5 Gate-level minimization

5.1 Principle of logic function minimization

Minimization of logic functions is based on the following two dual simplification rules:

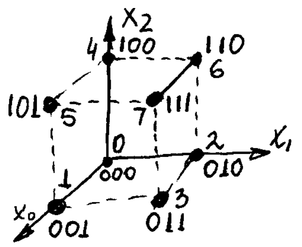
- If in a SoP expression two products differ only by a variable being in direct and complemented form then this variable can be deleted:

$$y = \bar{a} \cdot f(\mathbf{x}) + a \cdot f(\mathbf{x}) = f(\mathbf{x})$$

- If in a PoS expression two sums differ only by a variable being in direct and complemented form then this variable can be deleted:

$$y = (\bar{a} + f(\mathbf{x}))(a + f(\mathbf{x})) = f(\mathbf{x})$$

- Minterms (maxterms) can be visualised as 2^n vertices of an n-dimensional hyper-cube.
  For example consider a 3-D cube:



- Note that two connected vertices differ by a single variable being in a direct or complemented form
- For example consider the sum of two neighbouring minterms:

$$m_5 + m_1 = x_2 \cdot \bar{x}_1 \cdot x_0 + \bar{x}_2 \cdot \bar{x}_1 \cdot x_0 = \bar{x}_1 \cdot x_0$$

- Similarly $M_5 \cdot M_1 = (\bar{x}_2 + x_1 + \bar{x}_0)(x_2 + x_1 + \bar{x}_0) = x_1 + \bar{x}_0$

5.2 Karnaugh Maps

- Karnaugh maps aka K-maps can be thought of as a representation of sides of a n-dimensional hypercube representing logic functions.
- A K-map is a convenient tool of mimimizing logic functions up to 6 variables (practically 4)
- A K-map is a re-arrangement of a truth table such that the adjacent minterms/maxterms differ in one position only.

5.3 A 2-variable Karnaugh map

A 2-variable K-map template:
Minterms, m_i, can be replaced by respective maxterms, $M_i = \overline{m}_i$

$x_1 \backslash x_0$	0	1
0	m_0	m_1
1	m_2	m_3

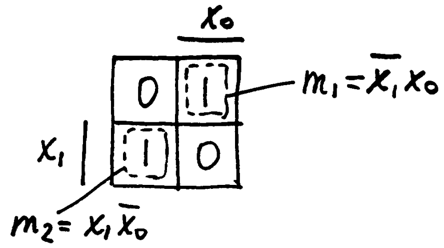
or

	x_0	
	m_0	m_1
x_1	m_2	m_3

Example 1

No simplification possible:

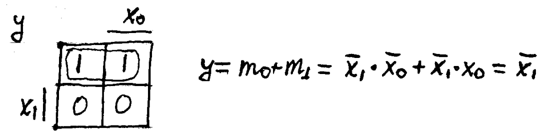
x_1	x_0	y	
0	0	0	m_0
0	1	1	m_1
1	0	1	m_2
1	1	0	m_3



$$y = m_1 + m_2 = \bar{x}_1 x_0 + x_1 \bar{x}_0$$

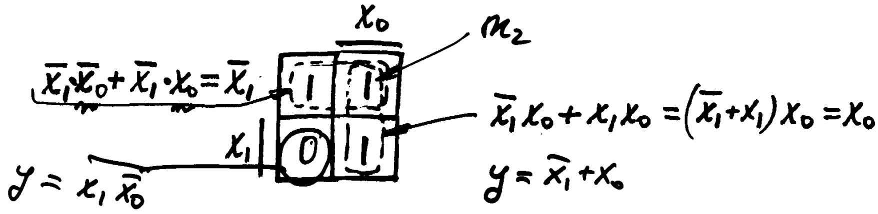
Example 2

Loop around a pair of adjacent minterms:
For Maxterms loop around zeros.



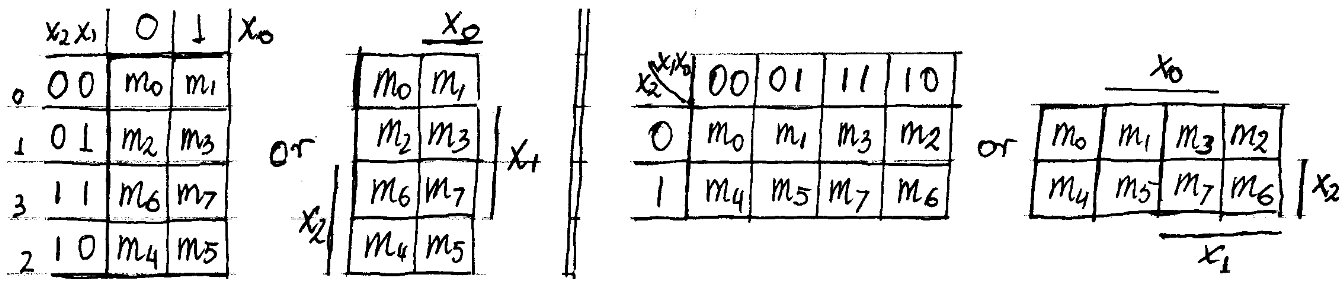
Example 3

Minterms can be "looped around" any number of times:



5.4 A 3-variable Karnaugh map

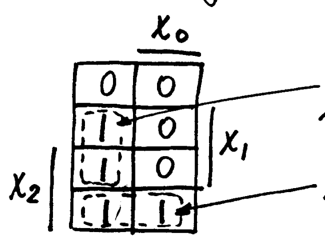
Cut a 3-D cube along one side and unfold it. Two templates are used:



The 8 minterms (or maxterms) of a 3-variable function are arranged in the K-map so that to preserve the property of adjacent squares being different in only a complement of single variable

EXAMPLE

Consider $y = f(x_2 x_1 x_0) = \Sigma(2,4,5,6)$

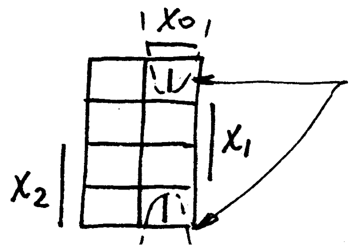


$$\bar{x}_2 x_1 \bar{x}_0 + x_2 x_1 \bar{x}_0 = x_1 \bar{x}_0$$

$$x_2 \bar{x}_1 \bar{x}_0 + x_2 \bar{x}_1 x_0 = x_2 \bar{x}_1$$

$$y = \Sigma(2,4,5,6) = m_2 + m_4 + m_5 + m_6 = x_1 \bar{x}_0 + x_2 \bar{x}_1$$

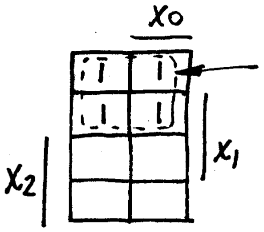
- "Adjacent" squares do not always touch each other:



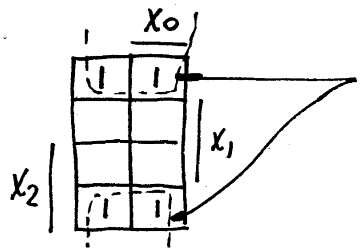
$$\bar{x}_2 \bar{x}_1 x_0 + x_2 \bar{x}_1 x_0 = \bar{x}_1 x_0$$

- **FOUR ADJACENT SQUARES**

form a $(n-2)$-variable product

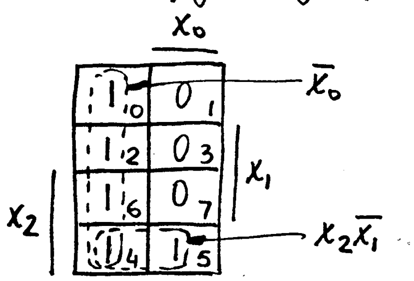


$$\bar{x}_2\bar{x}_1\bar{x}_0 + \bar{x}_2\bar{x}_1x_0 + \bar{x}_2x_1\bar{x}_0 + \bar{x}_2x_1x_0 = \bar{x}_2\bar{x}_1 + \bar{x}_2x_1 = \bar{x}_2$$



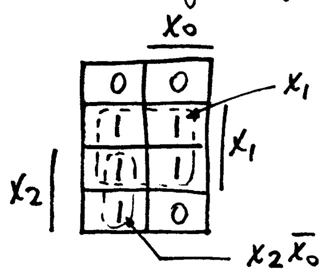
$\bar{x}_1$, because x_2 and x_0 vary

- Simplify $y = f(x_2 x_1 x_0) = \Sigma(0,2,4,5,6)$



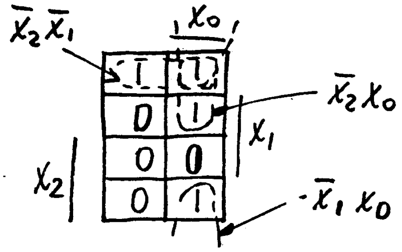
$y = \bar{x}_0 + x_2\bar{x}$

- Simplify $y = f(x_2 x_1 x_0) = \Sigma(2,3,4,6,7)$



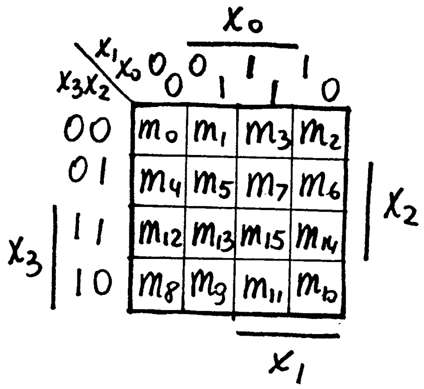
$y = x_1 + x_2\bar{x}_0$

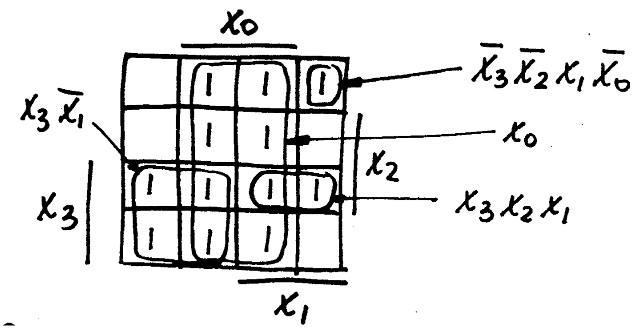
• simplify $y = f(x_2 x_1 x_0) = \Sigma(0,1,3,5)$



$$y = \bar{x}_2\bar{x}_1 + \bar{x}_2 x_0 + \bar{x}_1 x_0$$

5.5 A 4-variable Karnaugh map





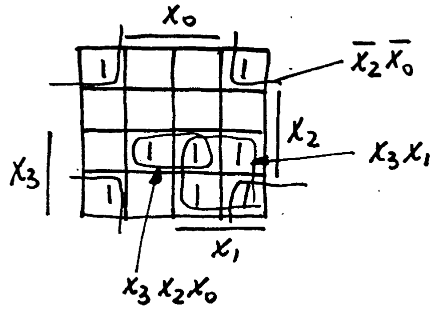
- 1 square represents 4-variable product (minterm)
- 2 squares represent 3-variable product
- 4 squares represent 2-variable product
- 8 squares represent a single variable

Consider that top and bottom, or left and right edges link adjacent vertices of a 4-dimensional cube representing a logic function of four variables.

Examples

Simplify the logic function

$$y = f(x_3 x_2 x_1 x_0) = \sum(0,2,8,10,11,13,14,15)$$

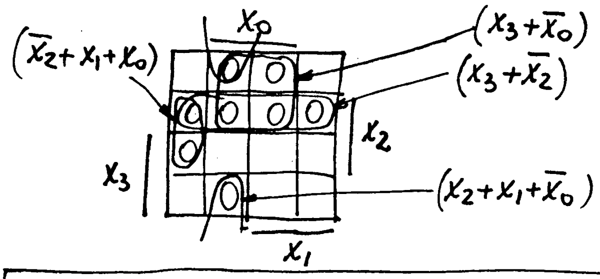


$$\boxed{y = x_3 x_2 x_0 + \bar{x}_2 \bar{x}_0 + x_3 x_1}$$ $\boxed{\text{SoP}}$ SUM-of-PRODUCTS form

In order to obtain a **simplified Product-of-Sums** form we loop around **zeros** and complement variables:

$$y = f(x_3 x_2 x_1 x_0) = \sum(0,2,8,10,11,13,14,15) \quad \boxed{\text{SoM}}$$

$$= \prod(1,3,4,5,6,7,9,12) \quad \boxed{\text{PoM}}$$



$$\boxed{y = (\bar{x}_2 + x_1 + x_0)(x_2 + x_1 + \bar{x}_0)(x_3 + \bar{x}_0)(x_3 + \bar{x}_2)}$$ PRODUCT-of-SUMS form $\boxed{\text{PoS}}$