The Maximum Number of Edges in a Three-Dimensional Grid-Drawing*

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Abstract

An exact formula is given for the maximum number of edges in a graph that admits a three-dimensional grid-drawing contained in a given bounding box.

A *three-dimensional* (*straight-line*) *grid-drawing* of a graph represents the vertices by distinct points in $\mathbb{Z}^3$, and represents each edge by a line-segment between its endpoints that does not intersect any other vertex, and does not intersect any other edge except at the endpoints. A folklore result states that every (simple) graph has a three-dimensional grid-drawing (see [2]). We therefore are interested in grid-drawings with small 'volume'.

The *bounding box* of a three-dimensional grid-drawing is the axis-aligned box of minimum size that contains the drawing. By an $X \times Y \times Z$ *grid-drawing* we mean a three-dimensional grid-drawing, such that the edges of the bounding box contain X, Y, and Z grid-points, respectively. The *volume* of a three-dimensional grid-drawing is the number of grid-points in the bounding box; that is, the volume of an $X \times Y \times Z$ grid-drawing is XYZ. (This definition is formulated to ensure that a two-dimensional grid-drawing has positive volume.) Our main contribution is the following extremal result.

Theorem 1. *The maximum number of edges in an $X \times Y \times Z$ grid-drawing is exactly*

$$(2X-1)(2Y-1)(2Z-1) - XYZ \ .$$

Proof. Consider an $X \times Y \times Z$ grid-drawing of a graph G with n vertices and m edges. Let P be the set of points (x, y, z) contained in the bounding box such that $2x$, $2y$, and $2z$ are all integers. Observe that $|P| = (2X-1)(2Y-1)(2Z-1)$. The midpoint of every edge of G is in P, and no two edges share a common midpoint. Hence $m \leq |P|$. In addition, the midpoint of an edge does not intersect a vertex. Thus

$$m \leq |P| - n \ . \tag{1}$$

A drawing with the maximum number of edges has no edge that passes through a grid-point. Otherwise, sub-divide the edge, and place the new vertex at that grid-point. Thus $n = XYZ$, and $m \leq |P| - XYZ$, as claimed.

This bound is attained by the following construction. Associate a vertex with each grid-point in an $X \times Y \times Z$ grid-box B. As illustrated in Figure 1, every vertex (x, y, z) is adjacent to each of $(x \pm 1, y, z)$, $(x, y \pm 1, z)$, $(x, y, z \pm 1)$, $(x \pm 1, y \pm 1, z)$, $(x \pm 1, y, z \pm 1)$, $(x, y \pm 1, z \pm 1)$, and $(x \pm 1, y \pm 1, z \pm 1)$, unless such a grid-point is not in B. It is easily seen that no two edges intersect, except at a common endpoint. Furthermore, every point in P is either a vertex or the midpoint of an edge. Thus the number of edges is $|P| - XYZ$. □

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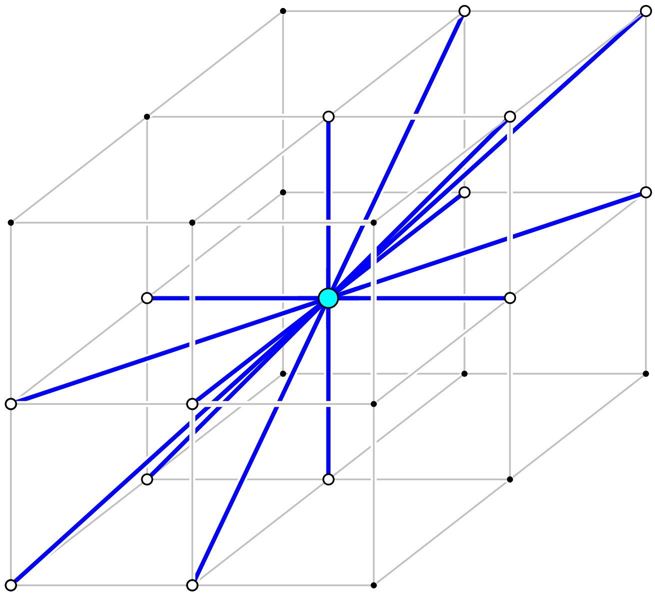

Figure 1: The neighbourhood of a vertex.

Theorem 1 can be interpreted as a lower bound on the volume of a three-dimensional grid-drawing of a given graph. Many upper bounds on the volume of three-dimensional grid-drawings are known [1–9]. There are two known non-trivial lower bounds for specific families of graphs. Cohen, Eades, Lin, and Ruskey [2] proved that the minimum volume of a three-dimensional grid-drawing of the complete graph K_n is $\Theta(n^3)$. The lower bound follows from the fact that K_5 is not planar, and hence at most four vertices can lie in a single grid-plane. The second lower bound is due to Pach, Thiele, and Tóth [7], who proved that the minimum volume of a three-dimensional grid-drawing of the complete bipartite graph $K_{n,n}$ is $\Theta(n^2)$. The proof of the lower bound is based on the observation that no two edges are parallel. The result follows since the number of distinct vectors between adjacent vertices is at most a constant times the volume. The following corollary of Theorem 1 generalises this lower bound for $K_{n,n}$ to all graphs.

Corollary 1. *A three-dimensional grid-drawing of a graph with n vertices and m edges has volume greater than $\frac{m+n}{8}$.*

Proof. Let v be the volume of an $X \times Y \times Z$ grid-drawing. By (1), $m \leq |P| - n < 8v - n$, and hence $v > \frac{m+n}{8}$. □

Obviously the bound in Corollary 1 can be slightly improved by considering the exact dimensions of the bounding box. Theorem 1 generalises to multi-dimensional polyline grid-drawings (where edges may bend at grid-points) as follows.

Theorem 2. *Let $B \subseteq \mathbb{R}^d$ be a convex set. Let $S = B \cap \mathbb{Z}^d$ be the set of grid-points in B. The maximum number of edges in a polyline grid-drawing with bounding box B is at most $(2^d - 1)|S|$. If B is an $X_1 \times \cdots \times X_d$ grid-box, then the maximum number of edges is exactly*

$$\prod_{i=1}^{d}(2X_i - 1) - \prod_{i=1}^{d} X_i \ .$$

Proof. Let $P = \{x \in B : 2x \in \mathbb{Z}^d\}$. Consider a polyline grid-drawing with bounding box B. The midpoint of every edge is in P, and no two edges share a common midpoint. A drawing with the maximum number of edges has no edge that passes through a grid-point. Otherwise, sub-divide the edge, and place the new vertex at that grid-point. Thus the number of edges is at most $|P| - |S| \leq (2^d - 1)|S|$.

If B is an $X_1 \times \cdots \times X_d$ grid-box, then $|P| - |S| = \prod_{i=1}^{d}(2X_i - 1) - \prod_{i=1}^{d} X_i$. To construct a grid-drawing with this many edges, associate one vertex with each grid-point in S. Observe that every point $x \in P \setminus S$ is in the interior of exactly one unit-sized d'-dimensional hypercube with corners in S, where $1 \leq d' \leq d$. For every point $x \in P \setminus S$, add an edge passing through x between opposite vertices of the unit-sized hypercube corresponding to x. Edges only intersect at common endpoints, since these unit-sized hypercubes only intersect along their boundaries. Every point in P contains a vertex or a midpoint of an edge. Thus the number of edges is precisely $|P| - |S|$. ☐

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