# Simultaneous Diagonal Flips in Plane Triangulations*

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**Abstract**

Simultaneous diagonal flips in plane triangulations are investigated. It is proved that every triangulation with at least six vertices has a simultaneous flip into a 4-connected triangulation, and that it can be computed in linear time. It follows that every triangulation has a simultaneous flip into a Hamiltonian triangulation. This result is used to prove that for any two $n$-vertex triangulations, there exists a sequence of $\mathcal{O}(\log n)$ simultaneous flips to transform one into the other. The total number of edges flipped in this sequence is $\mathcal{O}(n)$. The maximum size of a simultaneous flip is then studied. It is proved that every triangulation has a simultaneous flip of at least $\frac{1}{3}(n-2)$ edges. On the other hand, every simultaneous flip has at most $n-2$ edges, and there exist triangulations with a maximum simultaneous flip of $\frac{6}{7}(n-2)$ edges.

## 1 Introduction

A (*plane*) *triangulation* is a simple planar graph with a fixed (combinatorial) plane embedding in which every face is bounded by a *triangle* (that is, a 3-cycle). So that we can speak of the interior and exterior of a cycle, one face is nominated to be the *outerface*, although the choice of outerface will not be important for our results. Let $vw$ be an edge of a triangulation $G$. Let $(v, w, x)$ and $(w, v, y)$ be the faces incident to $vw$. Then $x$ and $y$ are distinct vertices, unless $G = K_3$. We say that $x$ and $y$ *see* $vw$. Let $G'$ be the embedded graph obtained from $G$ by deleting $vw$ and adding the edge $xy$, such that in the cyclic order of the edges incident to $x$ (respectively, $y$), $xy$ is added between $xv$ and $xw$ ($yw$ and $yv$). If $G'$ is a triangulation, then $vw$ is (*individually*) *flippable*, and $G$ is *flipped* into $G'$ by $vw$. This operation is called a (*diagonal*) *flip*, and is illustrated in Figure 1(a). If $G'$ is not a triangulation and $G \neq K_3$, then $xy$ is already an edge of $G$; we say that $vw$ is *blocked* by $xy$, and $xy$ is a *blocking* edge.

In 1936, Wagner [26] proved that a finite sequence of diagonal flips transform a given triangulation into any other triangulation with the same number of vertices. Since then diagonal flips in plane triangulations [10, 11, 14, 15, 17, 18, 20, 22, 24] and in triangulations of other surfaces [4, 6, 7, 16, 19–23, 27] have been studied extensively. The number of flips in Wagner's proof is $\mathcal{O}(n^2)$. Komuro [14] improved this bound to $\mathcal{O}(n)$. The best known bound is $\max\{6n-30, 0\}$ due to Mori et al. [17]. For labelled triangulations, Gao et al. [10] proved that $\mathcal{O}(n \log n)$ flips suffice.

Wagner [26] in fact proved that every $n$-vertex triangulation can be transformed by a sequence of flips into the so-called *standard* triangulation $\Delta_n$, which is illustrated in Figure 1(b), and is defined as the triangulation on $n$ vertices with two *dominant* vertices (adjacent to every other vertex). Clearly two $n$-vertex triangulations each with two dominant vertices are isomorphic. To transform one $n$-vertex triangulation $G_1$ into another $G_2$, first transform $G_1$ into $\Delta_n$, and then apply the flips to transform $G_2$ into $\Delta_n$ in reverse order. A similar approach is used in this paper in the context of simultaneous flips in triangulations.

Let $S$ be a set of edges in a plane triangulation $G$. The embedded graph obtained from $G$ by flipping every edge in $S$ is denoted by $G\langle S\rangle$. If $G\langle S\rangle$ is a triangulation, then $S$ is (*simultaneously*) *flippable* in $G$, and $G$ is *flipped* into $G\langle S\rangle$ by $S$. This operation is called a *simultaneous* (*diagonal*) *flip*. Note that it is possible for $S$ to be flippable, yet $S$ contains non-flippable edges, and it is possible for every edge in $S$ to be flippable, yet $S$ itself is not flippable. Simultaneous flips have only previously been studied in the more restrictive context of geometric triangulations of a point set [9]. Individual flips have also been studied in a geometric context [12, 13].

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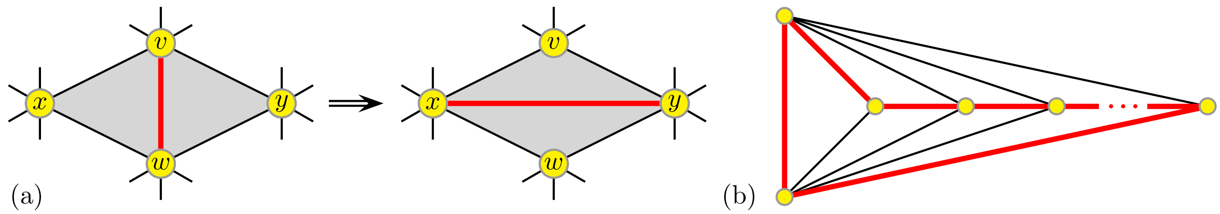


Figure 1: (a) Edge $vw$ is flipped into $xy$. (b) The standard triangulation and a Hamiltonian cycle.

In Section 2 we characterise flippable sets and give a number of introductory lemmas. Our first main result states that every triangulation with at least six vertices can be transformed by one simultaneous flip into a 4-connected (and hence Hamiltonian) triangulation. Moreover, this flip can be computed in linear time. These results are presented in Section 3. In Section 4 we study simultaneous flips in maximal outerplanar graphs. We prove that for any two $n$-vertex maximal outerplanar graphs, there exists a sequence of $\mathcal{O}(\log n)$ simultaneous flips to transform one into the other. The method used is the basis for the main result in Section 5, which states that for any two $n$-vertex triangulations, there exists a sequence of $\mathcal{O}(\log n)$ simultaneous flips to transform one into the other. This result is optimal for many pairs of triangulations. For example, if one triangulation has $\Theta(n)$ maximum degree and the other has $\mathcal{O}(1)$ maximum degree, then $\Omega(\log n)$ simultaneous flips are needed, since one simultaneous flip can at most halve the degree of a vertex. This also holds for diameter instead of maximum degree. Finally in Section 6 the maximum size of a simultaneous flip is studied. It is proved that every triangulation has a simultaneous flip of at least $\frac{1}{3}(n-2)$ edges. On the other hand, every simultaneous flip has at most $n-2$ edges, and there exist triangulations with a maximum simultaneous flip of $\frac{6}{7}(n-2)$ edges.

## 2 Basics

We start with a characterisation of flippable sets that is used throughout the paper. Two edges of a triangulation that are incident to a common face are *consecutive*. If two consecutive edges are simultaneously flipped, then the two new edges cross, as in Figure 2(a). Thus no two edges in a flippable set are consecutive. Two edges form a *bad pair* if they are seen by the same pair of vertices. If a bad pair of edges are simultaneously flipped, then the two new edges are parallel, as in Figure 2(b). Thus no two edges in a flippable set form a bad pair. If an edge $vw$ is blocked by an edge $pq$ as in Figure 2(c), then $vw$ is not individually flippable, but $vw$ can be in a flippable set $S$ as long as $pq$ is also in $S$. It is easily seen that these three properties characterise flippable sets.

Lemma 2.1. *A set of edges $S$ in a triangulation $G \neq K_3$ is flippable if and only if:*
(1) *no two edges in $S$ are consecutive,*
(2) *no two edges in $S$ form a bad pair, and*
(3) *for every edge $vw \in S$, either $vw$ is flippable or the edge that blocks $vw$ is also in $S$.*

The proofs of the following lemmas are elementary, and can be found in the full paper [2]. A cycle $C$ in a triangulation $G$ is *separating* if deleting the vertices of $C$ from $G$ produces a disconnected graph.

Lemma 2.2. *An edge in a separating triangle of a triangulation is individually flippable.*

Lemma 2.3. *Let $vw$ be an edge of a triangulation that is seen by vertices $p$ and $q$. Suppose that $p$ is inside some cycle $C$ and $q$ is outside $C$. Then $vw \in C$.*

Lemma 2.4. *A blocking edge is individually flippable in a triangulation $G \neq K_4$.*

Lemma 2.5. *Suppose that $vw$ and $xy$ are a bad pair in a triangulation $G$, both seen by vertices $p$ and $q$. Suppose that $vw$ blocks some edge $ab$. Then $xy$ and $ab$ are consecutive, and $vw$ and $xy$ are in a common triangle.*

## 3 Flipping into a 4-Connected Triangulation

The following is the main result of this section.

Theorem 3.1. *Every triangulation $G$ with $n \geq 6$ vertices has a simultaneous flip into a 4-connected triangulation that can be computed in $\mathcal{O}(n)$ time.*

The following sufficient condition will be used to prove Theorem 3.1.

Lemma 3.1. *Let $G$ be a triangulation with $n \geq 6$ vertices. Let $S$ be a set of edges in $G$ such that no two edges in $S$ are in a common triangle, every edge in $S$ is in a separating triangle, and every separating triangle contains an edge in $S$. Then $S$ is flippable and $G\langle S\rangle$ is 4-connected.*

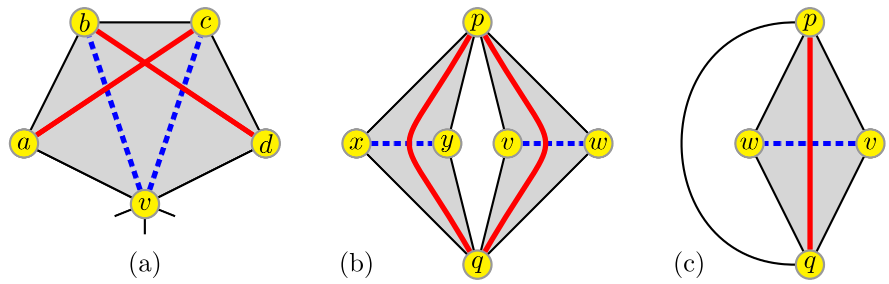


Figure 2: Obstacles to a flippable set. Dashed edges are flipped to create bold edges. Shaded regions are faces

*Proof.* We first prove that $S$ is flippable. By Lemma 2.2, every edge in $S$ is individually flippable. Thus, by Lemma 2.1, it suffices to prove that no two edges in $S$ form a bad pair. Suppose that $vw, xy \in S$ form a bad pair. Then $vw$ and $xy$ are seen by the same pair of vertices $p$ and $q$. Let $T$ be a separating triangle containing $vw$. Then one of $p$ and $q$ is inside $T$, and the other is outside $T$. By Lemma 2.3, $xy$ must be an edge of $T$, which implies that $vw$ and $xy$ are in a common triangle. This contradiction proves that $S$ is flippable.

Since a triangulation is 4-connected if and only if it has no separating triangle, it suffices to prove that $G\langle S\rangle$ contains no separating triangle. Suppose that $T = (u, v, w)$ is a separating triangle in $G\langle S\rangle$. Let $S'$ be the set of edges in $G\langle S\rangle$ that are not in $G$. We proceed by case-analysis on $|T \cap S'|$ (refer to Figure 3). Since every separating triangle in $G$ has an edge in $S$, $|T \cap S'| \geq 1$.

Case 1. $|T \cap S'| = 1$: Without loss of generality, $vw \in S'$, $uv \notin S'$, and $uw \notin S'$. Suppose $xy$ was flipped to $vw$. Then $xy$ is in a separating triangle $xyp$ in $G$. Any vertex adjacent to both $v$ and $w$ must be a vertex of the separating triangle $xyp$. Thus $p = u$. Since $G$ has at least six vertices, at least one of the triangles $\{(u, v, x), (u, v, y), (u, w, x), (u, w, y)\}$ is a separating triangle. Thus at least one of the edges in these triangles is in $S$. Since $xy \in S$, and no two edges of $S$ appear in a common triangle, $\{ux, uy, vx, vy, wx, wy\} \cap S = \emptyset$. Thus $uv$ or $uw$ is in $S$. But then $uvw$ is not a triangle in $G\langle S\rangle$, which is a contradiction.

Case 2. $|T \cap S'| = 2$: Without loss of generality, $uv \in S'$, $vw \in S'$, and $uw \notin S'$. Suppose $xy$ was flipped to $uv$, and $rs$ was flipped to $vw$. Without loss of generality, $y$ and $s$ are inside $uvw$ in $G\langle S\rangle$. Then in $G$, $xy$ was in a separating triangle $xyz$, and $rs$ was in a separating triangle $rst$. By an argument similar to that used to prove that $S$ is flippable, $z = w$ and $t = u$. But then the subgraph of $G$ induced by $\{u, v, w, x, y, r, s\}$ is not planar, or it contains parallel edges in the case that $x = r$ and $y = s$.

Case 3. $|T \cap S'| = 3$: Suppose $xy$ was flipped to $uv$, $rs$ was flipped to $vw$, and $ab$ was flipped to $uw$. Without loss of generality, $y$, $s$ and $b$ are inside $uvw$ in $G\langle S\rangle$. In $G$, $xy$ was in a separating triangle $xyz$, $rs$ was in a separating triangle $rst$, and $ab$ was in a separating triangle $(a, b, c)$. By an argument similar to that used to prove that $S$ is flippable, $z = w$, $t = u$, and $c = v$. But then the subgraph of $G$ induced by $\{u, v, w, x, y, r, s, a, b\}$ is non-planar, or contains parallel edges in the case that $y = s = b$ and $x = r = a$. □

Observe that the restriction in Lemma 3.1 to triangulations with at least six vertices is unavoidable. Every triangulation with at most five vertices has a vertex of degree three, and is thus not 4-connected. Now we consider how to determine a set of edges that satisfy Lemma 3.1.

Lemma 3.2. *Every $n$-vertex triangulation $G$ has a set of edges $S$ that contains a prespecified edge, every face of $G$ has exactly one edge in $S$, and can be computed in $\mathcal{O}(n)$ time.*

*Proof.* Biedl et al. [1] proved the following strengthening of Petersen's matching theorem: Every 3-regular bridgeless planar graph has a perfect matching that contains a prespecified edge and can be computed in linear time. The dual $G^*$ is a 3-regular bridgeless planar graph with $2n - 4$ vertices. A perfect matching in $G^*$ corresponds to the desired set $S$. □

Note that Lemma 3.2 only accounts for triangles of $G$ that are faces. It can be proved by induction on the number of separating triangles, applying Lemma 3.2 in the base case, that every triangulation $G$ has a set of edges $S$ that contain a prespecified edge and every triangle of $G$ has exactly one edge in $S$. By taking those edges in $S$ that are in some separating triangle, Lemma 3.1 implies that every triangulation with at least six vertices has a simultaneous flip into a 4-connected triangulation (Theorem 3.1). However, due to the

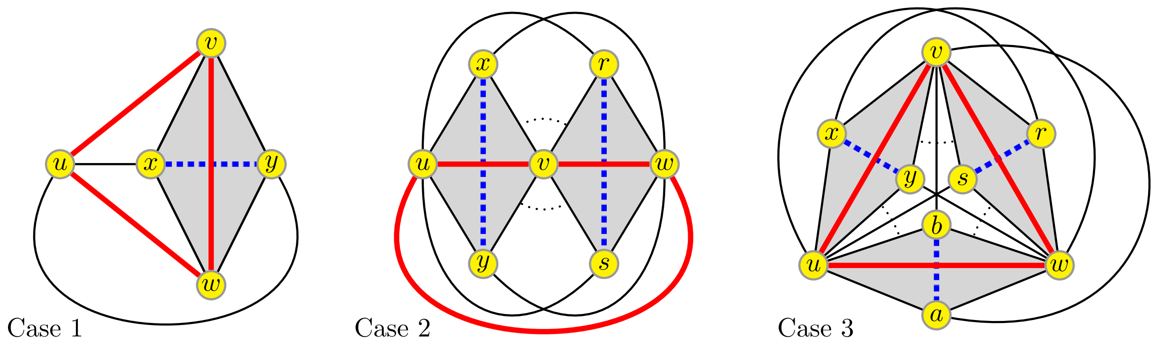


Figure 3: Dashed edges are flipped to create a bold separating triangle. Shaded regions are faces.

presence of separating triangles, it is not obvious how to implement this step in linear time. We now show how to do so.

Let $T$ be a separating triangle of a triangulation $G$. Thus $G \setminus T$ has two components, an *inner component* (containing no vertex on the outerface) and an *outer component*. Denote by $\mathsf{int}(T)$ and $\mathsf{ext}(T)$ the sets of vertices of the inner and outer components. For two separating triangles $T_1$ and $T_2$ of $G$, define $T_1 \preceq T_2$ whenever $\mathsf{int}(T_1) \subseteq \mathsf{int}(T_2)$. Clearly $\preceq$ is a partial order. We now describe how to compute a linear extension $R$ of $\preceq$ in linear time. The *canonical ordering* of de Fraysseix et al. [8] will be a useful tool. Let $G$ be a plane triangulation with outerface $(a, b, c)$. A linear ordering of the vertices $(v_1 = a, v_2 = b, v_3, \ldots, v_n = c)$ is *canonical* if for all $3 \leq i \leq n$:

- the subgraph $G_i$ induced by $\{v_1, v_2, \ldots, v_i\}$ is 2-connected, and the boundary of its outerface is a cycle $C_i$ containing the edge $ab$; and
- the vertex $v_i$ is in the outerface of $G_{i-1}$, and the neighbours of $v_i$ in $G_{i-1}$ form a subinterval of the path $C_{i-1} \setminus \{ab\}$ consisting of at least two vertices (and $v_3$ is adjacent to $v_1$ and $v_2$).

de Fraysseix et al. [8] proved that every triangulation has a canonical ordering. Define the *level* of a separating triangle $T$ as the largest index of a vertex of $T$ in a given canonical ordering.

Lemma 3.3. *For an $n$-vertex plane triangulation $G$, a linear extension $R$ of $\preceq$ can be computed in $\mathcal{O}(n)$ time.*

*Proof.* First note that a canonical ordering can be computed in $\mathcal{O}(n)$ time [8]. In the full paper [2] we prove that if all of the separating triangles of $G$ have different levels, then ordering them by increasing level gives the linear extension $R$. What remains is to order the separating triangles at the same level. These triangles share a common vertex $v_i$ that defines their level. The neighbours of $v_i$ in $G_{i-1}$ form a path $P = (p_1, p_2, \ldots, p_k)$ on the boundary of the outerface of $G_{i-1}$. Every separating triangle of $G$ at level $i$ consists of $v_i$ and two non-consecutive vertices of $P$. To establish the containment relation between these triangles, we simply need to look at the indices of the vertices of $P$. Let $T_1 = (v_i, p_a, p_b)$ and $T_2 = (v_i, p_c, p_d)$ be distinct separating triangles with $a < b$ and $c < d$. If $a < b \leq c < d$ or $c < d \leq a < b$ then $\mathsf{int}(T_1) \cap \mathsf{int}(T_2) = \emptyset$ by the canonical ordering. It is impossible for $a < c < b < d$ or $c < a < d < b$ since the graph induced on $P$ is outerplanar and this would violate planarity. If $a \leq c < d \leq b$ then $T_2 \preceq T_1$, and if $c \leq a < b \leq d$ then $T_1 \preceq T_2$. Since we can compute the graph induced by $\{p_1, p_2, \ldots, p_k\}$ in $\mathcal{O}(k)$ time, all of the separating triangles at level $i$ can be ordered in $\mathcal{O}(k)$ time by performing a breadth-first search on the graph induced on $P$; see [2] for more details. The result follows since the sum of the degrees of a plane graph is $\mathcal{O}(n)$. $\square$

Denote by FaceSet$(G, e)$ the set $S$ from Lemma 3.2; that is, every face of $G$ has exactly one edge in $S$, and if $e$ is specified then $e \in S$.

**Algorithm** TriangleSet$(G, R)$
*Input:* triangulation $G$, and
an ordered list $R$ of separating triangles of $G$.
*Output:* set $S$ of edges of $G$ such that
every triangle of $G$ has exactly one edge in $S$.

```
if R = ∅ then
  return FaceSet(G, unspecified);
else
  let T be the first triangle in R;
  let S := TriangleSet(G \ int(T), R \ T);
  let e be the edge in S ∩ T;
  return S ∪ FaceSet(G \ ext(T), e);
end if
```

LEMMA 3.4. *For every $n$-vertex triangulation $G$, the algorithm* TRIANGLESET$(G, R)$ *returns a set $S$ consisting of exactly one edge in every triangle of $G$. The running time is $\mathcal{O}(n)$.*

*Proof.* We proceed by induction on $|R|$. If $R = \emptyset$ then every triangle in $G$ is a face, and TRIANGLESET$(G, R)$ correctly computes $S$ with a call to FACESET$(G)$. Now assume that $R \neq \emptyset$. Let $T$ be the first triangle in $R$. Then $T$ is an innermost separating triangle of $G$, and $G \setminus \mathsf{ext}(T)$ has no separating triangle. Hence $R \setminus T$ is a linear extension of the containment relation $\preceq$ on the set of separating triangles of $G \setminus \mathsf{int}(T)$. By induction, $S :=$ TRIANGLESET$(G \setminus \mathsf{int}(T), R \setminus T)$ consists of exactly one edge in every triangle of $G \setminus \mathsf{int}(T)$. Thus there is exactly one edge $e \in S \cap T$. Every triangle in $G \setminus \mathsf{ext}(T)$ is a face. By Lemma 3.2, FACESET$(G \setminus \mathsf{ext}(T), e)$ consists of exactly one edge in every triangle of $G \setminus \mathsf{ext}(T)$ including $e$. Together with $S$ we have the desired set for $G$. The running time is described by the recurrence $X(n) = X(n - |\mathsf{int}(T)|) + \mathcal{O}(|\mathsf{int}(T)|) + \mathcal{O}(1)$, which solves to $\mathcal{O}(n)$. □

Note that Algorithm TRIANGLESET can be easily modified to guarantee that a prespecified edge is in $S$.

*Proof of Theorem 3.1.* By Lemma 3.4, there is a set of edges $S$ consisting of exactly one edge in every triangle of $G$. Let $S'$ be the set of edges in $S$ that that are in a separating triangle of $G$. Then $S'$ satisfies the requirements of Lemma 3.1. Thus $S'$ is flippable and $G\langle S' \rangle$ is 4-connected. □

We obtain a stronger result at the expense of a slower algorithm using the following well known corollary of the 4-colour theorem [25].

LEMMA 3.5. *Every $n$-vertex planar graph $G$ has an edge 3-colouring that can be computed in $\mathcal{O}(n^2)$ time, such that every triangle is trichromatic.*

THEOREM 3.2. *Let $G$ be a triangulation with $n \geq 6$ vertices. Then $G$ has three pairwise disjoint flippable sets of edges $S_1, S_2, S_3$ that can be computed in $\mathcal{O}(n^2)$ time, and each $G\langle S_i \rangle$ is 4-connected.*

*Proof.* By Lemma 3.5, $G$ has an edge 3-colouring such that every triangle is trichromatic. For any of the three colours, let $S$ be the set of edges receiving that colour and in a separating triangle. By Lemma 3.1, $S$ is flippable and $G\langle S \rangle$ is 4-connected. □

We have the following corollary of Theorems 3.1 and 3.2, since every triangulation on at most five vertices (that is, $K_3$, $K_4$ or $K_5 \setminus e$) is Hamiltonian, and every 4-connected triangulation has a Hamiltonian cycle [28] that can be computed in linear time [5].

THEOREM 3.3. *Every $n$-vertex triangulation $G$ has a simultaneous flip into a Hamiltonian triangulation that can be computed in $\mathcal{O}(n)$ time. Furthermore, $G$ has three disjoint simultaneous flips that can be computed in $\mathcal{O}(n^2)$ time, such that each transforms $G$ into a Hamiltonian triangulation.* □

## 4 Outerplane Graphs

A plane graph is *outerplane* if every vertex lies on the outerface. The other faces are *internal*. An edge that is not on the boundary of the outerface is *internal*. We consider $n$-vertex (edge-)maximal outerplane graphs $G$. Such graphs are 2-connected, every internal face is a triangle, and there are $2n - 3$ edges and $n - 2$ internal faces. The *dual tree* of $G$, denoted by $G^*$, is the dual graph of $G$ without a vertex corresponding to the outerface. Observe that $G^*$ is a tree with $n - 2$ vertices and maximum degree at most three. The notions of *diagonal flip* and *flippable set* for triangulations are extended to maximal outerplane graphs in the natural way, except that only internal edges are allowed to be flipped. We have the following simple characterisation of flippable sets in maximal outerplane graphs [2].

LEMMA 4.1. *A set $S$ of internal edges in a maximal outerplane graph is flippable if and only if no two edges in $S$ are consecutive.*

The following is the main result of this section. In the remainder of this paper all logarithms have base 2, and $c_1$ is the constant $2/\log\frac{6}{5}$ ($\approx 7.6$).

THEOREM 4.1. *Let $G_1$ and $G_2$ be (unlabelled) maximal outerplane graphs on $n$ vertices. There is a sequence of $4c_1 \log n$ simultaneous flips to transform $G_1$ into $G_2$.*

Two $n$-vertex maximal outerplane graphs both with a dominant vertex are isomorphic. Thus the following lemma proves Theorem 4.1 using the approach of Wagner described in Section 1.

LEMMA 4.2. *For every maximal outerplane graph $G$ on $n$ vertices, and for every vertex $v$ of $G$, there is a sequence of $2c_1 \log n$ simultaneous flips to transform $G$ into a maximal outerplane graph in which $v$ is dominant.*

Lemma 4.2 is implied by Lemmas 4.3 and 4.4 (with $k = c_1 \log n$) to follow. In Lemma 4.3 we reduce the diameter of the dual tree to $c_1 \log n$ using $c_1 \log n$ simultaneous flips. Then in Lemma 4.4 a dominant vertex is introduced using a further $c_1 \log n$ simultaneous flips.

Lemma 4.3. *Let $G$ be a maximal outerplane graph on $n$ vertices. Then $G$ can be transformed by a sequence of at most $c_1 \log n$ simultaneous flips into a maximal outerplane graph $X$ such that the diameter of the dual tree $X^*$ is at most $c_1 \log n$.*

*Proof.* We proceed by induction on $n$. The result holds trivially for $n = 3$. Let $G$ be a maximal outerplane graph on $n$ vertices. By a theorem of Bose et al. [3], $G$ has an independent set $I$ of at least $\frac{n}{6}$ vertices, and $\deg_G(v) \leq 4$ for every vertex $v \in I$. Obviously $\deg_G(v) \geq 2$. For $d \in \{2,3,4\}$, let $I_d$ be the set of vertices $v \in I$ with $\deg_G(v) = d$. For every vertex $v \in I_3 \cup I_4$, add one internal edge incident to $v$ to a set $S$. Since $I$ is independent, $|S| = |I_3| + |I_4|$. Suppose for the sake of contradiction that there are two consecutive edges $xu, xv \in S$. Then $x \notin I_3 \cup I_4$, which implies that $u, v \in I_3 \cup I_4$. Since every internal face of $G$ is a triangle, $uv$ is an edge of $G$, which contradicts the independence of $I$. Thus no two edges in $S$ are consecutive. By Lemma 4.1, $S$ is flippable in $G$. Let $G' := G\langle S\rangle$. Every vertex $v \in I_2 \cup I_3$ has $\deg_{G'}(v) = 2$, and every vertex $v \in I_4$ has $\deg_{G'}(v) = 3$. Since $I_4$ is an independent set of $G$, and any edge in $G'$ that is incident to a vertex in $I_4$ is also in $G$, $I_4$ is an independent set of $G'$. Let $S'$ be the set of internal edges of $G'$ incident to a vertex in $I_4$. Thus $|S'| = |I_4|$, and by the same argument used for $S$, no two edges in $S'$ are consecutive in $G'$. By Lemma 4.1, $S'$ is flippable in $G'$. Let $G'' := G'\langle S'\rangle$. Every vertex $v \in I$ has $\deg_{G''}(v) = 2$.

Thus $G$ can be transformed by two simultaneous flips into a maximal outerplane graph $G''$ containing at least $\frac{n}{6}$ vertices of degree two. Let $G'''$ be the maximal outerplane graph obtained from $G''$ by deleting the vertices of degree two. Then $G'''$ has at most $\frac{5}{6}n$ vertices. By induction, $G'''$ can be transformed by a sequence of at most $c_1 \log \frac{5}{6}n$ simultaneous flips into a maximal outerplane graph $X$ such that the diameter of $X^*$ is at most $c_1 \log \frac{5}{6}n$. Consider a vertex $v \in I$. Since $\deg_{G''}(v) = 2$, there is one internal face incident to $v$ in $G''$, which corresponds to a leaf in $G''^*$. Thus $X^*$ is obtained by adding leaves to $G''^*$. Hence the diameter of $X^*$ is at most the diameter of $G''^*$ plus two, which is $2 + c_1 \log \frac{5}{6}n = c_1 \log n$. We have used two simultaneous flips, $S$ and $S'$, to transform $G$ into $G''$, and then $c_1 \log \frac{5}{6}n$ simultaneous flips to transform $G''$ into $X$. The total number of flips is $2 + c_1 \log \frac{5}{6}n = c_1 \log n$. □

Lemma 4.4. *Let $v$ be a vertex of a maximal outerplane graph $G$ for which $G^*$ has diameter at most $k$. Then $G$ can be transformed by at most $k$ simultaneous flips into a maximal outerplane graph in which $v$ is dominant.*

*Proof.* Define the *distance* of each internal face $f$ of $G$ to be the minimum number of edges in a path of $G^*$ between the vertex that corresponds to $f$ and a vertex of $G^*$ that corresponds to a face incident with $v$. Since the diameter of $G^*$ is at most $k$, every internal face has distance at most $k$. Let $S$ be the set of internal edges that are seen by $v$. Clearly no two edges in $S$ are consecutive. By Lemma 4.1, $S$ is flippable. As illustrated in Figure 4, by flipping $S$, the distance of each internal face that is not incident with $v$ is reduced by one. By induction, at most $k$ simultaneous flips are required to reduce the distance of every internal face to zero, in which case $v$ is dominant. □

## 5 Simultaneous Flips Between Given Triangulations

In this section we prove the following theorem, which is an analogue of Theorem 4.1 for triangulations. Throughout, $c_1$ is the constant from Section 4, and $c_2$ is the constant $2/\log \frac{54}{53}$ ($\approx 74.2$).

Theorem 5.1. *Let $G_1$ and $G_2$ be (unlabelled) triangulations on $n$ vertices. There is a sequence of $2 + 4(c_1 + c_2)\log n$ simultaneous flips to transform $G_1$ into $G_2$.*

The proof of Theorem 5.1 uses the approach of Wagner described in Section 1. We first apply Theorem 3.3 to obtain a Hamiltonian triangulation with one simultaneous flip. Thus it suffices to prove that a Hamiltonian triangulation can be transformed into $\Delta_n$. A Hamiltonian cycle $H$ of a triangulation $G$ naturally divides $G$ into two maximal outerplane subgraphs: an 'inner' subgraph consisting of $H$ and the edges inside $H$, and an 'outer' subgraph consisting of $H$ and the edges outside of $H$. At this point, it is tempting to apply Lemma 4.2 twice, once on the inner subgraph to obtain one dominant vertex, and then on the outer subgraph to obtain a second dominant vertex, thus reaching the standard triangulation. However, Lemma 4.2 cannot be applied directly since we need to take into consideration the interaction between these two outerplane subgraphs. The main problem is that an internal edge in the inner subgraph may be blocked by an edge in the outer subgraph. The bulk of this section is dedicated to solving this impasse.

First some definitions. A *chord* of a cycle $C$ in a triangulation $G$ is an edge of $G$ that is not in $C$ and whose endpoints are both in $C$. A chord $e$ of $C$ is classified as *internal* or *external* depending on whether $e$ is contained in the interior or exterior of $C$ (with respect to the outerface of $G$). For our inductive arguments to work we need to consider a more general type of cycle than a Hamiltonian cycle. A cycle $C$ of a triangulation $G$ is *empty* if the interior of $C$ contains no vertices of

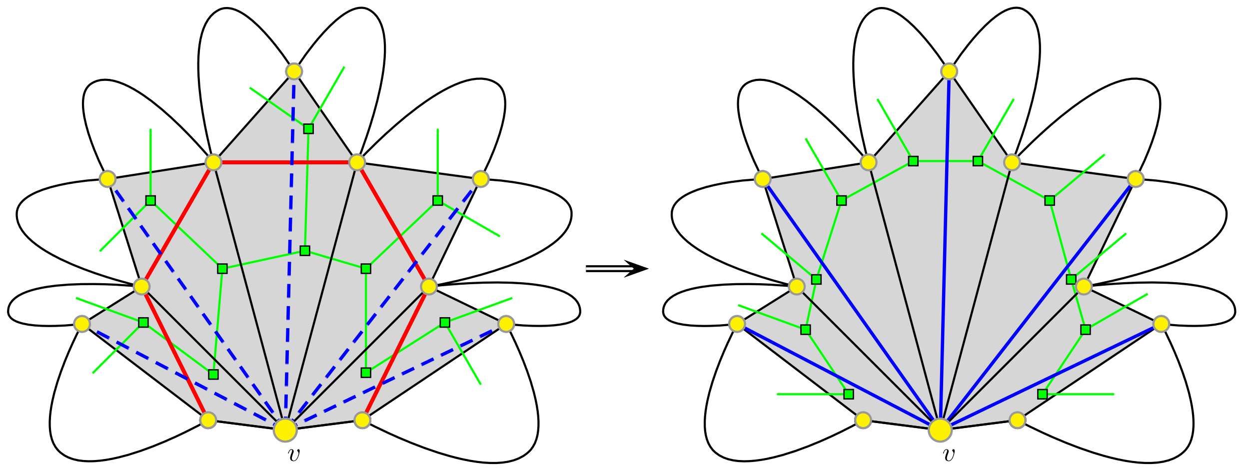


Figure 4: Making $v$ a dominant vertex in Lemma 4.4.

$G$. Obviously a Hamiltonian cycle is empty. For an empty cycle $C$ of a triangulation $G$, let $G\{C\}$ denote the subgraph of $G$ whose vertices are the vertices of $C$, and whose edges are the edges of $C$ along with the internal chords of $C$. Then $G\{C\}$ is a maximal outerplane graph, and the boundary of the outerface of $G\{C\}$ is $C$. The following elementary lemma is proved in the full paper [2].

Lemma 5.1. *Let $C$ be an empty cycle of a triangulation $G \neq K_4$. Let $vw$ be an internal chord of $C$ that is blocked by some edge $pq$. Then $pq$ is an external chord of $C$ that is flippable in $G$.*

Lemma 5.2. *Let $C$ be an empty cycle of a triangulation $G$. Let $S$ be a set of internal chords of $C$, no two of which are consecutive. Then there is a flippable set $T$ of edges in $G$ such that* (a) $T \cap C = \emptyset$, (b) $|S \cap T| \geq \frac{1}{3}|S|$, *and* (c) *every edge in $T \setminus S$ is an external chord of $C$ and $|T \setminus S| \leq |S \cap T|$.*

*Proof.* Let $S'$ be the set of edges in $S$ that are individually flippable in $G$. Let $S'' := S \setminus S'$. By Lemma 5.1, there is an external chord that blocks each edge $e \in S''$. Distinct edges $e_1, e_2 \in S''$ are blocked by distinct external chords, as otherwise $e_1$ and $e_2$ would be a bad pair, and the outerplane graph $G\{C\}$ would contain a subdivision of $K_4$. Let $B$ be this set of blocking external chords. Thus $|B| = |S''|$. By Lemma 3.5, $B$ can be 3-coloured such that no two monochromatic edges in $B$ are consecutive in $G$. Let $P$ be the largest set of monochromatic edges in $B$. Then $|P| \geq \frac{1}{3}|B|$. Let $Q$ be the set of edges in $S''$ that are blocked by edges in $P$. Then $|Q| = |P|$. Let $T := S' \cup P \cup Q$. It is straightforward to verify each of the conditions of Lemma 2.1 for $T$. Thus $T$ is flippable. Observe that $T \cap C = \emptyset$. This proves (a). Now $T \cap S = S' \cup Q$. Since $S' \cap Q = \emptyset$, we have $|T \cap S| = |S'| + |Q| = |S'| + |P| \geq |S'| + \frac{1}{3}|B| \geq \frac{1}{3}|S'| + \frac{1}{3}|S''| = \frac{1}{3}|S|$. This proves (b). Now $T \setminus S = P$, all of whose elements are external chords. Since $|S \cap T| = |S'| + |P|$, we have $|P| \leq |S \cap T|$. Since $T \setminus S = P$, we have $|T \setminus S| \leq |S \cap T|$. This proves (c). □

The following result extends Lemma 4.3 for outerplane graphs to the case of triangulations.

Lemma 5.3. *Let $G$ be a triangulation, and let $C$ be an empty cycle of $G$ with $n$ vertices. ($G$ may have more than $n$ vertices.) Then $G$ can be transformed by a sequence of at most $c_2 \log n$ simultaneous flips into a triangulation $X$ in which $C$ is an empty cycle and the diameter of $X\{C\}$ is at most $c_2 \log n$. Moreover, every edge of $G$ that is incident to a vertex not in $C$ remains in $X$.*

*Proof.* We proceed by induction on $n$. The result holds trivially for $n = 3$. Now $G\{C\}$ is maximal outerplane. As in Lemma 4.3, construct a degree-4 independent set $I$ of $G\{C\}$, define $I_d := \{v \in I : \deg_{G\{C\}}(v) = d\}$ for $d \in \{2, 3, 4\}$, and construct a set $S$ of non-consecutive internal edges of $G\{C\}$ such that there is exactly one internal edge in $S$ incident to every vertex $v \in I_3 \cup I_4$. By Lemma 5.2, there is a flippable set $T$ of edges in $G$, such that $T \cap C = \emptyset$ and $|S \cap T| \geq \frac{1}{3}|S| = \frac{1}{3}(|I_3| + |I_4|)$. Moreover, every edge in $T \setminus S$ is an external chord of $C$ in $G$. For $d \in \{3, 4\}$, let $I'_d$ be the set of vertices in $I_d$ incident to an edge in $S \cap T$. Thus $|I'_3| + |I'_4| \geq \frac{1}{3}(|I_3| + |I_4|)$.

Let $G' := G\langle T \rangle$. Since $T \cap C = \emptyset$, $C$ is an empty cycle of $G'$. Every vertex $v \in I_2 \cup I'_3$ has $\deg_{G'\{C\}}(v) =$

2. Every vertex $v \in I'_4$ has $\deg_{G'\{C\}}(v) = 3$. An edge in $G'\{C\}$ that is incident to a vertex in $I'_4$ is also in $G\{C\}$. Since $I'_4$ is an independent set of $G\{C\}$, it is also an independent set of $G'\{C\}$. Let $S'$ be the set of internal chords of $C$ in $G'$ that are incident to a vertex in $I'_4$. Thus $|S'| = |I'_4|$, and by the same argument used for $S$, no two edges in $S'$ are consecutive in $G'$. By Lemma 5.2, there is a flippable set of edges $T'$ in $G'$, such that $T' \cap C = \emptyset$ and $|S' \cap T'| \geq \frac{1}{3}|S'| = \frac{1}{3}|I'_4|$. Moreover, every edge in $T' \setminus S'$ is an external chord of $C$ in $G'$. Let $I''_4$ be the set of vertices in $I'_4$ incident to an edge in $S' \cap T'$. Thus $|I''_4| \geq \frac{1}{3}|I'_4|$. Let $G'' := G'\langle T' \rangle$. Since $T' \cap C = \emptyset$, $C$ is an empty cycle of $G''$. Every vertex $v \in I_2 \cup I'_3 \cup I''_4$ has $\deg_{G''\{C\}}(v) = 2$. Now $|I_2 \cup I'_3 \cup I''_4| \geq |I_2| + |I'_3| + \frac{1}{3}|I'_4| \geq |I_2| + \frac{1}{3}(|I'_3| + |I'_4|) \geq |I_2| + \frac{1}{9}(|I_3| + |I_4|) \geq \frac{1}{9}(|I_2| + |I_3| + |I_4|) = \frac{1}{9}|I| \geq \frac{n}{54}$.

In summary, $G$ can be transformed by two simultaneous flips into a triangulation $G''$ in which $C$ is an empty cycle, and $G''\{C\}$ has an independent set $L$ $(= I_2 \cup I'_3 \cup I''_4)$ such that $|L| \geq \frac{n}{54}$ and $\deg_{G''\{C\}}(v) = 2$ for every vertex $v \in L$. Consider a vertex $v \in L$. Say $(u, v, w)$ is the 2-edge path in $C$. Since $L$ is independent, $u \notin L$ and $w \notin L$. Since $\deg_{G''\{C\}}(v) = 2$, $uw$ is an internal chord of $C$ in $G''$. Let $D$ be the cycle of $G$ obtained by replacing the path $(u, v, w)$ in $C$ by the edge $uw$ (for all $v \in L$). Thus $D$ is an empty cycle of $G''$, and $|D| = n - |L| \leq \frac{53}{54}n$. By induction applied to $D$ and $G''$, $G''$ can be transformed by a sequence of at most $c_2 \log \frac{53}{54}n$ simultaneous flips into a triangulation $X$ in which $D$ is an empty cycle and the diameter of $X\{D\}^*$ is at most $c_2 \log \frac{53}{54}n$. Moreover, every edge of $G''$ that is incident to a vertex not in $D$ remains in $X$.

Consider a vertex $v \in L$. Say $(u, v, w)$ is the 2-edge path in $C$. Since $v$ is not in $D$, the edges $uv$ and $vw$ of $G$ are in $X$. Thus $C$ is an empty cycle of $X$. Since $uw$ is an edge of $D$, $uvw$ is a face of $X$. The vertex in the dual tree $X\{C\}^*$ that corresponds to $uvw$ is a leaf in $X\{C\}^*$. Thus the dual tree $X\{C\}^*$ is obtained by adding leaves to the dual tree $X\{D\}^*$. Hence the diameter of $X\{C\}^*$ is at most the diameter of $X\{D\}^*$ plus two, which is at most $2 + c_2 \log \frac{53}{54}n = c_2 \log n$. We have used two simultaneous flips, $T$ and $T'$, to transform $G$ into $G''$, and then $c_2 \log \frac{53}{54}n$ simultaneous flips to transform $G''$ into $X$. The total number of flips is $2 + c_2 \log \frac{53}{54}n = c_2 \log n$. Since every edge in $T$ is a chord of $C$ in $G$, and every edge in $T'$ is a chord of $C$ in $G'$, every edge of $G$ that is incident to a vertex not in $C$ remains in $X$. $\square$

*Proof of Theorem 5.1.* Let $G := G_1$. First apply Theorem 3.3 to transform $G$ with one flip into a triangulation containing a Hamiltonian cycle $H$. Then apply Lemma 5.3 (with $C = H$) to transform $G$ with $c_2 \log n$ flips into a triangulation in which $H$ is a Hamiltonian cycle and the diameter of $G\{H\}^*$ is at most $c_2 \log n$. There is a vertex $v$ of $G$ not incident to any external chords of $H$. By a similar proof to that of Lemma 4.4, we can make $v$ dominant in $G$ with $c_2 \log n$ flips. Observe that $G \setminus v$ is a maximal outerplane graph, in which the vertices are ordered on the outerface according to the cyclic order of the neighbours of $v$. Let $C$ be the cycle bounding the outerface of $G \setminus v$. By Lemma 4.2 there is a sequence of at most $2c_1 \log(n-1)$ simultaneous flips to transform $G \setminus v$ into a maximal outerplane graph with a dominant vertex. Each of these flips is valid in $G$ since $C$ has no internal chords (cf. Lemma 5.2). Thus $G$ now has two dominant vertices; that is, $G$ is the standard triangulation $\Delta_n$. The number of flips is at most $1+2(c_1+c_2)\log n$. Finally compute an analogous sequence of flips to transform $G_2$ into $\Delta_n$, and apply them in reverse order. $\square$

Although each of the $\mathcal{O}(\log n)$ simultaneous flips in Theorem 5.1 may involve a linear number of edges, the total number of flipped edges is linear [2].

## 6 Large Simultaneous Flips

Let $\mathsf{msf}(G)$ denote the maximum cardinality of a flippable set of edges in a triangulation $G$. In related work, Gao et al. [10] proved that every triangulation has at least $n-2$ (individually) flippable edges, and every triangulation with minimum degree four has at least $2n+3$ (individually) flippable edges. Galtier et al. [9] proved that every geometric triangulation has a set of at least $\frac{1}{6}(n-4)$ simultaneously flippable edges. In this section we prove bounds on $\mathsf{msf}(G)$. Our main contribution is the following lower bound.

THEOREM 6.1. *For every triangulation $G$ with $n \geq 4$ vertices,* $\mathsf{msf}(G) \geq \frac{1}{3}(n-2)$.

*Proof.* Let $G$ be a (vertex) minimum counterexample with $n$ vertices. It is easily seen that $n \geq 7$. By Lemma 3.5, there is a 3-colouring $\{E_1, E_2, E_3\}$ of the edges of $G$ such that every triangle is trichromatic. Let $S_i$ be set of edges in $E_i$ that are not in a bad pair with another edge in $E_i$. We claim that each $S_i$ is flippable. Since every triangle is trichromatic, no two edges in $S_i$ are consecutive. This is condition (1) in Lemma 2.1. Condition (2) in Lemma 2.1 holds by the definition of $S_i$. Suppose that an edge $ab \in S_i$ is blocked by an edge $vw$. To show that condition (3) of Lemma 2.1 is satisfied, we need to prove that $vw \in S_i$. First suppose that $vw \notin E_i$. Since $(v, a, w)$ is a triangle, one of $av$ and $bv$ is in $E_i$, which implies that this edge and $ab$ are consecutive and both in $E_i$. This contradiction proves that $vw \in E_i$. Now suppose that $vw$ and some edge $xy$

form a bad pair. By Lemma 2.5, $vw$ and $xy$ are in a common triangle. Thus $xy \notin E_i$ and $vw$ does not form a bad pair with another edge in $E_i$. Therefore $vw \in S_i$ as desired. By Lemma 2.1, $S_i$ is flippable.

We now prove that every face has at least one edge in $S_1 \cup S_2 \cup S_3$. The neighbours of a degree-3 vertex form a separating triangle. In the full paper [2] we prove that every edge in a separating triangle is in $S_1 \cup S_2 \cup S_3$. Thus every face incident to a degree-3 vertex has an edge in $S_1 \cup S_2 \cup S_3$. In the full paper [2] we prove that in a minimum counterexample, every edge seen by a degree-4 vertex is not in a bad pair. Thus every face incident to a degree-4 vertex has an edge in $S_1 \cup S_2 \cup S_3$. In the full paper [2] we prove that every face not incident to degree-3 or degree-4 vertex has an edge that is not in a bad pair, and is thus in $S_1 \cup S_2 \cup S_3$. Thus every face has an edge in $S_1 \cup S_2 \cup S_3$. There are $2(n-2)$ faces and every edge is in two faces. Thus $|S_1 \cup S_2 \cup S_3| \geq n-2$, and $|S_i| \geq \frac{1}{3}(n-2)$ for some $i$. Therefore $G$ is not a counterexample, and since $G$ was minimum, there are no counterexamples. □

It is easily seen that $\mathsf{msf}(G) \leq n-2$ for every $n$-vertex triangulation $G$. We have the following existential upper bound.

Lemma 6.1. *There exist $n$-vertex triangulations $G$ with $\mathsf{msf}(G) = \frac{6}{7}(n-2)$ for infinitely many $n$.*

*Proof.* Let $G$ be the triangulation obtained from an arbitrary triangulation $G_0$ by adding a triangle inside each face $(u, v, w)$, each vertex of which is adjacent to two of $\{u, v, w\}$. As in Figure 5(a)–(c), a straightforward case-analysis [2] shows that for every face of $G_0$, at least one of the seven corresponding faces of $G$ does not have an edge in any flippable set $S$. It follows that $|S| \leq \frac{6}{7}(n-2)$. As in Figure 5(d), a flippable set of $\frac{6}{7}(n-2)$ edges in $G$ is easily constructed [2]. □

An obvious open problem is to close the gap between the lower bound of $\frac{1}{3}(n-2)$ and the upper bound of $\frac{6}{7}(n-2)$ in the above results. In the full paper [2] we improve the lower bound for 5-connected triangulations $G$ to $\mathsf{msf}(G) = n-2$.

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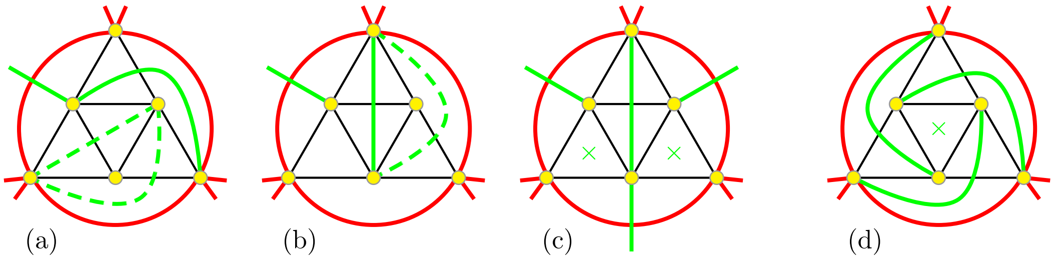

Figure 5: (a)–(c) For any number of flips in the outer triangle, at least one internal face does not have an edge in $S$. (d) How to construct a flip set for $G$.


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