2 Numbers in digital systems

2.1 Digits, numerals and numbers

- Assume that an integer number r is the **base** or **radix** of a number representation (or system), then
- integer numbers $a_i \in \{0, \ldots r-1\}$ are called **digits** . Each digit has its own symbol.
- We will be typically using bases $r = 2, 8, 10, 16$ and equivalent number systems are called: **binary, octal, decimal and hexadecimal**.
- Symbols for digits in the hexadecimal system are: $0, \ldots 9, A, B, C, D, E, F$.
- An ordered sequence, or **vector of digits** is called a **numeral**.
- Digits in a numeral are typically numbered from right to left, for example, a five-digit numeral

$$\mathbf{a} = a_4 a_3 a_2 a_1 a_0 = 23412$$

- In order to associate a number with a numeral we have to specify the radix r and assign a weight (a power of r), say r^i, to each ith digit. Then the integer number a equivalent to the n-digit numeral $\mathbf{a}$ can be found as:

$$a = (\mathbf{a})_r = (a_{n-1}, \ldots, a_1, a_0)_r = \sum_{i=0}^{n-1} a_i \cdot r^i = \mathbf{a} \cdot \mathbf{r}_n$$

where $\mathbf{r}_n = [r^{n-1}, \ldots r^1, r^0]$ is a vector of respective powers of the base (weights)

- A number is an **inner (dot) product** of a vector of digits (a numeral) and a vector of weights (powers of the radix).
- It is a common practice to call **numerals radix-r numbers**

  Hence we have binary, octal, decimal and hexadecimal numbers.

  Often, decimal numbers are simply called numbers

Example:

An octal numeral $(r = 8)$ expressed as a decimal numeral (number):

$$a = (3412)_8 = 3 \cdot 8^3 + 4 \cdot 8^2 + 1 \cdot 8^1 + 2 \cdot 8^0 = 3 \cdot 512 + 4 \cdot 64 + 8 + 2 = 1536 + 266 = (1802)_{10}$$

We can also write the following inner product expression

$$a = \begin{bmatrix} 3 & 4 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 8^3 \\ 8^2 \\ 8^1 \\ 8^0 \end{bmatrix} = 1802$$

- **Fractional numbers** need information about the position of the “radix point”.

  For example

$$(101.011)_2 = 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 + 0 \cdot 2^{-1} + 1 \cdot 2^{-2} + 1 \cdot 2^{-3} = 5 + 0.25 + 0.125 = (5.375)_{10}$$

2.2 Multiplication and division by a power of the radix, r^m

- **Multiplication** (of integers) by the power of the radix, r^m, appends m zeros to the right hand side of the respective numeral.
- This is equivalent to shifting the numeral m positions to the left,
  or moving the "**radix point**" m positions to the right.

  Example:

$$(243.)_{10} \cdot 10^2 = (24300.)_{10}$$

- **Division** by the power of the radix, r^m, (multiplication by the negative power of the radix) shifts the numeral m positions to the right,
  that is, moves the "**radix point**" m positions to the left.

  Example:

$$(243.)_{10} \cdot 10^{-2} = (2.43)_{10}$$

2.3 Basic arithmetic operations

2.3.1 Addition in radix r system

Given two n-digit numerals the sum is calculated digit by digit using additional carry digits:

$c_0 = 0;$
for $k = 0$ **to** $n - 1$
calculate c_{k+1}, s_k given a_k, b_k, c_k, so that:

$$r \cdot c_{k+1} + s_k = a_k + b_k + c_k;$$

or

$$\begin{aligned} s_k &= \text{mod}(a_k + b_k + c_k, r); \\ r \cdot c_{k+1} &= a_k + b_k + c_k - s_k \end{aligned}$$

$$\begin{array}{rcccccc} a & = & & a_{n-1} & \dots & a_1 & a_0 \\ b & = & & b_{n-1} & \dots & b_1 & b_0 \\ \hline & & c_n & c_{n-1} & \dots & c_1 & c_0 \\ s & = & s_n & s_{n-1} & \dots & s_1 & s_0 \end{array}$$

$$s = a + b; \quad (s_n = c_n \in \{0, 1\})$$

2.3.2 Multiplication by 10 in a binary system

Multiplication by 10 in a binary system is performed according to the formula

$$10 \cdot a = (2^2 \cdot a + a) \cdot 2$$

2.4 Radix conversion

Converting from radix r (source) to radix q (target)

Given an n-digit numeral in radix r system, find an equivalent m-digit numeral in radix q system,

$$a = \underbrace{(\; a_{n-1} \;\ldots\; a_1 \; a_0 \;)_r}_{\text{given}} = \underbrace{(\; b_{m-1} \;\ldots\; b_1 \; b_0 \;)_q}_{\text{unknown}}$$

e.g.

$$(253)_{10} = (\ldots\; b_1\; b_0)_2$$

2.4.1 Method 1: Division by target radix in the source system

The first method is based on the multi-step division of the number a by the target radix q, performed in the source radix r system. The remainders are unknown digits b_k.
The algorithm

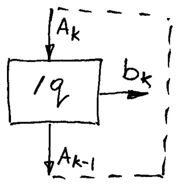
$A_0 = a;$
for $k = 0$ **to** $m - 1$

$$\frac{A_k}{q} = A_{k-1} + \frac{b_k}{q}$$

or

$$A_k = q \cdot A_{k-1} + b_k$$

A_k is a quotient
b_k is a reminder
$b_k \in \{0, \ldots, q-1\}$



Example:

$$\begin{array}{lllllllll}
A_0 &=& 253 &=& 2\cdot 126 &+& 1 &\rightarrow& b_0\\
A_1 &=& 126 &=& 2\cdot 63 &+& 0 &\rightarrow& b_1\\
A_2 &=& 63 &=& 2\cdot 31 &+& 1 &\rightarrow& b_2\\
A_3 &=& 31 &=& 2\cdot 15 &+& 1 &\rightarrow& b_3\\
A_4 &=& 15 &=& 2\cdot 7 &+& 1 &\rightarrow& b_4\\
A_5 &=& 7 &=& 2\cdot 3 &+& 1 &\rightarrow& b_5\\
A_6 &=& 3 &=& 2\cdot 1 &+& 1 &\rightarrow& b_6\\
A_7 &=& 1 &=& 2\cdot 0 &+& 1 &\rightarrow& b_7
\end{array}$$

$$\begin{array}{cccccccccc}
 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & \\
 & (1 & 1 & 1 & 1 & 1 & 1 & 0 & 1)_2 & \\
8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & \\
(1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1)_2 &
\end{array}$$

$$= 256 - 4 + 1 = (253)_{10}$$

Exercise:

Give an example of conversion from the binary to the decimal system using method 1.

2.4.2 Method 2: Multiplication by source radix in the target system

The second method is based on the multi-step multiplication of the source digits by the source radix r, performed in the target radix q system. The final product is the unknown numeral in the target system.

$$a = \underbrace{(\; a_{n-1} \;\dots\; a_1 \;\; a_0 \;)_r}_{\text{given}} = \underbrace{(\; b_{m-1} \;\dots\; b_1 \;\; b_0 \;)_q}_{\text{unknown}}$$

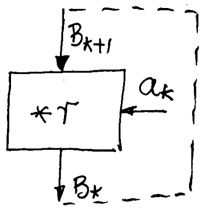
e.g.

$$(253)_{10} = (\dots\; b_1\; b_0)_2$$

The algorithm

$B_n = 0$;
for $k = n - 1$ **to** 0

$B_k = r \cdot B_{k+1} + a_k$ — B_k are partial products/results



$b = B_0$ is the result

Digits a_k needs to be expressed in the radix q system.

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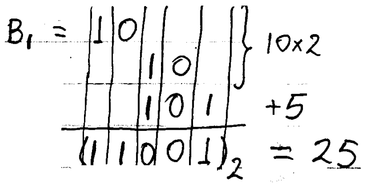
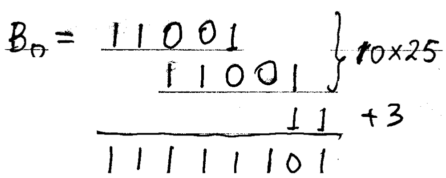
Example:

$$a = (253)_{10} = (b_{m-1}, \dots, b_1 b_0)_2$$

$$a_2 = (10)_2,\; a_1 = (101)_2,\; a_0 = (11)_2$$

$$\begin{aligned} B_3 &= 0 \\ B_2 &= 10 \cdot B_3 + a_2 = 10 \cdot 0 + 2 = 2 \\ B_1 &= 10 \cdot B_2 + a_1 = 10 \cdot 2 + 5 = 25 \\ B_0 &= 10 \cdot B_1 + a_0 = 10 \cdot 25 + 3 = 253 \end{aligned}$$





$$a = (11111101)_2 = 2^8 - 2^2 + 1 = (253)_{10}$$