Light Edges in Degree-Constrained Graphs*

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Abstract

A graph whose average degree is less than twice the minimum degree is called *degree-constrained*. An edge of a graph with bounded degree end-points is said to be *light*. The primary result of this paper is that every degree-constrained graph has a light edge, where the degree bound depends on the minimum and average degrees. All previous results in this direction have only been for planar graphs. This result is extended in a variety of ways. First it is proved that every degree-constrained graph has many light edges, where the degree bound now also depends on the number of edges. A further generalisation is that degree-constrained graphs have a large matching (that is, an independent set) of light edges. Finally the method is refined in the case of planar graphs to obtain improved degree bounds, and applications to terrain mapping are discussed.

1 Introduction

Throughout this paper we consider simple, undirected, and finite graphs $G = (V, E)$ with n vertices, m edges, minimum degree δ, and average degree $\alpha = \frac{2m}{n}$. All graphs are assumed to be non-empty and have no isolated vertices; that is, $n > 0$ and $\alpha \geq \delta \geq 1$. For $S \subseteq V$, we denote by $G[S]$ the subgraph of G induced by S.

Many results of the following form can be found in the literature (see [3]): *For every graph G in a certain family of planar graphs, there is a subgraph of G isomorphic to a given graph H (for example, a vertex, an edge, a path, a cycle or a star) such that every vertex in the subgraph has degree at most some constant.*

Such a graph H is said to be *light* in the particular family under consideration. Typical families include the planar triangulations, the 3-connected planar graphs, and the planar graphs with minimum degree at least 3, 4, or 5. A classical example of the above type of result, due to Lebesgue [5], is that every 3-connected planar graph with $\delta \geq 3$ has a light edge. In particular, Borodin [1] proved that every planar graph with $\delta \geq 3$ (respectively, 4 or 5) has an edge vw with $\deg(v)$ and $\deg(w)$ at most 10 (7 or 6). In this paper we prove analogous results for arbitrary; that is, not necessarily planar, graphs with $\delta \geq 1$ and $\alpha < 2\delta$. We say such a graph is *degree-constrained*. Our primary result is that every degree-constrained graph has an edge whose end-points both have degree bounded by a value inversely proportional to δ and proportional to α.

This result is then generalised in a variety of ways. First we show that every degree-constrained graph has $\epsilon \cdot n$ light edges for some given $\epsilon < 1 - \frac{\alpha}{2\delta}$. A further generalisation is that every degree-constrained graph has a large matching of light edges. These results are presented in Section 2. In Section 3 we refine the method to prove stronger results in the case of planar graphs.

An application of our result for light matchings in planar graphs can be found in terrain mapping. In particular, de Berg and Dobrindt [2] use the Kirkpatrick hierarchy [4] to model different views of a polyhedral domain with various levels of detail. To go to from one level to the next an independent set of vertices, each of bounded degree, is typically removed. To save memory the authors propose an alternative strategy in which the vertices in a matching of light edges are removed. Our results thus provide a method for the worst-case analysis of such an algorithm.

2 Degree-Constrained Graphs

For a given graph $G = (V, E)$ and integer $d \geq \delta$, define $V_d = \{v \in V : \deg(v) \leq d\}$. We will need the following elementary results.

Lemma 1. *For every graph $G = (V, E)$ with minimum degree δ and average degree α, and for every integer $d \geq \delta$,*

$$|V_d| \ \geq \ \frac{d + 1 - \alpha}{d + 1 - \delta} n \ .$$

Proof. Since vertices in V_d have degree at least δ and

*Research supported by NSERC.

vertices not in V_d have degree at least $d+1$,

$$|V_d|\delta + (n-|V_d|)(d+1) \;\le\; \sum_{v\in V_d}\deg(v) \;+\; \sum_{v\in V\setminus V_d}\deg(v) \;=\; \alpha\cdot n \;.$$

Thus $|V_d|(\delta - d - 1) \le (\alpha - d - 1)n$, and the result follows. □

Lemma 2. *An independent set I of a graph G with minimum degree $\delta \ge 1$ and average degree α has $|I| \le \frac{\alpha\cdot n}{2\delta}$.*

Proof. No two vertices in I are adjacent, and every vertex in I has degree at least δ, which implies that there are at least $\delta|I|$ edges in G. The number of edges in G is exactly $\frac{\alpha n}{2}$. Thus $\frac{\alpha n}{2} \ge \delta|I|$, and the result follows. □

Let $\lceil\lceil x\rceil\rceil$ be the minimum integer strictly greater than x. Note that $\lceil\lceil x\rceil\rceil = \lceil x\rceil$ if and only if x is not an integer.

Theorem 1. *Every graph $G=(V,E)$ with minimum degree $\delta \ge 1$ and average degree $\alpha < 2\delta$ has an edge $vw \in E$ with both $\deg(v)$ and $\deg(w)$ at most $d = \lceil\lceil\frac{\alpha(\delta+1)-2\delta}{2\delta-\alpha}\rceil\rceil$.*

Proof. Observe that since $\alpha < 2\delta$, d is well-defined. Let t be an integer with $t \ge \delta$. By Lemma 1, we have $|V_t| \ge \frac{t+1-\alpha}{t+1-\delta}\,n$, whereas by Lemma 2, any independent set of G has at most $\frac{\alpha}{2\delta}n$ vertices. Assume we choose t such that

$$\frac{t+1-\alpha}{t+1-\delta} \;>\; \frac{\alpha}{2\delta} \;. \tag{1}$$

Then V_t is not an independent set of G and, therefore, G contains an edge vw such that $\deg(v)$ and $\deg(w)$ are both at most t. Inequality (1) is equivalent to $t(2\delta-\alpha) > -2\delta+\alpha+\alpha\delta$, which, since $\alpha < 2\delta$, is equivalent to

$$t \;>\; \frac{\alpha(\delta+1)-2\delta}{2\delta-\alpha} \;. \tag{2}$$

The smallest integer t for which (2) holds is $t = \lceil\lceil d\rceil\rceil$. It remains to verify that $\lceil\lceil d\rceil\rceil \ge \delta$. The inequality $d \ge \delta - 1$ is equivalent to $\alpha \ge \delta$. Thus $\lceil\lceil d\rceil\rceil \ge \delta$, and the result follows. □

It is easily seen that Theorem 1 is tight for regular graphs. Note that as $\alpha \to 2\delta$ in Theorem 1, $d \to \infty$. This property is unavoidable as every edge of $K_{2,n-2}$ (which has $\delta = 2$ and $\alpha \to 4$) is incident to a vertex of degree $n-2$. As a further example of Theorem 1, every graph with minimum degree 2 and average degree less than 3 has an edge vw with both $\deg(v)$ and $\deg(w)$ at most 5. This shows that one does not have to assume minimum degree 3 to guarantee a light edge if the graph is sparse enough. (All known light edge results assume minimum degree at least 3.)

Theorem 2. *Let $G=(V,E)$ be an n-vertex graph with minimum degree $\delta \ge 1$ and average degree α. For every ϵ with $0 < \epsilon < 1 - \frac{\alpha}{2\delta}$, G has a set S of at least $\epsilon\cdot n$ edges such that every edge $vw \in S$ has both $\deg(v)$ and $\deg(w)$ at most $\lceil\lceil d\rceil\rceil$, where*

$$d = \frac{(1-\delta)(\alpha+2\delta\epsilon) - 2\delta(1-\alpha)}{2\delta(1-\epsilon)-\alpha} \;.$$

Proof. Since $\epsilon < 1 - \frac{\alpha}{2\delta}$, we have $2\delta(1-\epsilon)-\alpha > 0$, and thus, d is well-defined. If $\alpha = \delta$; that is, G is regular, then $d = \alpha - 1$ and $\lceil\lceil d\rceil\rceil = \alpha$, and any set of $\epsilon\cdot n$ edges satisfies the theorem. We now assume $\alpha > \delta$. Let t be an integer with $t \ge \delta$. By Lemma 1, we have $|V_t| \ge \frac{t+1-\alpha}{t+1-\delta}n$. Let I be a maximum independent set of $G[V_t]$. Then I is an independent set of G, and by Lemma 2, $|I| \le \frac{\alpha}{2\delta}n$. Assume we choose t such that

$$\frac{t+1-\alpha}{t+1-\delta} - \frac{\alpha}{2\delta} \;>\; \epsilon \;. \tag{3}$$

Then $|V_t \setminus I| > \epsilon\cdot n$, and every vertex in $V_t \setminus I$ is incident to at least one edge whose other end-point is in I (otherwise I is not maximum). Thus $G[V_t]$ contains more than $\epsilon\cdot n$ edges all of whose end-points have degree at most t. Inequality (3) is equivalent to $t > d$. The smallest t for which $t > d$ holds is $t = \lceil\lceil d\rceil\rceil$. It remains to verify that $\lceil\lceil d\rceil\rceil \ge \delta$. The inequality $d > \delta - 1$ is equivalent to $\alpha > \delta$, which we assume to be true. Thus $\lceil\lceil d\rceil\rceil \ge \delta$, and the proof is complete. □

Theorem 3. *Let $G=(V,E)$ be an n-vertex graph with minimum degree $\delta \ge 1$ and average degree α such that*

$$\frac{1}{\delta+1} \;\le\; \frac{\alpha}{2\delta} \;\le\; 1 - \epsilon(\delta-1) \;. \tag{4}$$

For every $\epsilon > 0$ such that

$$\epsilon \le \frac{3\delta\alpha + \alpha - 2\delta - 2\delta^2 - 2\sqrt{2\delta(\alpha-\delta)(\delta\alpha+\alpha-2\delta)}}{2\delta(\delta-1)^2} \;, \tag{5}$$

G has a matching M of at least $\epsilon\cdot n$ edges such that every edge $vw \in M$ has $\deg(v)$ and $\deg(w)$ at most $\lceil\lceil d\rceil\rceil$, where $d_0 = 4\epsilon^2\delta^4 - 8\epsilon^2\delta^3 - 12\epsilon\delta^2\alpha + 8\epsilon\delta^3 + 4\epsilon^2\delta^2 - 4\epsilon\delta\alpha + 8\epsilon\delta^2 + \alpha^2 - 4\delta\alpha + 4\delta^2$ and

$$d = \frac{1}{4\epsilon\delta}\left(2\delta(\epsilon\delta - \epsilon + 1) - \alpha - \sqrt{d_0}\right) \;. \tag{6}$$

Proof. Let t be an integer with $t \ge \delta$. By Lemma 1,

$$|V_t| \;\ge\; \frac{t+1-\alpha}{t+1-\delta}\,n \;.$$

Let I be a maximum independent set of $G[V_t]$. Then I is an independent set of G, and by Lemma 2, $|I| \le \frac{\alpha\cdot n}{2\delta}$. Thus

$$|V_t \setminus I| \;\ge\; \left(\frac{t+1-\alpha}{t+1-\delta} - \frac{\alpha\cdot n}{2\delta}\right) n \;.$$

Let S be a set of edges in $G[V_t]$ such that every vertex in $V_t \setminus I$ has exactly one incident edge in S. Such a set exists as otherwise I would not be maximum. Observe that $|S| = |V_t \setminus I|$. Let M be a subset of S such that every vertex in I which is incident to an edge in S, is incident to precisely one edge in M. M is a matching, all of whose end-vertices have degree at most t. Since every vertex $v \in I$ has $\deg(v) \leq t$,

$$|M| \geq \frac{|S|}{t} = \frac{|V_t \setminus I|}{t} \geq \frac{1}{t}\left(\frac{t+1-\alpha}{t+1-\delta} - \frac{\alpha}{2\delta}\right) n .$$

For a given ϵ, we must choose t such that $|M| \geq \epsilon \cdot n$; that is,

$$\epsilon \leq \frac{1}{t}\left(\frac{t+1-\alpha}{t+1-\delta} - \frac{\alpha}{2\delta}\right) . \qquad (7)$$

Inequality (7) is implied by $t \geq d$, subject to the contents of the square root in (6) being non-negative, which is implied by (5). The smallest integer t satisfying (7) is $\lceil\lceil d\rceil\rceil$. It remains to verify that $\lceil\lceil d\rceil\rceil \geq \delta$. Assuming (4), the inequality $d \geq \delta - 1$ is equivalent to $\alpha \geq \delta$. Thus $d \geq \delta - 1$, and hence $\lceil\lceil d\rceil\rceil \geq \delta$. This completes the proof. $\square$

Observe that the assumptions of Theorems 2 and 3 imply $\alpha < 2\delta$; that is, the graph is degree-constrained. As $\epsilon \to 0$, the maximum allowed value of $\alpha \to 2\delta$.

3 Planar Graphs

We now strengthen Lemma 2 in the case of planar graphs.

Lemma 3. *A maximum independent set I of an n-vertex planar graph $G = (V, E)$ with minimum degree δ has $|I| \leq \frac{2n-4}{\delta}$.*

Proof. Let $G' = (V, E')$ be the bipartite subgraph of G consisting of all edges from I to $V \setminus I$. Then $\delta|I| \leq |E'| \leq 2n - 4$, and hence $|I| \leq (2n-4)/\delta$. $\square$

Applying Lemma 3 with the arguments of Theorems 1, 2 and 3, we obtain the following stronger results for planar graphs (which have $\alpha < 6$). Complete details of these proofs can be found in the full version of the paper. Refer to Figures 1 and 2 for the actual degree bounds in particular cases of δ.

Theorem 4. *Every planar graph $G = (V, E)$ with minimum degree $\delta \geq 3$ has an edge $vw \in E$ with $\deg(v)$ and $\deg(w)$ at most $d = 3 + \lceil\frac{8}{\delta-2}\rceil$. For $\delta = 3$, 4 or 5, we have $d = 11$, 7 or 6, respectively.* $\square$

For $\delta \in \{4, 5\}$, Theorem 4 gives a new proof of the result by Borodin [1] mentioned in Section 1. For $\delta = 3$, the bound of Borodin [1] is 1 less than the bound in Theorem 4.

Theorem 5. *Let $G = (V, E)$ be an n-vertex planar graph with minimum degree $\delta \in \{3, 4, 5\}$. For every ϵ, $0 < \epsilon < 1 - \frac{2}{\delta}$, G has a set S of at least $\epsilon \cdot n$ edges such that every edge $vw \in S$ has $\deg(v)$ and $\deg(w)$ at most $\lceil d\rceil$, where*

$$d = \frac{2 + \delta(3 + \epsilon(1-\delta))}{\delta(1-\epsilon) - 2} . \qquad \square$$

Theorem 6. *Let $G = (V, E)$ be an n-vertex planar graph with minimum degree $\delta \in \{3, 4, 5\}$. Given ϵ such that*

$$0 < \epsilon \leq \frac{\delta(9-\delta) + 2 - 2\sqrt{\delta(6-\delta)(3\delta+2)}}{\delta(\delta-1)^2} , \qquad (8)$$

let $d_0 = \delta^2 - 4\delta - 18\epsilon\delta^2 + 2\epsilon\delta^3 + 4 - 4\epsilon\delta + \epsilon^2\delta^2 - 2\epsilon^2\delta^3 + \epsilon^2\delta^4$ *and*

$$d = \frac{1}{2\epsilon\delta}\left(\delta - 2 - \epsilon\delta + \epsilon\delta^2 - \sqrt{d_0}\right) . \qquad (9)$$

Then G has a matching M with at least $\epsilon \cdot n$ edges such that every edge $vw \in M$ has $\deg(v)$ and $\deg(w)$ at most $\lceil d\rceil$. $\square$

Finally, note that Lemma 3 can be generalised for graphs G with a 2-cell embedding on an arbitrary surface of Euler characteristic e. In particular, the number of edges in a bipartite subgraph of G is at most $2(n-e)$, and thus $|I| \leq \frac{2(n-e)}{\delta}$. Therefore analogous results to Theorems 4, 5 and 6 can easily be obtained.

References

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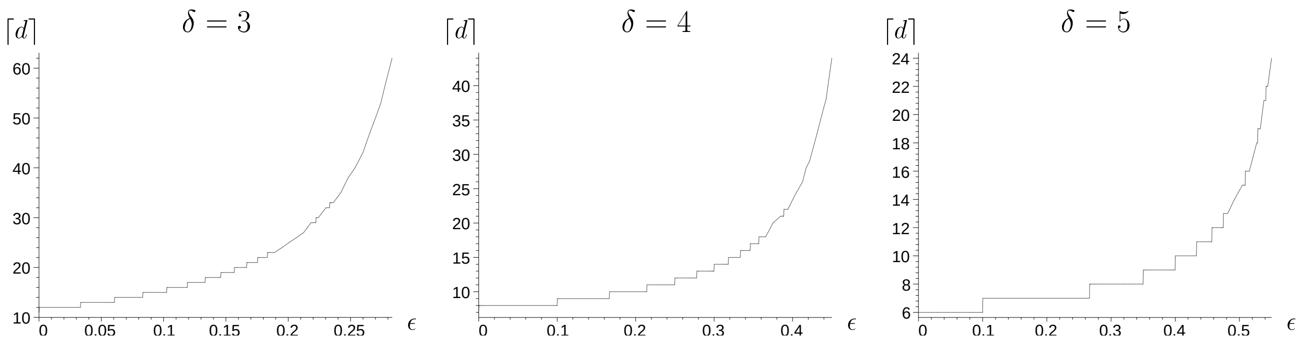

Figure 1: The degree bound in a set of $\epsilon \cdot n$ light edges in a planar graph (Theorem 5).

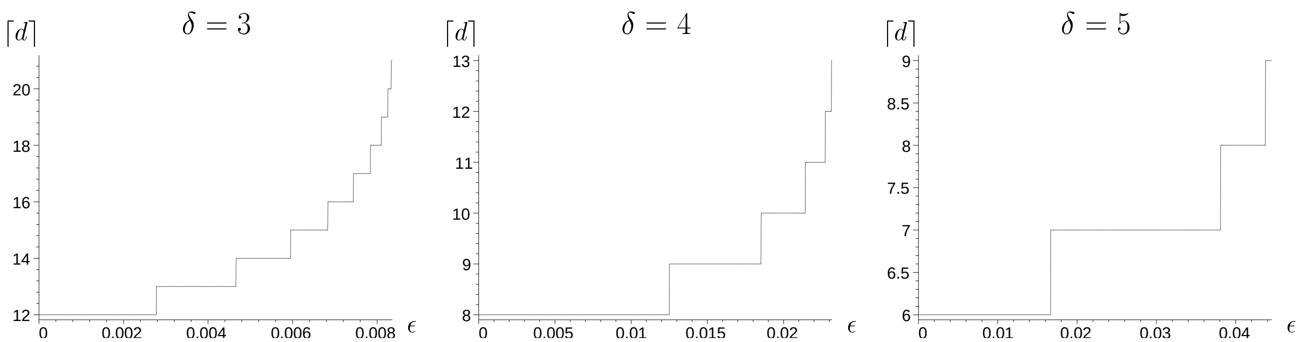

Figure 2: The degree bound in a light matching of $\epsilon \cdot n$ edges in a planar graph (Theorem 6).