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Partitions of complete geometric graphs into plane trees ☆

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Abstract

Consider the following question: does every complete geometric graph K_{2n} have a partition of its edge set into n plane spanning trees? We approach this problem from three directions. First, we study the case of convex geometric graphs. It is well known that the complete convex graph K_{2n} has a partition into n plane spanning trees. We characterise all such partitions. Second, we give a sufficient condition, which generalises the convex case, for a complete geometric graph to have a partition into plane spanning trees. Finally, we consider a relaxation of the problem in which the trees of the partition are not necessarily spanning. We prove that every complete geometric graph K_n can be partitioned into at most $n - \sqrt{n/12}$ plane trees. This is the best known bound even for partitions into plane subgraphs.

Keywords: Geometric graph; Complete graph; Plane tree; Convex graph; Book embedding; Book thickness; Crossing family

1. Introduction

A *geometric graph* G is a pair $(V(G), E(G))$ where $V(G)$ is a set of points in the plane in general position (that is, no three are collinear), and $E(G)$ is set of closed segments with endpoints in $V(G)$. Elements of $V(G)$ are *vertices* and elements of $E(G)$ are *edges*. An edge with endpoints v and w is denoted by $\{v, w\}$ or vw when convenient. A geometric graph can be thought of as a straight-line drawing of its underlying (abstract) graph. A geometric graph

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is *plane* if no two edges cross. A *tree* is an acyclic connected graph. A subgraph H of a graph G is *spanning* if $V(H) = V(G)$. We are motivated by the following question.

Problem 1. Does every complete geometric graph with an even number of vertices have a partition of its edge set into plane spanning trees?

Since K_n, the complete graph on n vertices, has $\frac{1}{2}n(n-1)$ edges and a spanning tree has $n-1$ edges, there are $\frac{1}{2}n$ trees in such a partition, and n is even. We approach this problem from three directions. In Section 2 we study the case of convex geometric graphs. We characterise the partitions of the complete convex graph into plane spanning trees. Section 3 describes a sufficient condition, which generalises the convex case, for a complete geometric graph to have a partition into plane spanning trees. In Section 4 we consider a relaxation of Problem 1 in which the trees of the partition are not necessarily spanning.

It is worth mentioning that decompositions of (abstract) graphs into trees have attracted much interest. In particular, Tutte [13] and Nash–Williams [11] independently obtained necessary and sufficient conditions for a graph to admit k edge-disjoint spanning trees, and Ringel's conjecture and the graceful tree conjecture about ways of decomposing complete graphs into trees are among the most outstanding open problems in the field. Nevertheless the non-crossing property that we require in our geometric setting changes the problems drastically.

2. Convex graphs

A *convex graph* is a geometric graph with the vertices in convex position. A *k-page book embedding* of a graph G consists of a representation of G as a convex graph, and a partition of $E(G)$ into k plane subgraphs called *pages*. The *book thickness* of G is the minimum integer k for which there is a k-page book embedding of G. See reference [6] for numerous references on this topic. Berhnart and Kainen [4] proved that the book thickness of K_{2n} equals n. In fact, they proved that the convex graph K_{2n} can be partitioned into n plane spanning paths, thus solving Problem 1 in the affirmative in the convex case (see Fig. 1).

In this section we characterise the solutions to Problem 1 in the convex case. In other words, we characterise the book embeddings of the complete graph in which every page is a spanning tree.

First some standard definitions and terms. We use the interval notation $[a, b]$ to denote the set $\{a, a+1, \ldots, b\}$ for all integers $a \leqslant b$.

An edge on the convex hull of a convex graph is called a *boundary edge*. Two convex graphs are *isomorphic* if the underlying graphs are isomorphic and the clockwise ordering of the vertices around the convex hull is preserved under this isomorphism. Suppose that G_1 and G_2 are isomorphic convex graphs. Then two edges cross in G_1 if and only if the corresponding edges in G_2 also cross. That is, in a convex graph, it is only the order of the vertices around the convex hull that determines edge crossings—the actual coordinates of the vertices are not important.

A *leaf* of a tree is a vertex of degree at most one. A *leaf-edge* of a tree is an edge incident to a leaf. A tree has exactly one leaf if and only if it is a single vertex with no edges. Every tree with at least one edge has at least two leaves. A tree has exactly two leaves if and only if it is a path with at least one edge. Let T be a tree. Let T' be the tree obtained by deleting the leaves and leaf-edges from T. Let $\ell(T)$ be the number of leaves in T'. A *star* is a tree with at most one non-leaf vertex. Clearly a tree T is a star if and only if $\ell(T) \leqslant 1$. A *caterpillar* is a tree T such that T' is a path. The path T' is called the *spine* of the caterpillar. Clearly T is a caterpillar if and only if $\ell(T) \leqslant 2$. Observe that stars are the caterpillars whose spines consist of a single vertex.

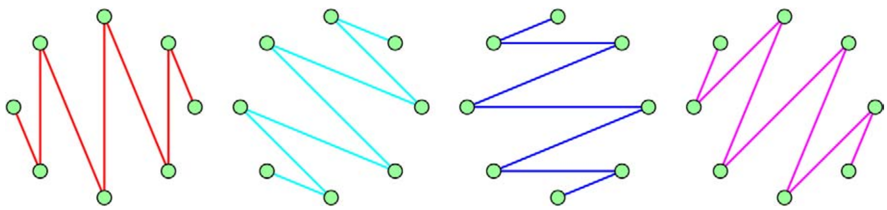

Fig. 1. Partition of the convex K_8 into four spanning paths.

We say a tree T is *symmetric* if there exists an edge vw of T such that if A and B are the components of $T \setminus vw$ with $v \in A$ and $w \in B$, then there exists a graph-isomorphism between A and B that maps v to w.

We can now state the main result of this section.

Theorem 2. *Let $T_1, T_2, \ldots, T_n$ be a partition of the edges of the convex complete graph K_{2n} into plane spanning trees. Then $T_1, T_2, \ldots, T_n$ are symmetric convex caterpillars that are pairwise isomorphic. Conversely, for any symmetric convex caterpillar T on $2n$ vertices, the edges of the convex complete graph K_{2n} can be partitioned into n plane spanning convex copies of T that are pairwise isomorphic.*

We prove Theorem 2 by a series of lemmas, starting with the following result of García et al. [9].

Lemma 3. [9] *Let T be a tree with at least two edges. In every plane convex drawing of T there are at least $\max\{2, \ell(T)\}$ boundary edges. Moreover, if T is not a star, then every plane convex drawing of T has at least two non-consecutive boundary edges.*

In what follows $\{0, 1, \ldots, 2n-1\}$ are the vertices of a convex graph G in clockwise order around the convex hull. All vertices are taken modulo $2n$. That is, vertex i refers to the vertex $i \bmod 2n$. Let $G[i, j]$ denote the subgraph of G induced by the vertices $[i, j]$ if $i < j$, and by $[j, 2n-1] \cup [0, i]$ if $j < i$.

Lemma 4. *For all $n \geqslant 2$, let $T_0, T_1, \ldots, T_{n-1}$ be a partition of the convex complete graph K_{2n} into plane spanning trees. Then (after relabelling the trees) for each $i \in [0, n-1]$,*

(1) *the edge $\{i, n+i\}$ is in T_i,*
(2) *T_i is a caterpillar with exactly two boundary edges, and*
(3) *for every non-boundary edge $\{a, b\}$ of T_i, there is exactly one boundary edge of T_i in each of $T_i[a, b]$ and $T_i[b, a]$.*

Proof. The edges $\{\{i, n+i\}: 0 \leqslant i \leqslant n-1\}$ are pairwise crossing. Thus each such edge is in a distinct tree. Label the trees such that each edge $\{i, n+i\}$ is in T_i. Since $n \geqslant 2$, each T_i has at least three edges, and by Lemma 3, has at least two boundary edges. There are $2n$ boundary edges in total and n trees. Thus each T_i has exactly two boundary edges, and by Lemma 3, $\ell(T_i) \leqslant 2$. For any tree T, $\ell(T) \leqslant 2$ if and only if T is a caterpillar. Thus each T_i is a caterpillar. Let $\{a, b\}$ be a non-boundary edge in some T_i. Then $T_i[a, b]$ has at least one boundary edge of T, as otherwise $T_i[a, b]$ would be a convex tree on at least three vertices with only one boundary edge (namely, $\{a, b\}$), which contradicts Lemma 3. Similarly $T_i[b, a]$ has at least one boundary edge of T. Thus each of $T_i[a, b]$ and $T_i[b, a]$ has exactly one boundary edge of T. □

Lemma 5. *Let $\{i, j\}$ be a non-boundary edge of a plane convex spanning tree T such that $T[i, j]$ has exactly one boundary edge of T. Then exactly one of $\{i, j-1\}$ and $\{j, i+1\}$ is an edge of T.*

Proof. If both $\{i, j-1\}$ and $\{j, i+1\}$ are in T then they cross, unless $j-1 = i+1$ in which case T contains a 3-cycle. Thus at most one of $\{i, j-1\}$ and $\{j, i+1\}$ is in T. Suppose, for the sake of contradiction, that neither $\{i, j-1\}$ nor $\{j, i+1\}$ are edges of T. Since T is spanning, there is an edge $\{i, a\}$ or $\{j, a\}$ in T for some vertex $i+1 < a < j-1$. Without loss of generality $\{i, a\}$ is this edge, as illustrated in Fig. 2.

The subtree $T[i, a]$ has at least three vertices i, $i+1$, and a. By Lemma 3, $T[i, a]$ has at least two boundary edges, one of which is $\{i, a\}$. Thus $T[i, a]$ has at least one boundary edge that is also a boundary edge of T. Now consider the subtree T' of T induced by $\{i\} \cup [a, j]$. Then T' has at least four vertices i, a, $j-1$, and j. Since $\{i, j-1\}$ is not an edge of T, and thus not an edge of T', the subtree T' is not a star. By Lemma 3, T' has at least two non-consecutive boundary edges, at most one of which is $\{i, j\}$ or $\{i, a\}$. Thus T' has at least one boundary edge that is also a boundary edge of T. No boundary edge of T can be in both $T[i, a]$ and T'. Thus we have shown that $T[i, j]$ has at least two boundary edges of T, which is the desired contradiction. □

In what follows we say an edge $e = \{i, j\}$ has *span*

$$\operatorname{span}(e) = \min\{(i-j) \bmod 2n, (j-i) \bmod 2n\}.$$

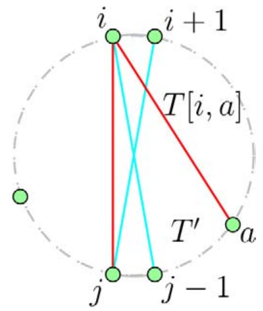

Fig. 2. One of $\{i, j-1\}$ and $\{j, i+1\}$ is an edge of T.

That is, span(e) is the number of edges in a shortest path between i and j that is contained in the boundary of the convex hull.

Lemma 6. *Let $\{i, j\}$ be an edge of a plane convex spanning tree T such that $1 \leqslant j-i \leqslant n$, and $T[i, j]$ has exactly one boundary edge of T. Then $T[i, j]$ has exactly one edge of span k, for each $k \in [1, j-i]$. Moreover, for each $k \in [2, j-i]$, the edge of span k has an endpoint in common with the edge of span $k-1$, and the other two endpoints are consecutive on the convex hull.*

Proof. If $j-i=1$ then $\{i, j\}$ is a boundary edge, and the result is trivial. Otherwise $\{i, j\}$ is not a boundary edge. By Lemma 5, exactly one of the edges $\{i, j-1\}$ and $\{j, i+1\}$ is in T. Without loss of generality $\{i, j-1\}$ is in T. Thus the edge of span $j-i$ has an endpoint in common with the edge of span $j-i-1$, and the other two endpoints are consecutive on the convex hull. The result follows by induction (on span) applied to the edge $\{i, j-1\}$. □

Let $e = \{a, b\}$ be an edge in the convex complete graph K_{2n}. Then $e+i$ denotes the edge $\{a+i, b+i\}$. For a set X of edges, $X + i = \{e + i\colon\ e \in X\}$, and $X^{(k)} = \{e \in X, \operatorname{span}(e) \geqslant k\}$.

Lemma 7. *Let $T_0, T_1, \dots, T_{n-1}$ be a partition of the edges of the convex complete graph K_{2n} into plane spanning convex trees. Then $T_0, T_1, \dots, T_{n-1}$ are pairwise isomorphic symmetric convex caterpillars.*

Proof. By Lemma 4, for each $i \in [0, n-1]$, T_i is a caterpillar with two boundary edges, the edge $\{i, n+i\}$ is in T_i, and for every non-boundary edge $\{a, b\}$ of T_i, there is exactly one boundary edge of T_i in each of $T_i[a, b]$ and $T_i[b, a]$.

Let $H = T_0[0, n]$. Since $\{0, n\}$ is an edge of H, by Lemma 6, H has exactly one edge of span k for each $k \in [1, n]$. Furthermore, for each $k \in [1, n-1]$, the edge of span k has an endpoint in common with the edge of span $k+1$, and the other two endpoints are consecutive on the convex hull. Let $h_k = \{x_k, x_k + k\}$ denote the edge of span k in H. For each $k \in [1, n-1]$, if $h_k \cap h_{k+1} = x_k + k$ $(= x_{k+1} + k + 1)$ then we say the *k-direction* is 'clockwise'. Otherwise, $h_k \cap h_{k+1} = x_k$ $(= x_{k+1})$, and we say the *k-direction* is 'anticlockwise', as illustrated in Fig. 3.

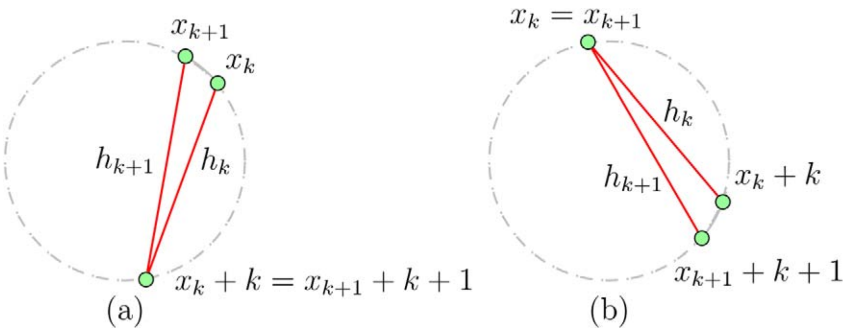

Fig. 3. k-direction is (a) clockwise and (b) anticlockwise.

We now prove that H determines the structure of all the trees $T_0, T_1, \ldots, T_{n-1}$. We proceed by downwards induction on $k = n, n-1, \ldots, 1$ with the hypothesis that for all $i \in [0, n-1]$,

$$T_i^{(k)} = (H^{(k)} + i) \cup (H^{(k)} + n + i). \tag{1}$$

Consider the base case with $k = n$. The only edge in H of span n is $\{0, n\}$. Thus $H^{(n)} = \{0, n\}$, which implies that $H^{(n)} + i = \{i, n+i\}$, and $H^{(n)} + n + i = \{n+i, 2n+i\} = \{i, n+i\}$. Thus the right-hand side of (1) is $\{i, n+i\}$. The only edge in T_i of span n is $\{i, n+i\}$. Thus $T_i^{(n)} = \{i, n+i\}$, and (1) is satisfied for $k = n$.

Now suppose that (1) holds for some $k+1 \geqslant 2$. We now prove that (1) holds for k. First suppose that the k-direction is clockwise. We proceed by induction on $j = 0, 1, \ldots, 2n-1$ with the hypothesis:

$$\text{the edge } \{x_k + j, x_k + k + j\} \text{ is in the tree } T_{j \bmod n}. \tag{2}$$

The base case with $j = 0$ is immediate since by definition, $\{x_k, x_k + k\} \in E(T_0)$. Suppose that $\{x_k + j, x_k + k + j\} \in E(T_{j \bmod n})$ for some $0 \leqslant j < 2n-1$. Consider the edge $e = \{x_k + j, x_k + k + j + 1\}$. Since the k-direction is clockwise, $x_k = x_{k+1} + 1$ and $x_k + k = x_{k+1} + k + 1$. Thus $e = \{x_{k+1} + 1 + j, x_{k+1} + k + 1 + j + 1\} = \{x_{k+1}, x_{k+1} + k + 1\} + j + 1 = h_{k+1} + j + 1$. Hence $e \in H + j + 1$, and since e has span $k+1$, we have $e \in H^{(k+1)} + j + 1$. By induction from (1), $e \in T^{(k+1)}_{(j+1) \bmod n}$, as illustrated in Fig. 4(a).

By Lemma 5 applied to e, which is a non-boundary edge of $T_{(j+1) \bmod n}$, exactly one of $\{x_k + j, x_k + k + j\}$ and $\{x_k + j + 1, x_k + k + j + 1\}$is an edge of $T_{(j+1) \bmod n}$. By induction from (2), $\{x_k + j, x_k + k + j\} \in T_{j \bmod n}$. Thus $\{x_k + j + 1, x_k + k + j + 1\} \in T_{(j+1) \bmod n}$. That is, (2) holds for $j+1$. Therefore for all $j \in [0, 2n-1]$, the edge $\{x_k + j, x_k + k + j\}$ is in $T_{j \bmod n}$. That is, $h_k + j$ is in $T_{j \bmod n}$. By (1) for $k+1$ we have that (1) holds for k. The case in which the k-direction is anticlockwise is symmetric; see Fig. 4(b).

By (1) with $k = 1$, each tree T_i can be expressed as $T_i = (H + i) \cup (H + n + i)$. Clearly $H \cup (H + n)$ is a symmetric convex caterpillar. Thus each T_i is a translated copy of the same symmetric convex caterpillar. Therefore $T_0, T_1, \ldots, T_{n-1}$ are pairwise isomorphic symmetric convex caterpillars. □

Fig. 5 illustrates the proof of Lemma 7.

Lemma 8. *For any symmetric convex caterpillar T on $2n$ vertices, the edges of the convex complete graph K_{2n} can be partitioned into n plane spanning pairwise isomorphic convex copies of T.*

Proof. Say $V(K_{2n}) = \{0, 1, \ldots, 2n-1\}$ in clockwise order around the convex hull. Let $\{0, n\}$ be the edge of T such that after deleting $\{0, n\}$, A and B are the components with $0 \in A$ and $n \in B$, and there exists a graph-isomorphism between A and B that maps 0 to n. It is easily seen that A has a plane representation on the vertices $[0, n-1]$. For each $i \in [0, n-1]$, let $T_i = (A + i) \cup (A + n + i)$ plus the edge $\{i, n+i\}$. Then as in Lemma 7, $T_0, T_1, \ldots, T_{n-1}$ is partition of K_{2n} into plane spanning pairwise isomorphic convex copies of T. □

Observe that Lemmas 7 and 8 together prove Theorem 2.

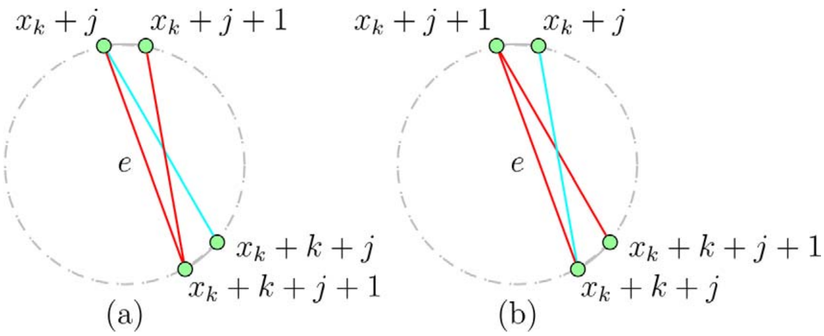

Fig. 4. k-direction is (a) clockwise and (b) anticlockwise.

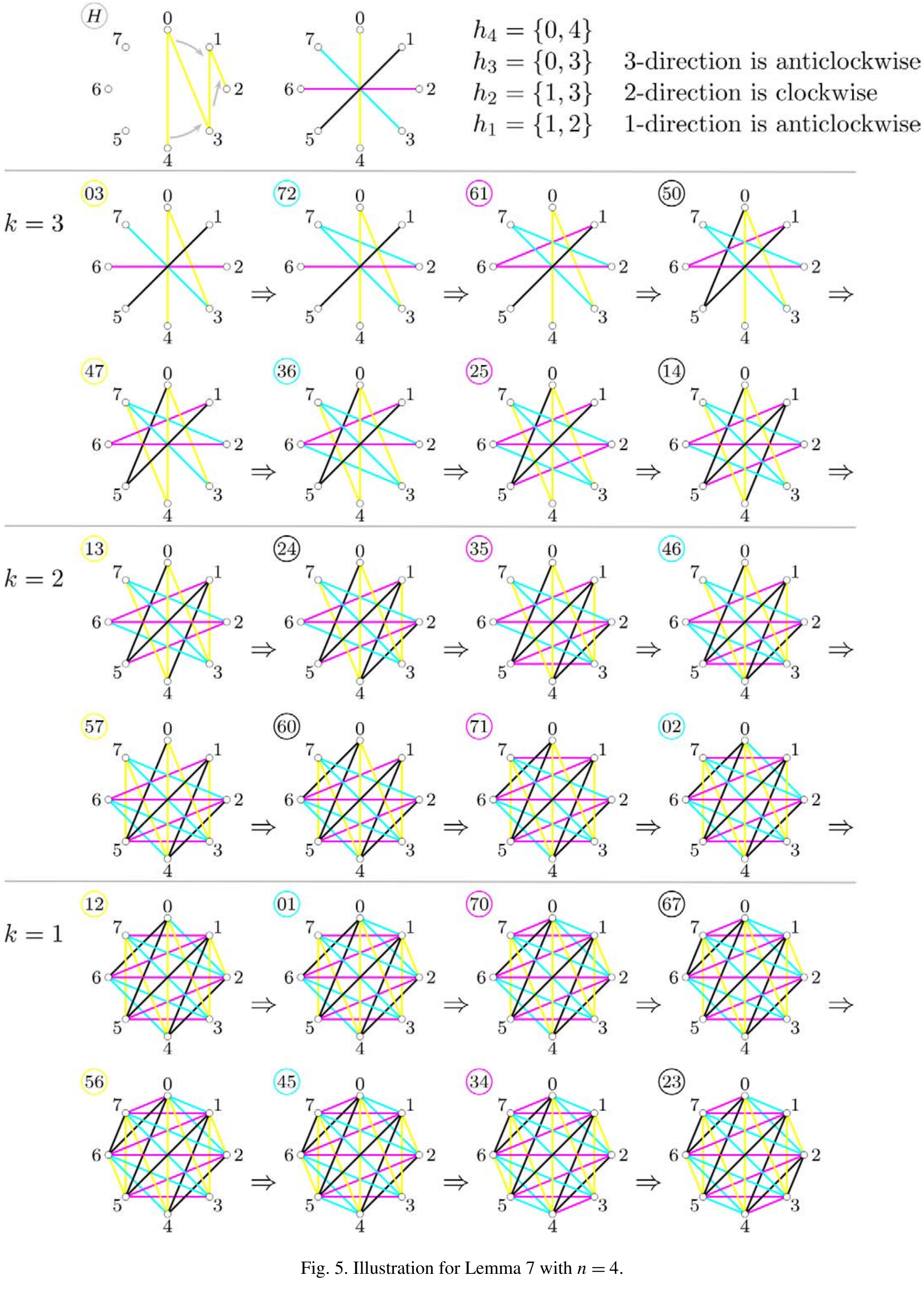

Fig. 5. Illustration for Lemma 7 with $n = 4$.

3. A sufficient condition

In this section we prove the following sufficient condition for a complete geometric graph to have an affirmative solution to Problem 1. A *double star* is a tree with at most two non-leaf vertices.

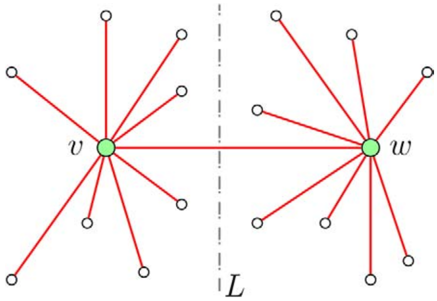

Fig. 6. Plane double star rooted at the edge vw and separated by the line L.

Theorem 9. *Let G be a complete geometric graph K_{2n}. Suppose that there is a set $\mathcal{L}$ of pairwise non-parallel lines with exactly one vertex of G in each open unbounded region formed by $\mathcal{L}$. Then $E(G)$ can be partitioned into n plane spanning double stars (that are pairwise graph-isomorphic).*

Observe that in a double star, if there are two non-leaf vertices v and w then they must be adjacent, in which case we say vw is the *root edge*.

Lemma 10. *Let P be a set of points in general position. Let L be a line with $L \cap P = \emptyset$. Let H_1 and H_2 be the half-planes defined by L. Let v and w be points such that $v \in P \cap H_1$ and $w \in P \cap H_2$. Let $T(P, L, v, w)$ be the geometric graph with vertex set P and edge set*

$$\{vw\} \cup \{vx\colon\ x \in (P \setminus \{v\}) \cap H_1\} \cup \{wy\colon\ y \in (P \setminus \{w\}) \cap H_2\}.$$

Then $T(P, L, v, w)$ is a plane double star with root edge vw.

Proof. The set of edges incident to v form a star. Regardless of the point set, a geometric star is always plane. Thus no two edges incident to v cross. Similarly no two edges incident to w cross. No edge incident to v crosses an edge incident to w since such edges are separated by L, as illustrated in Fig. 6. □

Lemma 11. *Let P be a set of points in general position. Let L_1 and L_2 be non-parallel lines with $L_1 \cap P = L_2 \cap P = \emptyset$. Let v, w, x, y be points in P such that v, w, x, y are in distinct quarter-planes formed by L_1 and L_2, with each pair (v, w) and (x, y) in opposite quarter-planes. (Note that this does not imply that vw and xy cross.) Let T_1 and T_2 be the plane double stars $T_1 = T(P, L_1, v, w)$ and $T_2 = T(P, L_2, x, y)$. Then $E(T_1) \cap E(T_2) = \emptyset$.*

Proof. Suppose, for the sake of contradiction, that there is an edge $e \in E(T_1) \cap E(T_2)$. All edges of T_1 are incident to v or w, and all edges of T_2 are incident to x or y. Thus $e \in \{vx, vw, vy, xw, xy, wy\}$. By assumption, v, w, x, y are in distinct quarter-planes formed by L_1 and L_2, with each pair (v, w) and (x, y) in opposite quarter-planes. Thus e crosses at least one of L_1 and L_2. Without loss of generality e crosses L_1. Since $e \in E(T_1)$, and the only edge of T_1 that crosses L_1 is the root edge vw, we have $e = vw$. Since all edges of T_2 are incident to x or y and v, w, x, y are distinct, we have $e \notin E(T_2)$, which is the desired contradiction. Therefore $E(T_1) \cap E(T_2) = \emptyset$, as illustrated in Fig. 7. □

We now prove the main result of this section.

Proof of Theorem 9. As illustrated in Fig. 8, let C be a circle such that the vertices of G and the intersection point of any two lines in $\mathcal{L}$ are in the interior of C. The intersection points of C and the lines in $\mathcal{L}$ partition C into $2n$ consecutive components $C_0, C_1, \ldots, C_{2n-1}$, each corresponding to a region containing a single vertex of G. Let i be the vertex in the region corresponding to C_i. Label the lines $L_0, L_1, \ldots, L_{n-1}$ so that for each $i \in [0, n-1]$, the components C_i and C_{i+n} run from $C \cap L_i$ to $C \cap L_{(i+1) \bmod n}$ in the clockwise direction.

For each $i \in [0, n-1]$, let T_i be the double star $T(V(G), L_i, i, i+n)$. By Lemma 10, each T_i is plane. Since $V(T_i) = V(G)$, T_i is a spanning tree of G. For all $i, j \in [0, n-1]$ with $i < j$, the points $i, i+n, j, j+n$ are in

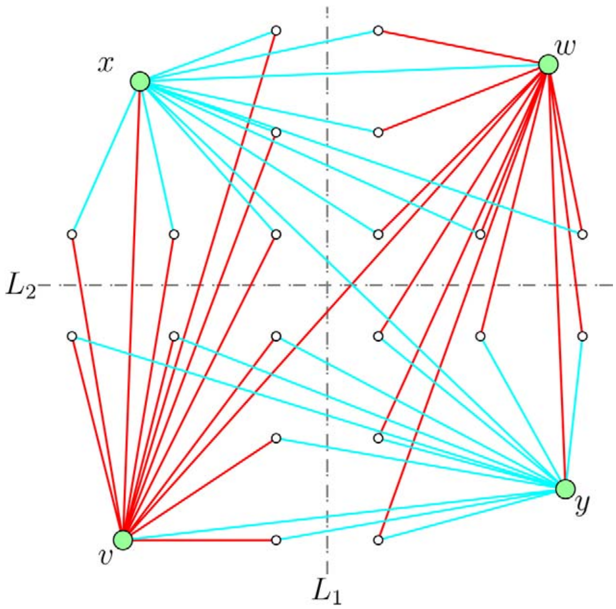

Fig. 7. Plane spanning double stars are edge-disjoint.

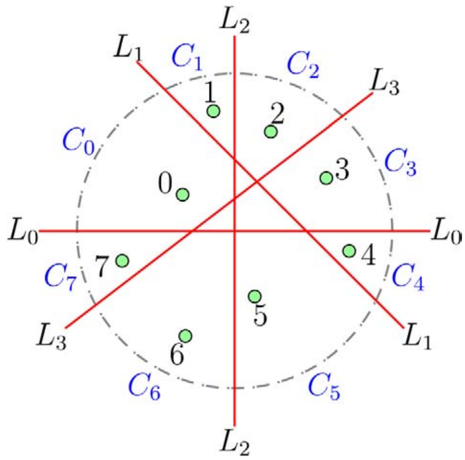

Fig. 8. Example of Theorem 9 with $n = 4$.

distinct quarter-planes formed by L_i and L_j, with each pair $(i, i+n)$ and $(j, j+n)$ in opposite quarter-planes. Thus, by Lemma 11, $E(T_i) \cap E(T_j) = \emptyset$. Since each T_i has $2n-1$ edges, and there are $n(2n-1)$ edges in total, $T_0, T_1, \dots, T_{n-1}$ is the desired partition of $E(G)$. $\square$

Note that each line in $\mathcal{L}$ in Theorem 9 is a halving line. Pach and Solymosi [12] proved a related result: a complete geometric graph on $2n$ vertices has n pairwise crossing edges if and only if it has precisely n halving lines.

4. Relaxations

We first drop the requirement that our plane trees be spanning. Thus we can consider complete graphs with any number of vertices.

Lemma 12. *Every complete geometric graph K_n can be partitioned into $n-1$ plane stars.*

Proof. Say $V(K_n) = [1, n]$. For each $i \in [1, n-1]$, let T_i be the star with edge set $\{ij\colon\ i < j \leqslant n\}$. Then T_i is plane regardless of the positions of the vertices. Clearly $\{T_1, T_2, \dots, T_{n-1}\}$ is a partition of $E(K_n)$. □

Lemma 12 can be strengthened by the following generalisation of Theorem 9.

Theorem 13. *Let G be a complete geometric graph K_n. Suppose that there is a set $\mathcal{L}$ of pairwise non-parallel lines with at least one vertex of G in each open unbounded region formed by $\mathcal{L}$. Then $E(G)$ can be partitioned into $n - |\mathcal{L}|$ plane trees.*

Proof. Let P be a set consisting of exactly one vertex in each open unbounded region formed by $\mathcal{L}$. Then $|P| = 2|\mathcal{L}|$. By Theorem 9, the induced subgraph $G[P]$ can be partitioned into $\frac{1}{2}|P|$ plane double stars. The edges incident to a vertex not in P can be covered by $n - |P|$ spanning stars, one rooted at each of the vertices not in P. Clearly a star is plane regardless of the vertex positions. Edges with both endpoints not in P can be placed in the star rooted at either endpoint. In total we have $\frac{1}{2}|P| + (n - |P|) = n - \frac{1}{2}|P| = n - |\mathcal{L}|$ plane trees. □

Lemma 14. *Every complete geometric graph K_n with k pairwise crossing edges can be partitioned into $n - k$ plane trees.*

Proof. Let $E = \{e_i\colon\ 1 \leqslant i \leqslant k\}$ be a set of k pairwise crossing edges. For each $i \in [1, k]$, let L_i be the line obtained by extending the segment e_i, and rotating it about the midpoint of e_i by some angle of ϵ degrees. Clearly there exists an ϵ such that each edge e_i crosses every line L_j, and there is one endpoint of an edge in E in each open unbounded region formed by $L_1, L_2, \dots, L_k$. The result follows from Theorem 13. □

Aronov et al. [2] proved that every complete geometric graph K_n has at least $\sqrt{n/12}$ pairwise crossing edges (called a *crossing family*). Thus we have the following corollary of Lemma 14.

Corollary 15. *Every complete geometric graph K_n can be partitioned into at most $n - \sqrt{n/12}$ plane trees.*

We now drop the requirement that our plane subgraphs by trees. The best known upper bound on the number of plane subgraphs in a partition of any geometric K_n is $n - \sqrt{n/12}$ (by Corollary 15). We have the following seemingly easier question than Problem 1.

Problem 16. Is there an $\epsilon > 0$, such that every complete geometric graph K_n can be partitioned into at most $(1-\epsilon)n$ plane subgraphs?

Of course $\epsilon \leqslant 1/2$ in Problem 16 since $\lfloor n/2 \rfloor$ edges can be pairwise crossing. An affirmative answer to Problem 16 is implied by Theorem 13 and an affirmative answer to the following question.

Problem 17. Is there an $\epsilon > 0$, such that for every set P of n points in general position, there is a set $\mathcal{L}$ of at least ϵn pairwise non-parallel lines, with at least one point of P in each open unbounded region formed by $\mathcal{L}$?

A famous conjecture by Aronov et al. [2] states that for some $\epsilon > 0$, every complete geometric graph K_n has at least ϵn pairwise crossing edges. This is considerably stronger than Problem 17.

Dillencourt et al. [5] defined the *geometric thickness* of an (abstract) graph G to be the minimum k such that G has a representation as a geometric graph whose edges can be partitioned into k plane subgraphs; also see [3,7,8,10]. They proved that the geometric thickness of K_n is between $\lceil (n/5.646) + 0.342 \rceil$ and $\lceil n/4 \rceil$. The difference between Problem 16 and determining the geometric thickness of K_n is that Problem 16 deals with all possible drawings of K_n, whereas geometric thickness asks for the best drawing.

As a final word, we refer the reader to reference [1] for more results and problems on the colouring of complete geometric graphs.

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