ON VISIBILITY AND BLOCKERS*

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Abstract. This expository paper discusses some conjectures related to visibility and blockers for sets of points in the plane.

1 Visibility Graphs

Let P be a finite set of points in the plane. Two distinct points v and w in the plane are *visible* with respect to P if no point in P is in the open line segment $\overline{vw}$. The *visibility graph* $\mathcal{V}(P)$ of P has vertex set P, where two distinct points $v, w \in P$ are adjacent if and only if they are visible with respect to P. In other words, $\mathcal{V}(P)$ is obtained by drawing a line through each pair of points in P, where two points are adjacent if they are consecutive on a such a line. See Figure 1 for an example.

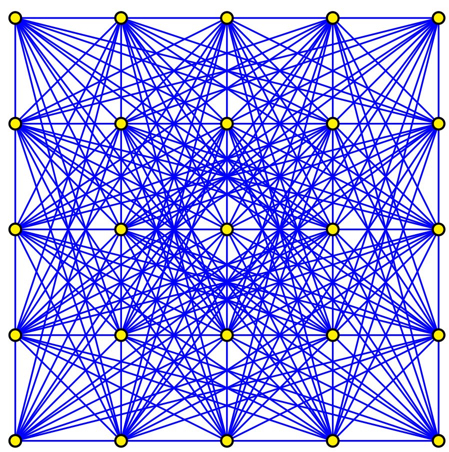

Figure 1: The visibility graph of the 5×5 grid.

Visibility graphs have many interesting properties. For example, if P is not collinear then $\mathcal{V}(P)$ has diameter at most two [24]. Consider the following Ramsey-theoretic conjecture by Kára et al. [24], which has recently received considerable attention [1, 2, 27].

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Conjecture 1 (Big-Line-Big-Clique Conjecture [24]). *For all integers $k \geq 2$ and $\ell \geq 2$ there is an integer n such that for every finite set P of at least n points in the plane:*

- *P contains ℓ collinear points, or*
- *P contains k pairwise visible points (that is, $\mathcal{V}(P)$ contains a k-clique).*

Conjecture 1 is true for $k \leq 5$ or $\ell \leq 3$ [1, 2, 24], and is open for $k = 6$ or $\ell = 4$. Note that the natural approach for attacking the Big-Line-Big-Clique Conjecture using extremal graph theory fails. Turán [47] proved that every n-vertex graph with more edges than the Turán graph $T_{n,k}$ contains K_{k+1} as a subgraph[1]. Thus the Big-Line-Big-Clique Conjecture would be proved if every sufficiently large visibility graph with no ℓ collinear points has more edges than $T_{n,k-1}$. However, Sylvester [42, 43, 44, 45] constructed a set P of n points with no four collinear, such that P determines $\frac{n^2}{6} - O(n)$ lines each containing three points[2]. Thus $\mathcal{V}(P)$ has $\frac{n^2}{3} + O(n)$ edges, which is less than the number of edges in $T_{n,k-1}$ for all $k \geq 5$ and large n. These examples show that the number of edges in a visibility graph with no four collinear points is not enough to necessarily imply the existance of a large clique via Turán's Theorem.

Consider the following weakening of Conjecture 1, due to Jan Kára Jan [private communication, 2005].

Conjecture 2. *For all integers $k \geq 2$ and $\ell \geq 2$ there is an integer n such that if P is a finite set of at least n points in the plane, and each point in P is assigned one of $k - 1$ colours, then:*

- *P contains ℓ collinear points, or*
- *some pair of visible points in P receive the same colour*
  (that is, the visibility graph $\mathcal{V}(P)$ has chromatic number $\chi(\mathcal{V}(P)) \geq k$).

Conjecture 1 implies Conjecture 2 since the chromatic number of any graph containing a k-clique is at least k. Thus Conjecture 2 is true for $k \leq 5$ or $\ell \leq 3$. See reference [3] for a study of a special case of Conjecture 2.

Consider a proper colouring of a visibility graph $\mathcal{V}(P)$. That is, visible points are coloured differently. In each colour class C, no two vertices are visible. So the vertices not in C 'block' the lines of visibility amongst vertices in C. This idea leads to the following definitions that were independently introduced by Matoušek [27] amongst others.

A point x in the plane *blocks* two points v and w if $x \in \overline{vw}$. Let P be a finite set of points in the plane. A set B of points in the plane *blocks* P if $P \cap B = \emptyset$ and for all distinct $v, w \in P$ there is a point in B that blocks v and w. That is, no two points in P are visible with respect to $P \cup B$, or alternatively, P is an independent set in $\mathcal{V}(P \cup B)$.

The purpose of this expository paper is to discuss some conjectures related to blocking sets. We remark that in the last few years, a number of researchers have started studying

[1] Let $T_{n,k}$ be the k-coloured graph with n_i vertices in the i-th colour class, where two vertices are adjacent if and only if they have distinct colours, $n = \sum_i n_i$, and $|n_i - n_j| \leq 1$ for all $i, j \in [k]$.

[2] While the proof by Sylvester is lacking details, subsequent proofs with improved $O(n)$ terms have been given by Burr et al. [9] and Füredi and Palásti [20]; also see [7, 8].

blocking sets around the same time (see [13, 27, 31] and the named researchers therein). So we expect that some of the observations in this paper have been independently discovered by others.

2 The Blocking Conjecture

Every set P of collinear points can be blocked by a set of $|P|-1$ points (for example, the midpoints of the consecutive pairs of points in P block P) . At the other extreme, how small can a blocking set be if P is in general position (that is, no three points are collinear)? Let $b(P)$ be the minimum size of a set of points that block P. Let $b(n)$ be the minimum of $b(P)$, where P is a set of n points in general position in the plane. We conjecture that every set of points in general position requires a super-linear number of blockers.

Conjecture 3. $\frac{b(n)}{n} \to \infty$ *as* $n \to \infty$.

In fact, Pinchasi [31] conjectured that $b(n) \in \Omega(n \log n)$. Linear lower bounds on $b(n)$ are known [13, 27]. Let P be a set of n points in the plane in general position with t vertices on the boundary of the convex hull. Each edge of a triangulation of P requires a distinct blocker, and every triangulation of P has $3n-3-t$ edges. So every blocking set of P has at least $3n-3-t \geq 2n-3$ vertices, and $b(n) \geq 2n-3$. Dumitrescu et al. [13] improved this bound to $b(n) \geq (\frac{25}{8}-o(1))n$.

3 Blocking Graph Drawings

A *drawing* of a graph[3] G represents each vertex of G by a distinct point in the plane, and represents each edge of G by a simple closed curve between its endpoints, such that a vertex v intersects an edge e if and only if v is an endpoint of e. We do not distinguish between graph elements and their representation in a drawing. Note that multiple edges may intersect at a common point. A drawing is *simple* if any two edges intersect at most once, at a common endpoint or as a proper crossing ("kissing" edges are not allowed). A drawing is *geometric* if each edge is a straight line-segment. Obviously, every geometric drawing is simple.

Blockers for point sets generalise for graph drawings as follows. A set of points B *blocks* a drawing of a graph G if no vertex of G is in B and every edge of G contains some point in B. Observe that if P is a set of points in general position, then B blocks P if and only if B blocks the geometric drawing of the complete graph with vertices drawn at P.

Some geometry is needed in Conjecture 3, in the sense that K_n has a simple (non-geometric) drawing that can be blocked by $2n-3$ blockers. As illustrated in Figure 2, if $V(K_n) = \{v_1, \dots, v_n\}$ then place v_i at $(i,0)$ and draw each edge v_iv_j with $i<j$ as a curve from v_i into the upper half-plane, through the point $(-i-j,0)$, into the lower half-plane, and across to v_j. As illustrated in Figure 2, the edges can be drawn so that two edges intersect at most once. Each edge is blocked by one of the $2n-3$ points in $\{(-k,0) : k \in [3, 2n-1]\}$.

[3]Throughout this paper, we consider graphs with no parallel edges and no loops.

This observation improves upon a $O(n \log n)$ upper bound on the number of blockers in a simple drawing of K_n, due to Dumitrescu et al. [13]. A similar construction is due to Harborth and Mengersen [22]; see Pach et al. [30]. Note that at least $n-1$ blockers are needed for every simple drawing of K_n (since each point can block at most $\frac{n}{2}$ edges).

Conjecture 4. *The minimum number of blockers in a simple drawing of K_n equals $2n-3$.*

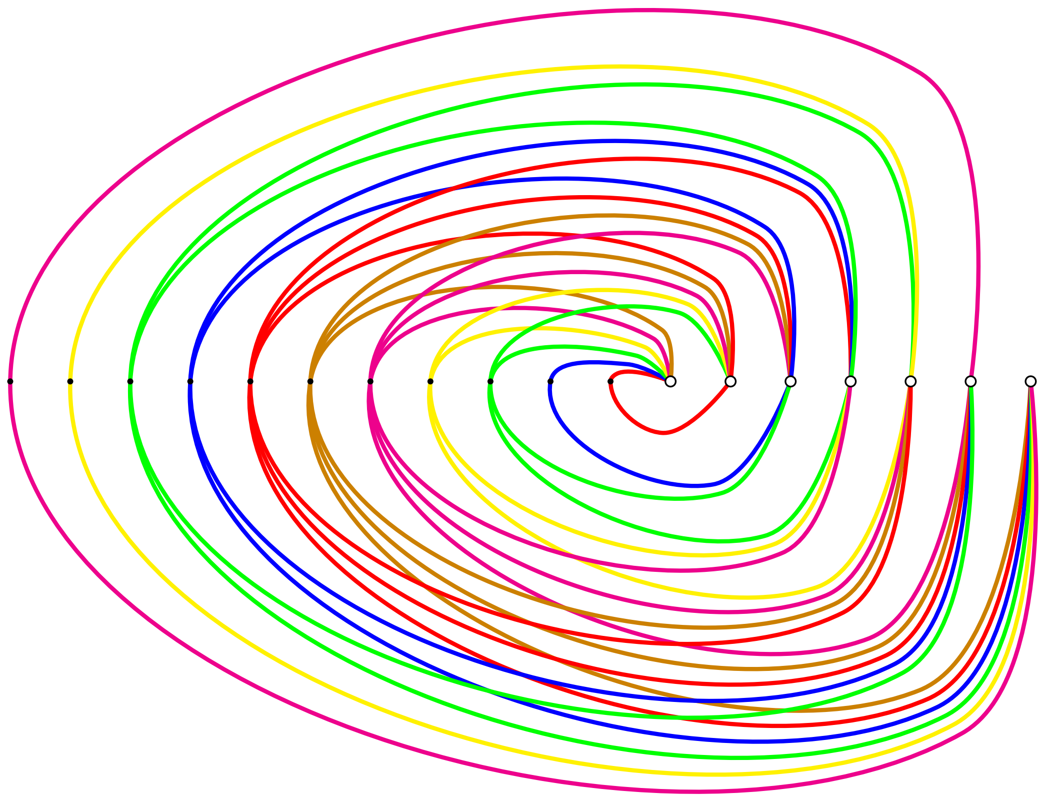

Figure 2: A drawing of K_7 blocked by 11 blockers.

While this example suggests that geometry is needed in Conjecture 3, Stefan Langerman [personal communication, 2009] proposed an alternative. A drawing of a graph is *extendable* if the edges are contained in a pseudoline arrangment; that is, for each edge e there is a simple unbounded curve C_e containing e, such that for all distinct edges e and e', the curves C_e and $C_{e'}$ intersect at most once. Observe that the above simple drawing that can be blocked by $O(n)$ blockers is not extendable. For extendible drawings we make the following conjecture:

Conjecture 5. *Eevery extendible simple drawing of K_n requires a super-linear number of blockers.*

4 Midpoints and Freiman's Theorem

Conjecture 3 is related to known results about midpoints. Hernández-Barrera et al. [23] introduced the following definitions[4]. For a set P of points in the plane, let $m(P)$ be the number of midpoints determined by distinct points in P; that is,

$$m(P) := |\{\tfrac{1}{2}(x+y) : x, y \in P, x \neq y\}| \ .$$

Let $m(n)$ be the minimum of $m(P)$, where P is a set of n points in general position in the plane. Since midpoints are also blockers, $b(n) \leq m(n)$. Hernández-Barrera et al. [23] constructed a set of n points in general position in the plane that determine at most $cn^{\log_2 3}$ midpoints for some contant $c > 0$. Thus

$$b(n) \leq m(n) \leq cn^{\log_2 3} = cn^{1.585\ldots} \ .$$

This upper bound was independently improved by Stanchescu [40] and Pach [29] (and later by Matoušek [27]) to

$$b(n) \leq m(n) \leq nc^{\sqrt{\log n}} \ .$$

(This function is between $n \log n$ and $n^{1+\epsilon}$ for large n.) Hernández-Barrera et al. [23] conjectured that $m(n)$ is super-linear, which was independently verified by Stanchescu [40] and Pach [29]; that is,

$$\frac{m(n)}{n} \to \infty \text{ as } n \to \infty \ . \tag{1}$$

Thus Conjecture 3 would stregthen this lower bound on $m(n)$. Pach's proof of (1) is based on Freiman's Theorem[5]:

Theorem 6 (Freiman's Theorem in the Plane [19])**.** *Let P be a set of n points in the plane (not necessarily in general position). If $m(P) = \alpha n$ then P is a subset of a d-dimensional progression of size at most βn, for some d and β depending only on α.*

Pach [29] concluded that at least $n^{1/d}/\beta$ points in P are collinear. Thus, assuming that P is in general position, n is bounded by a function of α. It follows that $\frac{m(n)}{n} \to \infty$. (This argument is generalised in Proposition 8 below.) Analogously, the following conjectured 'convex combination' version of Freiman's Theorem would establish Conjecture 3.

Conjecture 7. *Let P be a set of points in the plane with at most $\frac{1}{2}|P|$ points collinear. Suppose that P can be blocked by some set B with $|B| \leq \alpha|P|$. That is, for all distinct $x, y \in P$ there is a real number $\gamma \in (0,1)$, such that $\gamma x + (1-\gamma)y \in B$. Then P is a subset of a d-dimensional progression of size at most $\beta|P|$, for some d and β depending only on α.*

[4] These definitions and questions about midpoints are implicit in the literature on Freiman's Theorem, which pre-dates the study of midpoints in the combinatorial geometry literature.

[5] A d-dimensional *progression* in the plane is a set $\{v_0 + x_1 v_1 + \cdots + x_d v_d : x_i \in [1, n_i]\}$ for some vectors $v_0, \ldots, v_d \in \mathbb{R}^2$. Freiman's Theorem is usually stated in terms of the *sum set* $P + P := \{x + y : x, y \in P\}$, but this is not important since $m(P) \leq |P + P| \leq m(P) + |P|$. Freiman's Theorem actually applies in any abelian group; see [46]. See [18, 38, 39, 41] for more on Freiman's Theorem in the plane.

Note that some assumption on the number of collinear points is needed in Conjecture 7. For example, a set of n random collinear points can be blocked by $n-1$ points, but is not a subset of a progression of bounded dimension and linear size. This conjecture generalises Freiman's Theorem for the plane, which assumes $\gamma = \frac{1}{2}$ for all $x, y \in P$.

The proof of (1) by Stanchescu [40] gives an explicit lower bound on $m(n)$. In particular, for all $\epsilon > 0$ there is a constant $c_\epsilon > 0$ such that[6]

$$m(n) \geq c_\epsilon n(\log n)^{\frac{1}{8}-\epsilon} \;.$$

This bound was recently improved by Sanders [36] who proved the following more general result: If G is an abelian group and $P \subset G$ is finite and contains no non-trivial 3-term arithmetic progression, then $|P+P| \geq c_\epsilon |P|(\log|P|)^{\frac{1}{3}-\epsilon}$ for all $\epsilon > 0$. Consider this result with $G = \mathbb{R}^2$. The assumption that P contains no non-trivial 3-term arithmetic progression is equivalent to saying that the midpoint of distinct points in P is not in P, which is weaker than the assumption that P is in general position. Sander's theorem thus implies that for all $\epsilon > 0$,

$$m(n) \geq c_\epsilon n(\log n)^{\frac{1}{3}-\epsilon} \;. \tag{2}$$

While Freiman's Theorem applies in some sense for sum sets along the edges of any dense graph [15], it is worth noting that there is a geometric drawing of the complete bipartite graph $K_{n,n}$ that can be blocked by $O(n)$ blockers. Say the colour classes of $K_{n,n}$ are $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_n\}$. Position v_i at $(2i, 0)$, and w_j at $(2j, 2)$. Thus $v_i w_j$ is blocked by $(i+j, 1)$, and $\{(i,1) : i \in [2, 2n]\}$ is a set of $2n-1$ points blocking every edge. In fact, there is a geometric drawing of $K_{n,n}$ with its vertices in general position that can be similarly blocked. Position v_i at $(-2^i, 2^{2i})$ and w_j at $(2^j, 2^{2j})$. These points lie on opposite sides of the parabola $y = x^2$. The edge $v_i w_j$ is blocked by $(0, 2^{i+j})$, and $\{(0, 2^i) : i \in [2, 2n]\}$ is a set of $2n-1$ points blocking every edge.

In general, say $S = \{s_1, \dots, s_n\}$ is a set of n positive integers. Draw $K_{n,n}$ by positioning each v_i at $(-s_i, s_i^2)$ and each w_j at (s_j, s_j^2) (again on opposite sides of the parabola $y = x^2$). Say we block every edge by a point on the y-axis. The edge $v_i w_j$ crosses the y-axis at $(0, s_i s_j)$. Thus to have few blockers, S should be chosen so that the product set $S \cdot S := \{ab : a, b \in S\}$ is small. Geometric progessions, such as $2^1, 2^2, \dots, 2^n$, minimise the size of the product set (leading to the construction of $K_{n,n}$ above). It is interesting that both sum sets (that is, midpoints) and product sets appear to be related to blocking sets. There is a known trade-off between the sizes of sum sets and product sets (so-called *sum-product estimates*). In particular, $|S+S|$ or $|S \cdot S|$ is at least $c|S|^{1+\epsilon}$ for some $c > 0$ and $\epsilon > 0$; see [11, 12, 14, 17, 37]. Especially given that geometric methods based on the Szemerédi-Trotter theorem can be used to prove such a result [14], it is plausible that sum-product estimates might shed some light on Conjecture 3.

[6] Stanchescu's result is stated for points with integer coordinates, but by the well-known Freiman isomorphism [46], the result also applies for general point sets.

5 Point Sets with Bounded Collinearities

Now consider midpoints and blocking sets for point sets with a bounded number of collinear points. Let $m_\ell(n)$ be the minimum number of midpoints determined by some set of n points in the plane with no ℓ collinear points. Thus $m_3(n) = m(n)$. Pach's proof of (1) generalises as follows. Here we use a recent result of Bourgain [6] to improve upon the bound in (2).

Proposition 8. *For all $\epsilon > 0$, $\ell \geq 3$ and sufficiently large $n > n(\ell, \epsilon)$,*

$$m_\ell(n) \geq n(\log n)^{\frac{4}{11}-\epsilon} \ .$$

Proof. Let P be a set of n points in the plane with no ℓ collinear, such that $m(P) = m_\ell(n) = \alpha n$. As observed by Pach [29], Freiman's Theorem implies that at least $n^{1/d}/\beta$ points in P are collinear; see Theorem 6. Thus $n < (\beta\ell)^d$ and $\log n < d\log\beta + d\log\ell$. Bourgain [6] proved that, for some absolute constant $c > 0$, one can take $d = \lfloor\alpha - 1\rfloor$ and $\log\beta = c\alpha^{7/4}\log^c\alpha$ in Freiman's Theorem; also see [10, 35]. Thus $\log n < c\alpha^{11/4}\log^c\alpha + \alpha\log\ell$. Since $n \geq n(\epsilon, \ell)$, we have $c\log^c\alpha + \log\ell \leq \alpha^\epsilon$. Thus $\log n < \alpha^{11/4+\epsilon}$. Therefore $m_\ell(n) = \alpha n > n(\log n)^{1/(11/4+\epsilon)} \geq n(\log n)^{4/11-\epsilon}$. □

Analogous to the definition of $m_\ell(n)$, let $b_\ell(n)$ be the minimum integer such that every set of n points in the plane with no ℓ collinear points is blocked by some set of $b_\ell(n)$ points. Thus $b_3(n) = b(n)$. We conjecture that $b_\ell(n)$ is also super-linear in n for fixed ℓ.

Conjecture 9. *For all fixed ℓ, we have $\frac{b_\ell(n)}{n} \to \infty$ as $n \to \infty$.*

Proposition 10. *Conjecture 9 implies Conjecture 2.*

Proof. Suppose on the contrary that Conjecture 9 holds but Conjecture 2 does not. Then there are constants ℓ and k, and there are arbitrarily large point sets P containing no ℓ collinear points, and with $\chi(\mathcal{V}(P)) \leq k$. Conjecture 9 implies that $b_\ell(n) \geq n \cdot g_\ell(n)$ for some non-decreasing function g_ℓ for which $g_\ell(n) \to \infty$ as $n \to \infty$. Thus there is an integer n' such that $g_\ell(n) > k - 1$ for all $n \geq n'$. Let P be a set of $n \geq kn'$ points, containing no ℓ collinear points, and with $\chi(\mathcal{V}(P)) \leq k$. Let S be the largest colour class in a k-colouring of $\mathcal{V}(P)$. Thus S has no ℓ collinear points and $P - S$ blocks S. That is, there is a set of $s = \lceil\frac{n}{k}\rceil$ points blocked by a set of $n - s$ points. Thus $b_\ell(s) \leq n - s \leq n(1 - \frac{1}{k})$. On the other hand, $b_\ell(s) \geq s \cdot g_\ell(s) \geq \frac{n}{k} \cdot g_\ell(s)$. Hence $\frac{n}{k} \cdot g_\ell(s) \leq n(1 - \frac{1}{k})$ and $g_\ell(s) \leq k - 1$. Since $n' \leq s$ and g is non-decreasing, $g_\ell(n') \leq k - 1$, which is the desired contradiction. □

6 Colouring Edges and Points in Convex Position

Now consider edge-colourings of graph drawings, such that if two edges have the same colour, then they cross. This idea is related to blockers, since if a graph drawing can be blocked by b blockers, then it can be coloured with b colours. Let $t(n)$ be the minimum integer such that the edges in some geometric drawing of K_n can be coloured with $t(n)$ colours such that every monochromatic pair of edges cross. Each colour class is called a *crossing family* [4]. Hence $t(n) \leq b(n)$. We conjecture the following strengthening of Conjecture 3.

Conjecture 11. $\frac{t(n)}{n} \to \infty$ *as* $n \to \infty$.

The analogous conjecture could be made for extendible simple drawings of K_n.

For point sets in convex position, the above edge-colouring problem is equivalent to covering a circle graph[7] by cliques. It follows from a result by Kostochka [26] (see [25]) that the minimum number of colours is at least $n \ln n - c$ and at most $n \ln n + cn$, for some constant c. Thus the number of blockers for a point set in convex position is at least $n \ln n - c$. We conjecture that the answer is quadratic.

Conjecture 12. *Every set of* n *points in convex position requires* $\Omega(n^2)$ *blockers.*

For n equally spaced points around a circle, at least $\frac{n^2}{14} - O(n)$ blockers are required, since except for the point in the centre, at most 7 edges intersect at a common interior point [33]. This property does not hold for arbitrary points in convex position, since as described in Section 4, for the point set $P = \{(-2^i, 2^{2i}), (2^i, 2^{2i}) : i \in [1, n]\}$, the point $(0, 2^k)$ blocks each edge $(-2^i, 2^{2i})(2^j, 2^{2j})$ for which $k = i + j$. Thus $\Omega(n)$ points on the y-axis each block $\Omega(n)$ edges.

Note that Erdős et al. [16] proved that the minimum number of midpoints for a set of n points in convex position is between $0.8\binom{n}{2}$ and $0.9\binom{n}{2}$.

7 A Final Conjecture

We finish the paper with a strengthening of Conjecture 2.

Conjecture 13. *For all integers* $k \geq 1$ *and* $\ell \geq 2$ *there is an integer* n *such that if* P *is a set of at least* n *points in the plane, and each point in* P *is assigned one of* k *colours, then:*

- P *contains* ℓ *collinear points, or*
- P *contains a monochromatic line*
  (that is, a maximal set of collinear points, all receiving the same colour).

Conjecture 13 is trivially true for $k = 1$ and $n = 2$, or $\ell \leq 3$ and $n = k + 1$. The Motzkin-Rabin Theorem says that it is true for $k = 2$ with $n = \ell$; see [5, 28, 34]. Conjecture 13 is related to the Hales-Jewett Theorem [21, 32], which states that for sufficiently large d, every k-colouring of the grid $[1, \ell - 1]^d$ contains a monochromatic "combinatorial" line of length $\ell - 1$.

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[7] A *circle graph* is the intersection graph of a set of chords of a circle.

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