Orthogonal Drawings with Few Layers*

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Abstract. In this paper, we study 3-dimensional orthogonal graph drawings. Motivated by the fact that only a limited number of layers is possible in VLSI technology, and also noting that a small number of layers is easier to parse for humans, we study drawings where one dimension is restricted to be very small. We give algorithms to obtain point-drawings with 3 layers and 4 bends per edge, and algorithms to obtain box-drawings with 2 layers and 2 bends per edge. Several other related results are included as well. Our constructions have optimal volume, which we prove by providing lower bounds.

1 Introduction

Motivated by experimental evidence suggesting that displaying a graph in three dimensions is better than in two [22,23], there is a growing body of research in 3-dimensional graph drawing. Orthogonal drawings, in which edges are drawn as axis-parallel polylines, is a popular layout style with applications in VLSI circuit layout. Since present-day VLSI technology limits circuits to a few layers, consideration of the number of layers for orthogonal drawings are important. In this paper we present bounds on the volume and the number of bends in orthogonal graph drawings with only a few (2 or 3) layers. As well as VLSI concerns, we are motivated by the effective visualisation of 3-D orthogonal drawings in which one wishes to minimise the depth of a 3-D drawing displayed on a screen.

The (*3-dimensional*) *orthogonal grid* is the cubic lattice, consisting of *grid points* with integer coordinates, together with the axis-parallel *grid lines* determined by these points. We use the word *box* to mean an axis-parallel box with integral boundaries. At each grid point in a box B that is extremal in some direction $d \in \{\pm X, \pm Y \pm Z\}$, we say there is *port* on B in direction d. One grid point

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can thus define up to six incidents ports. For each dimension $I \in \{X, Y, Z\}$, an *I-line* is a line parallel to the I-axis, an *I-segment* is a line-segment within an I-line, and an *I-plane* is a plane perpendicular to the I-axis.

Let $G = (V, E)$ be an undirected graph without loops, $n = |V|$ and $m = |E|$. Let Δ be the maximum degree of G; a graph with maximum degree at most Δ is called a *Δ-graph.* An *orthogonal (box-)drawing* of G represents vertices by pairwise non-intersecting boxes and edges by pairwise disjoint grid paths connecting the endpoints of the edge. An orthogonal drawing with a particular shape of box representing every vertex, e.g., point or cube, is called an orthogonal *shape*-drawing. An orthogonal point-drawing can only exist for 6-graphs.

From now on, we use the term *drawing* to mean a 3-dimensional orthogonal drawing. Furthermore, the graph-theoretic terms 'vertex' and 'edge' also refer to their representation in a drawing. The *size* of a vertex v is denoted by $X(v) \times Y(v) \times Z(v)$, where for each $I \in \{X, Y, Z\}$, $I(v)$ is the number of I-planes intersecting v. The number of ports of v is called its *surface*, denoted by *surface*(v). The number of grid points in a box is called its *volume.*

Various criteria have been proposed in the literature to evaluate the aesthetic quality of a particular drawing. The primary criterion considered in this paper is that one dimension of the bounding box should be very small (2 or 3 units). For convenience we choose this to be the Z-dimension, and refer to the Z-planes of such a drawing as *layers.* We also consider the following secondary criteria.

First, the volume of a drawing should be small, where the *volume* of a drawing is that of the smallest axis-aligned box, called the *bounding box*, which encloses the drawing. Minimising the number of bends is also an important aesthetic criterion for orthogonal drawings. A drawing with no more than b bends per edge is called a *b-bend drawing.* Minimising either the volume or the total number of bends in a drawing is NP-hard [11].

For box-drawings the size and shape of a vertex with respect to its degree are also considered an important measures of aesthetic quality. A vertex v is *α-degree-restricted* if $\text{surface}(v) \leq \alpha \cdot \deg(v) + o(\deg(v))$. If for some constant α, every vertex v is α-degree-restricted, then the drawing is said to be *degree-restricted*; we use the term α-degree-restricted if we want to specify constant α. A drawing is said to be *strictly* α-degree-restricted if $\text{surface}(v) \leq \alpha \cdot \deg(v)$ for all vertices v, that is, no smaller-order terms are allowed.

The aspect ratio of a vertex v is normally defined to be the ratio between its largest and smallest side; that is, $\max\{X(v), Y(v), Z(v)\} / \min\{X(v), Y(v), Z(v)\}$. Since we are primarily concerned with drawings in a constant number of layers we define the *aspect ratio* of a vertex v to be $\max\{X(v), Y(v)\} / \min\{X(v), Y(v)\}$. We say that a drawing has *bounded aspect ratios* if there exists a constant r such that all vertices have aspect ratio at most r.

1.1 Box-Drawings

Algorithms to produce orthogonal box-drawings have been studied in [1,4,7,8,14, 19,28,29]. Lower bounds for the volume of orthogonal box-drawings have been presented in [1,7,8,14]. Table 1 summarises the known bounds on the volume and maximum number of bends per edge with various aesthetic criteria. We include the number of layers in each construction as well.

Table 1. Bounds on the volume, the number of layers and the maximum number of bends in box-drawings (assuming $m \in \Omega(n)$).

Lower Bound volume	reference	Upper Bound volume	layers	bends	graphs	reference
bounded aspect ratio / degree-restricted						
$\Omega(m^{3/2})$	[8]	$O(nm\sqrt{\Delta})$	$O(\sqrt{\Delta})$	2	simple	[4]
$\Omega(m^{3/2})$	[8]	$O(m^2)$	$O(\sqrt{m})$	5	multigraphs	[8]
$\Omega(m^{3/2})$	[8]	$O(m^{3/2})$	$O(\sqrt{m})$	6	multigraphs	[8]
bounded aspect ratio / not necessarily degree-restricted						
$\Omega(m^2)$	Thm. 7	$O(m^2)$	2	2	multigraphs	Thm. 3
no bounds on aspect ratio / degree-restricted						
$\Omega(m^{3/2})$	[8]	$O(n^2\Delta)$	$O(\Delta)$	2	simple	[4]
$\Omega(m^2)$	Thm. 7	$O(m^2)$	2	3	multigraphs	Thm. 5
$\Omega(m^{3/2})$	[8]	$O(m^{3/2})$	$O(\sqrt{m})$	6	multigraphs	[8]
no bounds on aspect ratio / not necessarily degree-restricted						
$\Omega(mn)$	[1]	$O(n^3)$	$O(n)$	1	simple	[7]
$\Omega(m\sqrt{n})$	[8]	$O(mn^{3/2})$	$O(\min\{n, m/\sqrt{n}\})$	1	simple	[29]
$\Omega(mn)$	Thm. 7	$O(mn)$	2	2	multigraphs	Thm. 2
$\Omega(m\sqrt{n})$	[8]	$O(n^{5/2})$	$O(\sqrt{n})$	3	simple	[7]
$\Omega(m\sqrt{n})$	[8]	$O(m\sqrt{n})$	$O(\sqrt{n})$	4	simple	[8]

Orthogonal box-drawings with a constant number of layers have not been studied previously. In this paper we prove the following results.

- Every graph has a 2-bend box-drawing in an $m \times n \times 2$ grid. This volume bound matches the best known upper bound for 2-bend box-drawings [29], but uses only 2 layers, as opposed to $O(m/n)$ layers in [29].
- Every graph has a degree-restricted 3-bend drawing in a $(m+n) \times (m + \frac{3}{2}n + 1) \times 2$ grid. This volume bound shows that 3 bends per edge suffice for degree-restricted drawings with $O(m^2)$ volume; previously 5 bends per edge were needed [8]. Additionally, we use only 2 layers, as opposed to $O(\sqrt{m})$ layers in [8].
- Every graph has a 2-bend drawing with bounded aspect ratios in a $(\frac{3}{4}m + \frac{1}{2}n) \times (\frac{3}{4}m + \frac{1}{2}n) \times 2$ grid. This volume bound shows that 3 bends per edge suffice for bounded aspect ratio drawings with $O(m^2)$ volume; previously 5 bends per edge were needed [8]. Additionally, we use only 2 layers, as opposed to $O(\sqrt{m})$ layers in [8].

We prove the following lower bounds on the volume of drawings with k layers (assuming that no vertices are "above" each other; see Section 5).

- There are graphs that need $\Omega(mn/k^5)$ volume in any drawing with at most k layers.
- There are graphs that need $\Omega(m^2/k^5 r)$ volume in any drawing with at most k layers and aspect ratios at most r.
- There are graphs that need $\Omega(m^2/k^5)$ volume in any drawing with at most k layers which is strictly α-degree-restricted for some $\alpha \in o(n/k^3)$. Typically, α is a small constant, so the assumption $\alpha \in o(n/k^3)$ is reasonable.
- No drawing with a constant number of layers can be both degree-restricted and have bounded aspect ratios.

1.2 Point-Drawings

Algorithms for producing point-drawings have been presented in [3,9,10,11,13, 15,17,19,26,25,27]. A lower bound of $\Omega(n^{3/2})$ for the volume of point-drawings was established by Kolmogorov and Barzdin [15]. Lower bounds for the number of bends in point-drawings were established by Wood [30].

Table 2. Upper Bounds for 3-Dimensional Orthogonal Point-Drawing

Graphs	Max. (Avg.) Bends	Bounding Box	Volume	Reference
multigraph	7	$O(\sqrt{n}) \times O(\sqrt{n}) \times O(\sqrt{n})$	$\Theta(n^{3/2})$	[12,13]
multigraph	6	$O(\sqrt{n}) \times O(\sqrt{n}) \times O(n)$	$O(n^2)$	[13]
multigraph	5	$O(n) \times O(n) \times O(1)$	$O(n^2)$	[9]
multigraph	4	$O(n) \times O(n) \times O(1)$	$O(n^2)$	Thm. 1
multigraph $\Delta \leq 4$	3	$O(n) \times O(n) \times O(1)$	$O(n^2)$	[13]
multigraph	5	$O(\sqrt{n}) \times O(n) \times O(n)$	$O(n^{5/2})$	[13]
simple	4 $(2\frac{2}{7})$	$O(n) \times O(n) \times O(n)$	$2.13n^3$	[25,27]
multigraph	3	$O(n) \times O(n) \times O(n)$	$8n^3$	[12,13]
multigraph	3	$O(n) \times O(n) \times O(n)$	$4.63n^3$	[19]
multigraph	3	$O(n) \times O(n) \times O(n)$	$n^3 + o(n^3)$	[26]
simple $\Delta \leq 5$	2	$O(n) \times O(n) \times O(n)$	n^3	[25,27]

Point-drawings with a constant number of layers were first studied by Eades *et al.*[13] for 4-graphs. Their algorithm produces a 3-bend point-drawing in a $2n \times (n+2) \times 3$ grid. A corresponding lower bound is $\Omega(n^2/k)$ volume for point-drawings with at most k layers [3], hence $\Omega(n^2)$ volume is necessary for drawings with a constant number of layers. Closson *et al.* [9] were the first to give drawings with a constant number of layers for any 6-graph; their algorithm produces a 5-bend point-drawing of a 6-graph in a $7n \times 5n \times 5$ grid. At the expense of allowing one more bend per edge, the the authors present a fully dynamic algorithm that supports the on-line insertion and deletion of vertices and edges in $O(1)$ time.

In this paper, we describe an algorithm to produce a 4-bend point-drawing of a 6-graph in a $3n \times 2n \times 3$ grid. Thus, we establish that with only 4 bends per edge, $O(n^2)$ volume can be obtained (the previous best volume bound for 4-bend point-drawings was $O(n^3)$ [13]). Also, our volume is $24n^2$, improving on the bound of $175n^2$ for the algorithm in [9]. If the graph has maximum degree 5, we can obtain drawings with only two layers.

2 Toolkit

In this section we give a number of introductory results which will be employed by our algorithms to follow. They can be considered to be an orthogonal graph drawer's toolkit.

A *cycle cover* of a directed graph is a spanning subgraph consisting of directed cycles. The following result, which can be considered as three applications of the classical result of Petersen that "every regular graph of even degree has a 2-factor" [21], has an algorithmic proof by Eades *et al.* [13].

Lemma 1 ([13]). *If G is an n-vertex 6-graph then there exists a directed graph G' (possibly with loops) such that:*

1. *G is a subgraph of the underlying undirected graph of G'.*
2. *Each vertex of G' has in-degree 3 and out-degree 3.*
3. *G' can be partitioned into three arc-disjoint cycle covers.*

G' and the cycle covers can be computed in $O(n)$ time. □

We will need the following lemma which slightly strengthens a previous result [8,13].

Lemma 2. *The edges of a graph G can be coloured red and blue so that the number of monochromatic edges incident to each vertex v is at most $\frac{1}{2}\deg(v)+1$.*

Proof. Pair the odd degree vertices in G, and add an edge between the paired vertices. All vertices now have even degree. In particular, the degree of a vertex v is now $2\lceil\frac{1}{2}\deg(v)\rceil$. Alternately colour the edges red and blue by following an Eulerian tour of G starting at an inserted edge (if any). Thus, there are at most $\lceil\frac{1}{2}\deg(v)\rceil + 1$ monochromatic edges incident to v. In fact, all vertices v, except the starting vertex in the Eulerian tour, have at most $\lceil\frac{1}{2}\deg(v)\rceil$ monochromatic incident edges. If there were no inserted edges then every vertex has even degree in the original G, and the number of monochromatic edges incident to v is at most $\frac{1}{2}\deg(v) + 1$. If there were some inserted edges then, since we started the Eulerian tour at an inserted edge, the number of monochromatic edges incident to a vertex v is at most $\lceil\frac{1}{2}\deg(v)\rceil \le \frac{1}{2}\deg(v) + \frac{1}{2}$. □

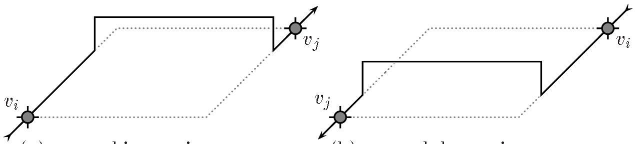
3 Point Drawings

In this section we describe an algorithm for producing point-drawings in a constant number of layers. Our algorithm is based on the decomposition of a 6-graph into three cycle covers, and the classification of edges according to the relative positions of the endpoints in an arbitrary ordering of the vertices along a 2-dimensional diagonal. This approach was first introduced in the 3-BENDS algorithm of Eades *et al.* [13]. The difference between the 3-BENDS algorithm and the algorithm which follows is that we use a 2-dimensional diagonal layout of the vertices, whereas the 3-BENDS algorithm uses a 3-dimensional diagonal layout. A 2-dimensional diagonal vertex layout is also used by Closson *et al.* [9].

Theorem 1. *Every 6-graph $G = (V, E)$ has a 4-bend point-drawing in a $3n \times 2n \times 3$ grid.*

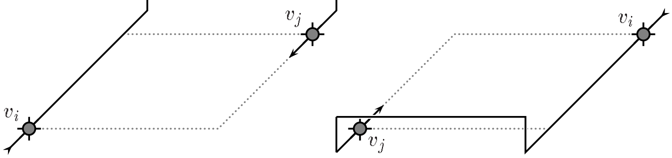
Proof. Consider the following algorithm.

1. Compute G' and a cycle cover decomposition of G'; see Lemma 1. Label the cycle covers and the arcs in G' *red*, *green* and *blue*.
2. Let $V = (v_1, \dots, v_n)$ be an arbitrary linear ordering of the vertices.
3. For each directed cycle C in the cycle decomposition, and for each arc $\overrightarrow{v_i v_j} \in C$ with $\overrightarrow{v_j v_k}$ the next arc in C, classify $\overrightarrow{v_i v_j}$ as follows, depending on the relative values of i, j and k. If $i < j < k$ then $\overrightarrow{v_i v_j}$ is *normal increasing*. If $i > j > k$ then $\overrightarrow{v_i v_j}$ is *normal decreasing*. If $i < j > k$ then $\overrightarrow{v_i v_j}$ is *increasing to a local maximum*. If $i > j < k$ then $\overrightarrow{v_i v_j}$ is *decreasing to a local minimum*.
4. Position vertices considering the red cycle cover as follows: For each vertex v_j, $1 \leq j \leq n$, suppose $\overrightarrow{v_i v_j}$ is the red arc entering v_j. If $\overrightarrow{v_i v_j}$ is normal increasing or decreasing to a local minimum then set $Y_j = 2j$. Otherwise $\overrightarrow{v_i v_j}$ is normal decreasing or increasing to a local maximum; set $Y_j = 2j - 1$. Position v_j at $(3j, Y_j, 0)$.
5. For each arc $\overrightarrow{v_i v_j}$ in the red cycle cover of G' (with $v_i v_j$ in G), route $v_i v_j$ using a Y-port at v_i and a Y-port at v_j. Whether a Y^+-port or a Y^- port is used depends on the classification of $\overrightarrow{v_i v_j}$. More precisely:
   a) If $\overrightarrow{v_i v_j}$ is *normal increasing*, as in Fig. 1(a), route $v_i v_j$ with the 4-bend edge: $(3i, Y_i, 0) \to (3i, Y_j - 1, 0) \to (3i, Y_j - 1, 1) \to (3j, Y_j - 1, 1) \to (3j, Y_j - 1, 0) \to (3j, Y_j, 0)$.
   b) If $\overrightarrow{v_i v_j}$ is *normal decreasing*, as in Fig. 1(b), route $v_i v_j$ with the 4-bend edge: $(3i, Y_i, 0) \to (3i, Y_j + 1, 0) \to (3i, Y_j + 1, 1) \to (3j, Y_j + 1, 1) \to (3j, Y_j + 1, 0) \to (3j, Y_j, 0)$.
   c) If $\overrightarrow{v_i v_j}$ is *increasing to a local maximum*, as in Fig. 2(a), route $v_i v_j$ with the 4-bend edge: $(3i, Y_i, 0) \to (3i, Y_j + 1, 0) \to (3i, Y_j + 1, 1) \to (3j, Y_j + 1, 1) \to (3j, Y_j + 1, 0) \to (3j, Y_j, 0)$.
   d) If $\overrightarrow{v_i v_j}$ is *decreasing to a local minimum*, as in Fig. 2(b), route $v_i v_j$ with the 4-bend edge: $(3i, Y_i, 0) \to (3i, Y_j - 1, 0) \to (3i, Y_j - 1, 1) \to (3j, Y_j - 1, 1) \to (3j, Y_j - 1, 0) \to (3j, Y_j, 0)$.



(a) normal increasing (b) normal decreasing

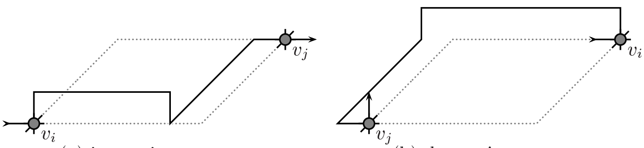
Fig. 1. Normal edge routes in the red cycle cover.



(a) increasing to max (b) decreasing to min

Fig. 2. Local min/max edge routes in the red cycle cover.

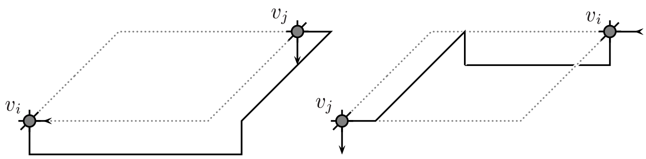
6. For each arc $\overrightarrow{v_i v_j}$ in the green cycle cover of G' (with $v_i v_j$ in G), route $v_i v_j$ using the Z^+-port at v_i and a X^--port at v_j. More precisely, route $v_i v_j$ with the 4-bend edge. $(3i, Y_i, 0) \to (3i, Y_i, 1) \to (3j-1, Y_i, 1) \to (3j-1, Y_i, 0) \to (3j-1, Y_j, 0) \to (3j, Y_j, 0)$. See Fig. 3.



(a) increasing (b) decreasing

Fig. 3. Edge routes in the green cycle cover.

7. For each arc $\overrightarrow{v_i v_j}$ in the blue cycle cover of G' (with $v_i v_j$ in G) route $\overrightarrow{v_i v_j}$ using the Z^--port at v_i and the X^+-port at v_j. More precisely, route $v_i v_j$ with the 4-bend edge: $(3i, Y_i, 0) \to (3i, Y_i, -1) \to (3j+1, Y_i, -1) \to (3j+1, Y_i, 0) \to (3j+1, Y_j, 0) \to (3j, Y_j, 0)$. See Fig. 4.



(a) increasing (b) decreasing

Fig. 4. Edge routes in the blue cycle cover.

A proof that there are no crossings is sketched as follows: Observe first that all Z-segments have unit length, and hence cannot cause a crossing. The (Z=0)-plane contains only Y-segments (except at vertices), whereas the (Z=1)-plane and the (Z= − 1)-plane contain only X-segments (except at vertices). Finally, no crossings happen at vertices, as illustrated in Figure 5. □

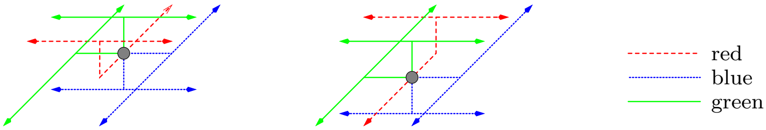


Fig. 5. Edges incident to a vertex, in the two possible positions of a vertex.

For 5-graphs, one can save one layer by rerouting the blue edge that uses the bottom port to another free port. Details are omitted.

4 Box Drawing Algorithms

Now we turn to graphs with arbitrarily high degrees.

A simple algorithm: The following simple algorithm produces a box-drawing with two layers. Given a graph $G = (V, E)$, let $V = (v_1, \ldots, v_n)$ and $E = (e_1, \ldots, e_m)$. Represent each vertex v_i, $1 \leq i \leq n$, by the line-segment with endpoints $(1, i, 0)$ and $(m, i, 0)$. As shown in Fig. 6, draw the edge $e_k = v_i v_j$ with the 2-bend edge route

$$(k, i, 0) \to (k, i, 1) \to (k, j, 1) \to (k, j, 0) \ .$$

Clearly there are no edge crossings. We thus have the following result.

Theorem 2. *Every graph has a 2-bend box-drawing in an $m \times n \times 2$ grid.* □

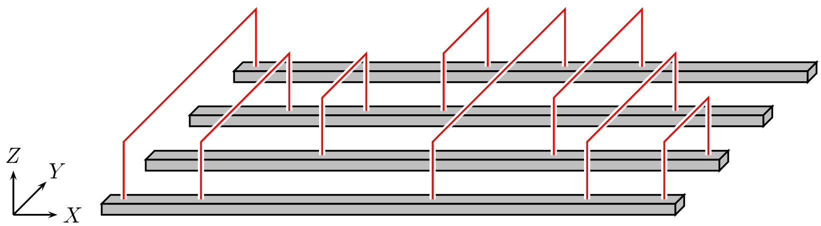


Fig. 6. A box-drawing in an $m \times n \times 2$ grid.

Lifting 2-dimensional drawings: Another method for producing box-drawings with two layers is to start with a 2-dimensional drawing with crossings. Vertices are then represented by boxes of height two, and edges are constructed by routing the X-segment of a 2-dimensional edge in the $(Z{=}0)$-plane and the Y-segment in the $(Z{=}1)$-plane connected by unit-length Z-segments at each bend. The resulting 3-dimensional edge has twice as many bends as the original 2-dimensional edge. This method was called *lifting half-edges* in [4]. We apply this method to the algorithms of Biedl and Kaufmann [6], Papakostas and Tollis [20], and Wood [28].

These algorithms all produce degree-restricted 2-dimensional drawings, hence in the resulting 3-dimensional drawings we have $X(v)+Y(v) \in O(\deg(v))$ for all vertices v. While this is not necessarily a degree-restricted drawing, we will see in Lemma 3 that $X(v) + Y(v) \in \Omega(\deg(v))$ is required for all drawings with a constant number of layers. Two of the algorithms also produce bounded aspect ratios, which is transferred while lifting them to the third dimension.

Theorem 3. *Every graph has:*

(a) a 2-bend drawing in a $\frac{m+n}{2} \times \frac{m+n}{2} \times 2$ *grid [6] (see also [4, Theorem 3]),*
(b) a 2-bend drawing in a $(m-1) \times (\frac{m}{2}+2) \times 2$ *grid [20],*
(c) a 2-bend drawing in a $(\frac{3}{4}m+\frac{1}{2}n) \times (\frac{3}{4}m+\frac{1}{2}n) \times 2$ *grid such that each vertex* v *has aspect ratio at most* $2+O(\frac{1}{deg(v)})$ *[6],*
(d) a 2-bend drawing in a $(\frac{3}{4}m+\frac{5}{8}n) \times (\frac{3}{4}m+\frac{5}{8}n) \times 2$ *grid such that every vertex has aspect ratio one [28].* □

4.1 Degree-Restricted Box-Drawings

We obtain results for degree-restricted box-drawings in two ways. One possible drawing is obtained by lifting the 2-dimensional drawing by Biedl and Kant [5]. This yields drawings with 4 bends per edge. Next, we describe an algorithm that uses only 3 bends per edge.

Lifting the drawing by Biedl/Kant: In [5], the first author and Kant gave an algorithm for 2-dimensional point-drawings, which can be extended to give 2-dimensional box-drawings of graphs with arbitrarily high degrees. The resulting grid size is $(m-n+1) \times (m-n/2+n_2/2)$, where n_2 is the number of vertices of degree 2. This drawing also has the property that vertex v is drawn as a $1 \times \lceil \deg(v)/2 \rceil$ line segment.

We can obtain a 3-dimensional drawing by applying the lifting half-edges technique. Every vertex now becomes a $1 \times \lceil \deg(v)/2 \rceil \times 2$ box; hence the drawing is 2-degree restricted. Since every edge in the 2-dimensional drawing has at most 2 bends, every edge in the resulting 3-dimensional drawing has at most 4 bends.

Theorem 4. *Every graph has a 4-bend box-drawing in a* $(m-n+1) \times m \times 2$*-grid such that every vertex is 2-degree restricted.* □

A drawing with 3 bends per edge: In this section we describe an algorithm for producing degree-restricted box-drawings with only 3 bends per edge.

Theorem 5. *Every graph has a 3-bend box-drawing in an* $(m+n)\times(m+\frac{3}{2}n+1)\times 2$ *grid such that every vertex is 3-degree restricted.*

Proof. Let $(v_1,\ldots,v_n)$ be a an arbitrary linear ordering of the vertices of G. As described in Lemma 2, colour the edges red and blue such that at most $\frac{1}{2}\deg(v)+1$ edges of the same colour are incident to v. Define $W(v)=\lfloor\frac{1}{2}\deg(v)\rfloor+1$.

Represent v_i as a $W(v_i)\times 1\times 2$-box, and place it such that its leftmost points have X-coordinate $\sum_{j<i}W(v_j)$ and Y-coordinate $Y_i=\sum_{j\le i}W(v_j)$. All vertex boxes share the (Z=0)-plane and the (Z=1)-plane; see also Fig. 7.

Assign unique Y^- ports at v_i with $Z=0$ (respectively, $Z=1$) coordinates to the red successor (predecessor) edges of v_i. For each vertex v_i, denote by X_i the largest X-coordinate of the box of v_i. We route a red edge $e=v_iv_j$ $(i<j)$ as follows: Assume that e was assigned the port with X-coordinate $X_i-\alpha$ at v_i and $X_j-\beta$ at v_j. Then we draw v_iv_j with the 3-bend edge route: $(X_i-\alpha,\, Y_i,\, 0)\to(X_i-\alpha,\, Y_i-\alpha-1,\, 0)\to(X_j-\beta,\, Y_i-\alpha-1,\, 0)\to(X_j-\beta,\, Y_i-\alpha-1,\, 1)\to(X_j-\beta,\, Y_j,\, 1)$, as illustrated in Fig. 7.

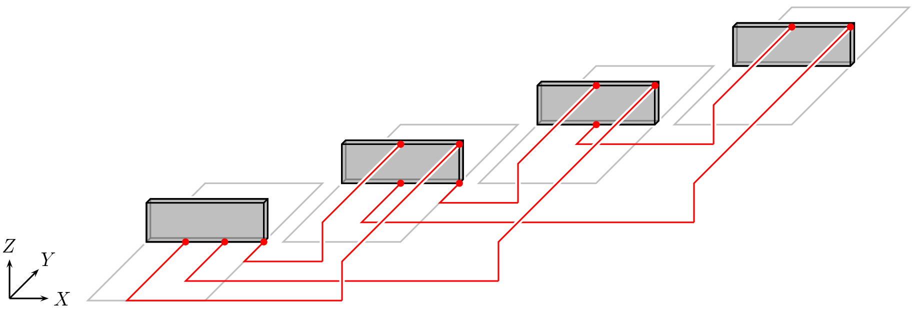


Fig. 7. A selection of red edge routes in a degree-restricted box-drawing.

Blue edges are routed similarly using Y^+ ports. For space reasons we omit the proof that edges do not cross.

The width of the drawing is $\sum_{i=1}^n W(v_i)\le\sum_{i=1}^n(\frac{1}{2}\deg(v_i)+1)\le m+n$, and the depth is at most $\sum_{i=1}^n W(v_i)+W(v_n)\le m+n+\frac{1}{2}\deg(v_n)+1\le m+\frac{3}{2}n+1$. $\square$

Note that we can obtain a smaller volume by using a vertex ordering and edge colouring that guarantees that the number of red predecessors and successors is not too unbalanced. If we use the *median placement heuristic* (see [2]) to obtain such an ordering, one can show that the width reduces to $\frac{3m}{4}+\frac{9n}{8}$, and the depth becomes $\frac{3m}{4}+\frac{13n}{8}+1$.

5 Box Drawing Lower Bounds

In this section, we give lower bounds. These lower bounds hold under the assumption that no vertices are "above each other", defined precisely as follows: The *airspace* of a vertex v are all those points that have a common X-coordinate and a common Y-coordinate with v; see also Fig. 8. We say that a drawing has *no vertices above each other* if the airspaces of any two vertices are distinct.

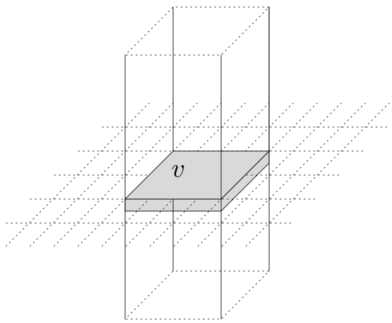


Fig. 8. The airspace of a vertex v.

Note that all our algorithms produce drawings without vertices above each other. Also, in VLSI design, vertices would normally not be put above each other to avoid interference. Likewise, for visualisation purposes vertices above each other might easily obscure each other, which should be avoided. Hence, assuming that no vertices are above each other is not an unreasonable assumption.

We start with a lemma discussing the dimensions of each vertex.

Lemma 3. *Assume that Γ is a drawing with k layers, with no vertices above each other. Then $X(v) + Y(v) \geq \deg(v)/2k$ for every vertex v.*

Proof. Let v be an arbitrary vertex. Let P_1 be the set of X-ports and Y-ports of v, and let P_2 be those points on the boundary of the airspace of v that are not ports of v. Note that $|P_1 \cup P_2| \leq k \cdot 2(X(v) + Y(v))$, since there are k layers.

We claim that each incident edge e of v must use one element of P_1 or P_2, without counting elements in P_1 or P_2 repeatedly. This holds if the port of e at v is an X-port or a Y-port, because then we assign to e the port in P_1. If the port of e at v is a Z-port, then e must somewhere enter the airspace of v. (Note that e must be outside v's airspace at the other endpoint, since no two vertices have intersecting airspaces.) We assign this point of entry into the airspace of v to e. No two such edges can use the same point because edge routes are disjoint.

Thus, we must have $k \cdot 2(X(v) + Y(v)) \geq \deg(v)$, which yields the claim. □

As a consequence of this lemma, no drawing with $O(1)$ layers can be both degree-restricted and have bounded aspect ratios at the same time.

Theorem 6. *Let Γ be a drawing in k layers without vertices above each other. If Γ is strictly α-degree-restricted and has aspect ratios at most r, then*

$$\alpha \geq \Delta/8k^2r,$$

where Δ is the maximum degree of the graph. In particular, not all of α, k, r can be constant unless Δ is a constant.

Proof. Let v be a vertex of maximum degree, and assume v is represented by an $X \times Y \times Z$-box. Without loss of generality we may assume $X \geq Y$. Since $X + Y \geq \Delta/2k$ by the previous lemma, we have $X \geq \Delta/4k$. Since the aspect ratio of v is at most r, we have $Y \geq \frac{1}{r}X \geq \Delta/4kr$. The surface of v hence satisfies $\alpha \cdot \Delta \geq \text{surface}(v) \geq 2XY \geq \Delta^2/8k^2r$, which yields the claim. □

Now we proceed to prove lower bounds.

Theorem 7. *For every $k \geq 1$, there exist an infinite number of graphs G such that for any drawing Γ with k layers without vertices above each other*

- *Γ has volume $\Omega(mn/k^5)$.*
- *if Γ has aspect ratios at most r, then Γ has volume $\Omega(m^2/k^5r)$.*
- *if Γ is strictly α-degree-restricted, where $\alpha \in o(n/k^3)$, then Γ has volume $\Omega(m^2/k^5)$.*

Proof. We use as graphs the so-called Ramanujan-graphs; see [16] for their definition and [8,1] for some of their properties. For our proof, all we need to know is that for a fixed k, there exists an infinite number of Ramanujan-graphs such that for any two vertex sets V_1, V_2 with $|V_1|, |V_2| \geq n/4k$ there are at least $C \cdot m/k^2$ edges between V_1 and V_2, for some constant C independent of k. Let G_k be such a graph; we know that G_k is d-regular for some constant d, so $m = dn/2$.

Consider an arbitrary drawing Γ of G_k without vertices above each other, and assume it is contained in an $X \times Y \times k$-grid. Similar as in lower bound proofs in [7,8], we show a lower bound by distinguishing whether many vertices are intersected by one grid-line or not. For space reasons we omit rounding details and assume that n is divisible by $4k$.

Case 1. One grid line intersects at least $n/2k$ vertices:

Assume that there exists a grid line, say an X-line, that intersects at least $n/2k$ vertices. Let $v_1, \ldots, v_t$ be the vertices intersected by the X-line, listed in order of occurrence along the line. Let X_0 be a not necessarily integer X-coordinate such that the ($X{=}X_0$)-plane intersects none of these t vertices and separates the first $n/4k$ of them from the remaining ones, of which there are at least $n/4k$. We will refer to the first set as V_+ and the second set as V_-.

By assumption at least $C \cdot m/k^2$ edges connect V_+ and V_-. These edges cross the ($X{=}X_0$)-plane, which thus must contain at least $C \cdot m/k^2$ points having integer Y- and Z-coordinates. Hence $Yk \geq C \cdot m/k^2$. The three different claims are now proved as follows:

- The X-line intersects the vertices $v_1, \ldots, v_t$, $t \geq n/2k$, hence $X \geq t \geq n/2k$ and the volume of Γ is $XYk \geq Cmn/2k^3 \in \Omega(mn/k^3)$.
- If we know a bound r on the aspect ratio, then $Y(v_i) \leq rX(v_i)$, and therefore by Lemma 3, $X(v_i) \geq d/2k(1+r)$. Therefore $X \geq X(v_1) + \cdots + X(v_t) \geq dt/2k(1+r) \geq dn/4k^2(1+r) = m/2k^2(1+r)$ which yields $XYk \geq Cm^2/3k^4(1+r) \in \Omega(m^2/k^4 r)$.
- Assume that Γ is strictly α-degree-restricted; we may assume $\alpha \leq Cn/4k^3$ by $\alpha \in o(n/k^3)$. Let Y_0 be the Y-coordinate of the X-line, and define $Y_+ = Y_0 + \alpha d/2$ and $Y_- = Y_0 - \alpha d/2$. Note that $v_1, \ldots, v_t$ are contained in the range $Y_- \leq Y \leq Y_+$, since a vertex with surface at most αd extends at most $\alpha d/2$ in Y-direction. Define P to be the grid points (see also Figure 9)

$$P = \{(X,Y,Z) : X{<}X_0, Y{=}Y_-\} \cup \{(X,Y,Z) : X{<}X_0, Y{=}Y_+\}$$
$$\cup \{(X,Y,Z) : X{=}X_0, Y_- \leq Y \leq Y_+\}.$$

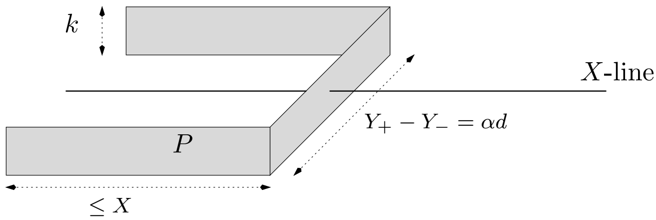


Fig. 9. The set P separates V_- from V_+.

The points in P separate the vertices in V_- from the vertices V_+. Hence, the Cm/k^2 edges between V_- and V_+ must use a grid point in P, so $|P| \geq Cm/k^2$. Note that $|P| \leq 2Xk + k\alpha d \leq 2Xk + Cdn/4k^2 = 2Xk + Cm/2k^2$ by $\alpha \leq Cn/4k^3$. Therefore $X \geq Cm/2k^3$, and $XYk \geq C^2m^2/2k^5 \in \Omega(m^2/k^5)$.

Case 2: No grid line intersects many vertices:

Now assume that no grid line intersects at least $n/2k$ vertices. Since there are at most k Z-planes, there must exist a $(Z{=}Z_0)$-plane that intersects at least n/k vertices.

As an $(X{=}X_0)$-plane is swept from smaller to larger values of X_0, the Y-line determined by the intersection of this $(X{=}X_0)$-plane with the $(Z{=}Z_0)$-plane sweeps the $(Z{=}Z_0)$-plane. At any time, this Y-line intersects at most $n/2k$ vertices by assumption. We can therefore find a (not necessarily integral) value X_0 such that there are at least $n/4k$ vertices to the left of the $(X{=}X_0)$-plane and not intersected by it, and there are at least $n/4k$ vertices to the right of the $(X{=}X_0)$-plane and not intersected by it. (See [7] for details of finding X_0.)

By assumption at least Cm/k^2 edges connect the vertices to the left and to the right of the $(X{=}X_0)$-plane. These edges cross the $(X{=}X_0)$-plane at a grid point, hence $Yk \geq Cm/k^2$. Similarly one shows $Xk \geq Cm/k^2$, therefore $XYk \geq C^2m^2/k^5 \in \Omega(m^2/k^5)$. This proves all claims. □

6 Conclusion

In this paper, we have studied 3-dimensional orthogonal graph drawings with few layers. We gave algorithms both for point-drawings (using 3 layers) and for box-drawings (using 2 layers). Note that one cannot hope for fewer layers, unless one allows edges to overlap each other, or crossings to occur. Our constructions are optimal with respect to the volume, as they match (within a constant) the lower bounds, some of which were provided in this paper as well. Some open problems that deserve attention are outlined in the following:

– What results can be shown for dynamic drawings with few layers? The algorithm in [9] can be extended to a dynamic setting by adding one more bend per edge. Is this possible for our algorithms as well? Note that in a dynamic setting we cannot rely on the cycle-decomposition of Lemma 1, as updating a cycle-decomposition appears to be impossible in constant time.
– We suspect that no 1-bend drawing of K_n with a constant number of layers exist. How can this be shown? If it holds, is it true that any 1-bend drawing of K_n needs $\Omega(n)$ layers, or is a drawing with, say, $O(\log n)$ layers possible?
– We gave 4-bend point-drawings with 3 layers. Can the number of bends per edge be reduced to 3 or even 2? Or is there a 6-graph that does not have a 2-bend point-drawing with 3 layers? Note that answering this question would yield progress on the *2-bend problem*: does every 6-graph have a 2-bend point-drawing? (See [13,25,30]).

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