', the h being fixed, we have a system of homogenous linear equations in bijection with the elements of the h^{th} row, with the result that every element is proportional to the conjugate of the quantity of the corresponding minor. Formulae for adjoint determinants then imply that minors of order k are proportional to the complementary minors of order $n - k$.

We are thus led to the determinants called *inverse orthogonal* by Sylvester[2] and given a simple and abundant set of examples, for any choice of n, by the Vandermonde determinant formed with the roots of the binomial equation $x^n = 1$.

6. For $n = 3$, as was remarked by Sylvester, all of the other solutions reduce to this one given, considering the following changes as insignificant: permutation of rows or columns, multiplication of all of the elements in a single row or all of the elements in a single column by by the same factor. But this is not the case for values of n greater than 3 and the construction of a maximal determinant is more involved than was supposed by the English geometer.

Using in effect the method given in his memoir[3] to construct a maximal determinant of order $n_1 n_2$ when

[2]Philosophical Magazine, XXXIV, p. 461-475, 1867
[3]ibid. 465, no. 6

we take as known maximal determinants Δ_1 and Δ_2, of orders n_1 and n_2 respectively: we write the determinant Δ_1 a total of n_2^2 times, in a tableau of n_2 rows and n_2 columns, denoted $(\mathcal{E})$. Now, the determinant Δ_1 occupying the h^{th} row and k^{th} column of this tableau is multiplied by the element of the determinant Δ_2 given by the indices h and k. Thus modified, the tableau $(\mathcal{E})$ gives a maximum determinant, since the relations $s_{h,h'} = 0$ hold.

But these relations do not cease to hold if in each of the determinants Δ_1 of the premier column we multiply a given row by any number of modulus 1. The new determinant obtained through this operation (which evidently we could vary in many ways) is not reducible to the preceding by the insignificant operations previously discussed.

For example, when $n = 4$, Sylvester has indicated the two types

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{vmatrix} \tag{4}$$

and

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{vmatrix} \tag{5}$$

Our method leads to the determinant

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & a & -a \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -a & a \end{vmatrix} \qquad (a = e^{i\theta})$$

which gives the determinants (4) and (5) for $a = 1$ and $a = i$, but is essentially distinct for any distinct value of θ.

7. In the case that n is a power of 2, the preceding gives us a method to obtain a maximum determinant all of whose elements are real.

For what other values of n can such determinants be found?

In such a case, every element must be equal to ± 1, in such a way as that, considering two distinct rows and comparing the corresponding elements, one has an equal number of concordances and discordances of signs.[4]

One sees easily that this cannot occur except when n is a multiple of 4. In effect, if we force the elements of the first row to be 1, the second row must contain as many $+1$s as -1s, which shows already that n must be even: $n = 2n'$. If we suppose now that the first n' elements of the second row are positive and that the remainder are negative, the sum of the of the first n' elements of the third row must be zero; so n' is for its part even.

In fact, there exist real maximal determinants for values of n that are not powers of 2. For $n = 12$, for example we arrive at a result by the following method: We group the columns 3 by 3 into four sets. The first row is composed of 1s, the second of 1s in the first two sets, and -1s in the last two; the third row has its four sets composed of alternately 1s and -1s. In the 9 following rows the first and the last set each contain 2 positive elements and 1 negative element; the seond and third sets 2 negative elements and 1 positive, as in the following table:

1	1	1	1
1	2	2	2
1	3	3	3
2	1	2	3
2	2	3	1
2	3	1	2
3	1	3	2
3	2	1	3
3	3	2	1

the numbers indicate in each set the position of the element which is unique of its sign.

[4]Sylvester, Anallagmatic pavement.

There also exists a real maximal determinant for $n = 20$. To obtain it, we again partition the columns into 4 sets of 5 each, we compose the first three rows as in the previous case. The fourth row will be, in the first and last sets, composed entirely of positive elements except the first in each case; in the second and third sets, all the elements will be negative except the first. Each of the remaining rows will comprise in the first and last sets, two negative elements; in the second and third, two positive elements, as in the following table:

12	23	23	23
13	23	45	45
14	45	23	45
15	45	45	23
45	12	24	24
45	13	35	35
23	14	24	35
23	15	35	24
34	34	34	12
34	25	25	13
25	34	25	14
25	25	34	15

Thus it is appropriate to ask for which values of n there exist maximal determinants with real elements.[5] Furthermore, one can research, for other values what is the greatest modulus that can be attained by a determinant if we impose on the elements the condition of being real.

[5]The maximal determinants that we have given for $n = 12$ and $n = 20$ prove further evidence to support the current question; it is clear that these new maximal determinants cannot be deduced from the procedure given in no. 6.

COMMENTS

Hadamard's paper originally appeared in volume 17 of the *Bulletin des sciences mathématiques* in 1893.

I have attempted to translate the paper as faithfully as possible from the original French. For the most part, I have retained his terminology and notation (the main changes being *row* for *ligne* and *set* for *séries*). I have also introduced the standard notation for the factorial function. Page breaks have been retained as in the original, as has the page numbering. While the paper remains as fresh as the day it was written, some of the language is slightly at odds with modern usage, and may benefit from a word of explanation.

(1) Nowhere in either the original paper does Hadamard use the word *matrix*. By *déterminant* Hadamard normally means the homogeneous polynomial of degree n in n^2 variables obtained by calculating the determinant of an $n \times n$ matrix (given in (1)) whose entries are commuting indeterminates. That is

$$\Delta = \sum_{\sigma \in S_n} \chi(\sigma) a_{1^\sigma} b_{2^\sigma} \dots l_{n^\sigma},$$

where S_n is the symmetric group acting naturally on $\{1, 2, \dots, n\}$ and χ is the alternating character on S_n.

(2) In Section 2, Hadamard intends that T_0 be a tableau whose entries are the conjugates of the entries of T. He then multiplies *the determinants* to obtain P_p. It maybe more natural to think of T_0 as the Hermitian transpose of T, and of P_p as the determinant of the *matrix product*.

(3) In Section 3 Hadamard takes for granted an *identité bien connue* on the minors of a matrix. Since the theory of determinants (as distinct from linear algebra) is no longer a part of mainstream mathematics, I have provided a proof of the required result. I am grateful for Michael Tuite for bringing the following lemma to my attention, and for placing it in historical context as the key step in *Dodgson condensation*. Dodgson's method is described in a paper of 1866. The lemma seems to be due originally to Jacobi. Some historical remarks are contained in Section 2.3 of [1]. (Dodgson is perhaps better known under the psedonym Lewis Carroll, as the author of Alice in Wonderland.)

(4) The relationship between complementary minors is given by the Jacobi determinant identity (see below). In particular, if M is a minor of order

k in a Hadamard matrix of order n, and M' is its complementary minor, then $n^{k-\frac{n}{2}}\det(M') = \det(M)$.

(5) In Section 6, there was a misprint in the matrix labelled (5), which we have corrected.

Lemma. *Let M be an $n \times n$ invertible matrix, and let $M^{-1} = N$. We write M and N as block matrices:*

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \qquad N = \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix}$$

Then $\det(M)\det(N_1) = \det(M_4)$.

Proof. We observe that

$$\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}\begin{pmatrix} N_1 & 0 \\ N_3 & I \end{pmatrix} = \begin{pmatrix} I & M_2 \\ 0 & M_4 \end{pmatrix},$$

where I is an identity matrix of suitable dimension. It suffices to take determinants on both sides. □

Note that the lemma holds more generally for any submatrix M_4 indexed by $I \subseteq \{1, \ldots, n\}$.

Hadamard's identity is obtained as follows:

(1) Let N_4 be the 2×2 submatrix of Q_p indexed by h and p, and N_1 be complementary $p-2 \times p-2$ submatrix.

(2) Set $M = Q_{p-1}^{-1}$. Then $\det(M)\det(N_1) = \det(M_4)$.

(3) But note that the elements of M are cofactors of Q_{p-1}, thus in Hadamard's notation:

$$Q_p^{-1}P_{p-2} = Q_{p-1}P_{p-1} - \left[\frac{\partial Q_p}{\partial(h,p)}\right]^2$$

(4) Now, for any matrix M, $\det(M)$ has the same sign as $\det(M^{-1})$. Hadamard's induction hypothesis is that Q_t is negative for all $t \leq p-1$. Since $\det(P_t)$ is positive for any subscript t, we obtain by induction that $\det(Q_p)$ is negative.

Padraig Ó Catháin

References

[1] C. Krattenthaler. Advanced determinant calculus. *Sém. Lothar. Combin.*, 42:Art. B42q, 67 pp. (electronic), 1999. The Andrews Festschrift (Maratea, 1998).