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Light edges in degree-constrained graphs☆

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Abstract

Let α denote the average degree, and δ denote the minimum degree of a graph. An edge is *light* if both its endpoints have degree bounded by a constant depending only on α and δ. A graph is *degree-constrained* if $\alpha < 2\delta$. The primary result of this paper is that every degree-constrained graph has a light edge. Most previous results in this direction have been for embedded graphs. This result is extended in a variety of ways. First it is proved that there exists a constant $c(\alpha, \delta)$ such that for every $0 \leqslant \varepsilon < c(\alpha, \delta)$, every degree-constrained graph with n vertices has at least $\varepsilon \cdot n$ light edges. An analogous result is proved guaranteeing a matching of light edges. The method is refined in the case of planar graphs to obtain improved degree bounds.

Keywords: Graph; Light edge; Matching

1. Introduction

Throughout this paper we consider finite undirected graphs $G = (V, E)$ with n vertices, m edges, minimum degree δ, and average degree $\alpha = 2m/n$. All graphs are assumed to be non-empty and have no isolated vertices; that is, $n > 0$ and $\alpha \geqslant \delta \geqslant 1$. For $S \subseteq V$, we denote by $G[S]$ the subgraph of G induced by S.

Many results of the following form can be found in the literature (see [6] for a recent survey): *For every graph G in a certain family of graphs, there is a subgraph of G isomorphic to a given graph H (for example, a vertex, an edge, a path, a cycle or a star) such that every vertex in the subgraph has degree at most some constant.*

Such a graph H is said to be *light* in the particular family under consideration. Typical families include the planar triangulations, the 3-connected planar graphs, and the planar graphs with $\delta \geqslant 3, 4$ or 5. A classical example of the above type of result, due to Lebesgue [9], is that every 3-connected planar graph has a light edge. Kotzig [8] proved that every 3-connected planar graph has an edge vw with $\deg(v) + \deg(w) \leqslant 13$, and if $\delta \geqslant 4$ then $\deg(v) + \deg(w) \leqslant 11$. Borodin [1] extended these results to planar graphs with $\delta \geqslant 3$.

In this paper we prove analogous results for (not necessarily planar) graphs with $\delta \geqslant 1$ and $\alpha < 2\delta$. We say such a graph is *degree-constrained*. Our primary result is that every degree-constrained graph has an edge whose endpoints both have degree bounded by a value inversely proportional to δ and proportional to α. Since there exist graphs with $\alpha = 2\delta$ which contain no light edge (see Section 2), the restriction to degree-restricted graphs is necessary. Most previous results in this direction have been for embedded graphs. A notable exception is due to Jendrol' and Schiermeyer [5], who determined the minimum c such that every n-vertex m-edge graph has an edge vw such that $\deg(v) + \deg(w) \leqslant c$.

Our primary result is generalised in a variety of ways. First we prove that every degree-constrained graph has $\varepsilon \cdot n$ light edges for any given $\varepsilon < 1 - \alpha/2\delta$. A further generalisation is that every degree-constrained graph has a large light *matching* (that is, a set of edges with no common endpoint). These results are presented in Section 2. In Section 3, we

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refine the method to prove stronger results in the case of planar graphs. Note that results by Borodin [2] and Borodin and Sanders [3] also provide a large, but constant sized, set of light edges in a planar graph.

An application of our result for light matchings in planar graphs can be found in terrain mapping. In particular, de Berg and Dobrindt [4] use the Kirkpatrick hierarchy [7] to model different views of a polyhedral domain with various levels of detail. To go from one level to the next, an independent set of vertices, each of bounded degree, is typically removed. To save memory the authors propose an alternative strategy in which the vertices in a matching of light edges are removed. Our results thus provide a method for the worst-case analysis of such an algorithm.

2. Degree-constrained graphs

For a given graph $G=(V,E)$ and integer $d \geqslant \delta$, define $V_d=\{v\in V:\ \deg(v)\leqslant d\}$. We will need the following elementary results.

Lemma 1. *For every graph $G=(V,E)$ with minimum degree δ and average degree α, and for every integer $d\geqslant\delta$,*

$$|V_d| \geqslant \frac{d+1-\alpha}{d+1-\delta}\, n.$$

Proof. Since vertices in V_d have degree at least δ and vertices not in V_d have degree at least $d+1$,

$$|V_d|\delta + (n-|V_d|)(d+1) \leqslant \sum_{v\in V_d} \deg(v) + \sum_{v\in V\setminus V_d} \deg(v) = \alpha\cdot n.$$

Thus $|V_d|(\delta-d-1)\leqslant(\alpha-d-1)n$, and the result follows. □

Lemma 2. *Every independent set I of a graph G with minimum degree $\delta\geqslant 1$ and average degree α has $|I|\leqslant \alpha\cdot n/2\delta$.*

Proof. No two vertices in I are adjacent, and every vertex in I has degree at least δ, which implies that there are at least $\delta|I|$ edges in G. The number of edges in G is exactly $\alpha n/2$. Thus $\alpha n/2\geqslant\delta|I|$, and the result follows. □

Theorem 3. *Every graph $G=(V,E)$ with minimum degree $\delta\geqslant 1$ and average degree $\alpha<2\delta$ has an edge $vw\in E$ with both $\deg(v)$ and $\deg(w)$ at most $\lfloor d\rfloor$, where*

$$d=\frac{\alpha\delta}{2\delta-\alpha}.$$

Proof. Observe that since $\alpha<2\delta$, d is well-defined.

Case 1: $\alpha=\delta$ (that is, G is regular): Then $d=\alpha$, and every edge vw of G has $\deg(v)=\deg(w)=\lfloor d\rfloor$.

Case 2: $\alpha>\delta$: Let t be an integer with $t\geqslant\delta$. By Lemma 1, we have $|V_t|\geqslant(t+1-\alpha)n/(t+1-\delta)$, whereas by Lemma 2, any independent set of G has at most $\alpha/2\delta n$ vertices. Assume we choose t such that

$$\frac{t+1-\alpha}{t+1-\delta} > \frac{\alpha}{2\delta}. \tag{1}$$

Then V_t is not an independent set of G and, therefore, G contains an edge vw such that $\deg(v)$ and $\deg(w)$ are both at most t. Inequality (1) is equivalent to $t(2\delta-\alpha)>-2\delta+\alpha+\alpha\delta$, which, since $\alpha<2\delta$, is equivalent to

$$t > \frac{\alpha(\delta+1)-2\delta}{2\delta-\alpha}. \tag{2}$$

The smallest integer t for which (2) holds is $t=\lfloor(\alpha(\delta+1)-2\delta)/(2\delta-\alpha)\rfloor+1=\lfloor d\rfloor$. It remains to verify that $\lfloor d\rfloor\geqslant\delta$. Since $\alpha>\delta$, we have that $d>\delta$. Thus $\lfloor d\rfloor\geqslant\delta$, and the result follows. □

Observe that Theorem 3 is tight for regular graphs. As $\alpha\to 2\delta$ in Theorem 3, $d\to\infty$. This property is unavoidable since, for every $\delta\geqslant 1$, every edge of the complete bipartite graph $K_{\delta,n-\delta}$ (which has minimum degree δ and average degree $\alpha\to 2\delta$) is incident to a vertex of unbounded degree.

Theorem 4. *Let $G=(V,E)$ be an n-vertex graph with minimum degree $\delta \geqslant 1$ and average degree α. For every ε with $0 \leqslant \varepsilon < 1-\alpha/2\delta$, there is a set S of edges of G such that $|S| > \varepsilon \cdot n$, and every edge $vw \in S$ has both* $\deg(v)$ *and* $\deg(w)$ *at most $\lfloor d \rfloor$, where*

$$d = \frac{\delta(\alpha - 2\varepsilon\delta)}{2\delta(1-\varepsilon)-\alpha}.$$

Proof. Since $\varepsilon < 1-\alpha/2\delta$, we have $2\delta(1-\varepsilon)-\alpha > 0$, and thus, d is well-defined.

Case 1: $\alpha = \delta$ (that is, G is regular): Then $d = \alpha$ and any set of more than $\varepsilon \cdot n$ edges satisfies the theorem. There is always more than $\varepsilon \cdot n$ edges, since $\varepsilon \cdot n < (1-\alpha/2\delta)n = \frac{1}{2}n \leqslant |E|$.

Case 2: $\alpha > \delta$: Let t be an integer with $t \geqslant \delta$. By Lemma 1, we have $|V_t| \geqslant (t+1-\alpha)n/(t+1-\delta)$. Let I be a maximum independent set of $G[V_t]$. Then I is an independent set of G, and by Lemma 2, $|I| \leqslant (\alpha/2\delta)n$. Assume we choose t such that

$$\frac{t+1-\alpha}{t+1-\delta} - \frac{\alpha}{2\delta} > \varepsilon. \tag{3}$$

Then $|V_t \backslash I| > \varepsilon \cdot n$, and every vertex in $V_t \backslash I$ is incident to at least one edge whose other endpoint is in I (otherwise I is not maximum). Thus $G[V_t]$ contains more than $\varepsilon \cdot n$ edges all of whose endpoints have degree at most t. Inequality (3) is equivalent to

$$t > \frac{2\varepsilon\delta - 2\varepsilon\delta^2 - 2\delta + \delta\alpha + \alpha}{2\delta(1-\varepsilon)-\alpha}. \tag{4}$$

The smallest integer t for which inequality (4) holds is

$$\left\lfloor \frac{2\varepsilon\delta - 2\varepsilon\delta^2 - 2\delta + \delta\alpha + \alpha}{2\delta(1-\varepsilon)-\alpha} + 1 \right\rfloor = \lfloor d \rfloor.$$

It remains to verify that $\lfloor d \rfloor \geqslant \delta$. Since $\alpha > \delta$, we have that $d > \delta$. Thus $\lfloor d \rfloor \geqslant \delta$, and the proof is complete. □

Note that Theorem 4 with $\varepsilon = 0$ is the same as Theorem 3.

Theorem 5. *Let $G=(V,E)$ be an n-vertex graph with minimum degree $\delta \geqslant 2$ and average degree α. For every $\varepsilon > 0$ such that*

$$\varepsilon < \frac{1}{2\delta^2(\delta-2)}\left(\alpha(3\delta+1) - 2\delta(1+\delta) - \sqrt{(8\delta^2+8\delta+1)\alpha^2 - 4\delta(2\delta^2+6\delta+1)\alpha + (1+4\delta)4\delta^2}\right) \tag{5}$$

and

$$0 < \varepsilon \leqslant \frac{2\delta-\alpha}{2\delta(\delta-1)}, \tag{6}$$

G has a matching M such that $|M| > \varepsilon \cdot n$, and every edge $vw \in M$ has $\deg(v)$ *and* $\deg(w)$ *at most $\lfloor d \rfloor$, where*

$$d = \frac{1}{4\varepsilon\delta}\left(2\delta(\varepsilon\delta+\varepsilon+1) - \alpha - \sqrt{4(\delta-1)^2\delta^2\varepsilon^2 + 4\delta(2\delta^2 - 3\delta\alpha + 2\delta - \alpha)\varepsilon + (\alpha-2\delta)^2}\right). \tag{7}$$

Proof. Let t be an integer with $t \geqslant \delta$. By Lemma 1,

$$|V_t| \geqslant \frac{t+1-\alpha}{t+1-\delta}\,n.$$

Let I be a maximum independent set of $G[V_t]$. Then I is an independent set of G, and by Lemma 2, $|I| \leqslant \alpha \cdot n/2\delta$. Thus

$$|V_t \backslash I| \geqslant \left(\frac{t+1-\alpha}{t+1-\delta} - \frac{\alpha}{2\delta}\right)n.$$

Let S be a set of edges in $G[V_t]$ such that every vertex in $V_t \backslash I$ has exactly one incident edge in S. Such a set exists as otherwise I would not be maximum. Observe that $|S| = |V_t \backslash I|$. Let M be a subset of S such that every vertex in I that is incident to an edge in S, is incident to precisely one edge in M. M is a matching, all of whose endpoints have degree at most t. Since every vertex $v \in I$ has $\deg(v) \leqslant t$,

$$|M| \geqslant \frac{|S|}{t} = \frac{|V_t \backslash I|}{t} \geqslant \frac{1}{t}\left(\frac{t+1-\alpha}{t+1-\delta} - \frac{\alpha}{2\delta}\right)n.$$

For a given ε, we must choose t such that $|M| > \varepsilon \cdot n$; that is,

$$\varepsilon > \frac{1}{t}\left(\frac{t+1-\alpha}{t+1-\delta} - \frac{\alpha}{2\delta}\right). \tag{8}$$

This is equivalent to finding an integer solution to the quadratic inequality

$$(2\varepsilon\delta)t^2 + (2\delta(\varepsilon - \varepsilon\delta - 1) + \alpha)t + (\alpha(\delta+1) - 2\delta) < 0. \tag{9}$$

The real-valued solutions to (9) are $\{t\colon t^- < t < t^+\}$, where

$$t^{\pm} = \frac{1}{4\varepsilon\delta}\left(2\delta(\varepsilon\delta - \varepsilon + 1) - \alpha \pm \sqrt{4(\delta-1)^2\delta^2\varepsilon^2 + 4\delta(2\delta^2 - 3\delta\alpha + 2\delta - \alpha)\varepsilon + (\alpha - 2\delta)^2}\right). \tag{10}$$

The minimum integer strictly greater than t^- is $\lfloor t^- \rfloor + 1 = \lfloor d \rfloor$. We now show that $\lfloor d \rfloor < t^+$. Now $d < t^+$ if and only if

$$0 < 4(\delta - 2)\delta^3\varepsilon^2 + 4\delta(2\delta^2 - 3\delta\alpha + 2\delta - \alpha)\varepsilon + (\alpha - 2\delta)^2. \tag{11}$$

Solving (11) for ε, we find that condition (5) ensures that (11) is satisfied. Thus $d < t^+$, and hence $t^- < \lfloor d \rfloor < t^+$. Therefore $t = \lfloor d \rfloor$ is an integer-valued solution of (9).

Note that condition (5) ensures that the content of the square root in the definition of d is non-negative. Thus d is well-defined. Also observe that since $\alpha \geqslant \delta$, the content of the square root in (5) is non-negative, and the maximum allowed value of ε is well-defined.

It remains to verify that $\lfloor d \rfloor \geqslant \delta$. First suppose that $\alpha = \delta$; that is, G is regular. Then

$$d = \frac{1}{4\varepsilon\alpha}\left(2\alpha(\varepsilon\alpha + \varepsilon + 1) - \alpha - \alpha\sqrt{(2(\alpha-1)\varepsilon - 1)^2}\right).$$

By (6), we have that $2(\alpha - 1)\varepsilon - 1 \leqslant 0$. Thus,

$$d = \frac{1}{4\varepsilon\alpha}(2\alpha(\varepsilon\alpha + \varepsilon + 1) - \alpha + \alpha(2(\alpha-1)\varepsilon - 1)) = \alpha,$$

and $\lfloor d \rfloor \geqslant \delta$ in the case that $\alpha = \delta$. Now suppose $\alpha > \delta$. Then $d > \delta$ if and only if

$$2\delta(\varepsilon - \varepsilon\delta + 1) - \alpha > \sqrt{4(\delta-1)^2\delta^2\varepsilon^2 + 4\delta(2\delta^2 - 3\delta\alpha + 2\delta - \alpha)\varepsilon + (\alpha - 2\delta)^2}.$$

By (6), we have that $2\delta(\varepsilon - \varepsilon\delta + 1) - \alpha \geqslant 0$. Thus $d > \delta$ if and only if

$$(2\delta(\varepsilon - \varepsilon\delta + 1) - \alpha)^2 > 4(\delta-1)^2\delta^2\varepsilon^2 + 4\delta(2\delta^2 - 3\delta\alpha + 2\delta - \alpha)\varepsilon + (\alpha - 2\delta)^2,$$

which reduces to $\alpha > \delta$. Thus $d > \delta$ and hence $\lfloor d \rfloor \geqslant \delta$. This completes the proof. □

Observe that the assumptions of Theorems 4 and 5 imply $\alpha < 2\delta$; that is, the graph is degree-constrained.

3. Planar graphs

Every planar graph G has average degree $\alpha < 6$, and thus, if the minimum degree $\delta \geqslant 3$, then G is degree-constrained. Using the following analogue of Lemma 2 for planar graphs, we now strengthen our results from the previous section in the case of planar graphs.

Lemma 6. *A maximum independent set I of an n-vertex planar graph $G = (V, E)$ with minimum degree $\delta \geqslant 1$ has $|I| \leqslant (2n - 4)/\delta$.*

Proof. Let $G' = (V, E')$ be the bipartite subgraph of G consisting of all edges between I and $V \backslash I$. Then $\delta|I| \leqslant |E'| \leqslant 2n - 4$, and hence $|I| \leqslant (2n-4)/\delta$. □

Theorem 7. *Every planar graph $G = (V, E)$ with minimum degree $\delta \geqslant 3$ has an edge $vw \in E$ with $\deg(v)$ and $\deg(w)$ at most $d = 3 + \lceil 8/(\delta - 2) \rceil$. For $\delta = 3, 4$ or 5, we have $d = 11, 7$ or 6, respectively.*

Proof. Observe that since $\delta \geqslant 3$, d is well-defined. Let t be an integer with $t \geqslant \delta$. For a planar graph, $\alpha < 6$. Thus, by Lemma 1, we have $|V_t| > (t-5)n/(t+1-\delta)$, whereas by Lemma 6, any independent set of G has less than $2n/\delta$ vertices. Assume we choose t such that

$$\frac{t-5}{t+1-\delta} \geqslant \frac{2}{\delta}. \tag{12}$$

Then V_t is not an independent set of G and, therefore, G contains an edge vw such that $\deg(v)$ and $\deg(w)$ are both at most t. Inequality (12) is equivalent to

$$t \geqslant \frac{2+3\delta}{\delta-2} = 3 + \frac{8}{\delta-2}. \tag{13}$$

The smallest integer t satisfying (13) is $t = d$. Since $d \geqslant \delta$ the result follows. □

It follows from the result of Borodin [1] mentioned in Section 1 that every planar graph with $\delta \geqslant 3, 4$ or 5 has an edge whose endpoints both have degree at most 10, 7 or 6, respectively. For $\delta \in \{4,5\}$, Theorem 7 gives a new proof of this result.

Theorem 8. *Let $G=(V,E)$ be an n-vertex planar graph with minimum degree $\delta \in \{3,4,5\}$. For every ε, $0 \leqslant \varepsilon < 1-2/\delta$, G has a set of edges S such that $|S| > \varepsilon \cdot n$, and every edge $vw \in S$ has* $\deg(v)$ *and* $\deg(w)$ *at most $\lceil d \rceil$, where*

$$d = \frac{2+\delta(3+\varepsilon(1-\delta))}{\delta(1-\varepsilon)-2}.$$

Proof. Since $\varepsilon < 1-2/\delta$ we have $\delta(1-\varepsilon)-2 > 0$, and d is well-defined. Let t be an integer with $t \geqslant \delta$. For a planar graph, $\alpha < 6$. Thus, by Lemma 1, we have $|V_t| > (t-5)n/(t+1-\delta)$, whereas by Lemma 6, a maximum independent set I of $G[V_t]$ has less than $2n/\delta$ vertices (since I is an independent set of G). Assume we choose t such that

$$\frac{t-5}{t+1-\delta} - \frac{2}{\delta} \geqslant \varepsilon. \tag{14}$$

Then $|V_t \backslash I| > \varepsilon \cdot n$, and every vertex in $V_t \backslash I$ is incident to at least one edge whose other endpoint is in I (otherwise I is not maximum). Thus $G[V_t]$ contains a set S of edges, all of whose endpoints have degree at most t, and $|S| > \varepsilon \cdot n$. Inequality (14) is equivalent to

$$t(\delta(1-\varepsilon)-2) \geqslant (1-\delta)(2+\varepsilon\delta)+5\delta.$$

Since $\varepsilon < 1-2/\delta$, it follows that (14) is equivalent to

$$t \geqslant \frac{2+\delta(\varepsilon(1-\delta)+3)}{\delta(1-\varepsilon)-2}. \tag{15}$$

The smallest t for which (15) holds is $t = \lceil d \rceil$. It remains to verify that $\lceil d \rceil \geqslant \delta$. The inequality $d \geqslant \delta$ is equivalent to $2+5\delta+\varepsilon\delta \geqslant \delta^2$, which holds since $3 \leqslant \delta \leqslant 5$ and $\varepsilon \geqslant 0$. □

Note that Theorem 8 with $\varepsilon = 0$ is the same as Theorem 7. Fig. 1 illustrates the degree bound in Theorem 8.

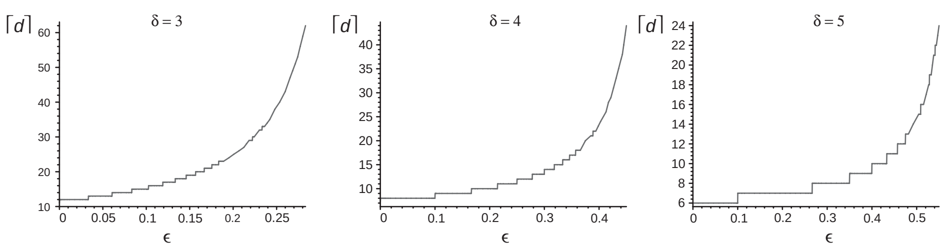

Fig. 1. The degree bound in a set of $\varepsilon \cdot n$ light edges in a planar graph.

Theorem 9. *Let $G=(V,E)$ be an n-vertex planar graph with minimum degree $\delta\in\{3,4,5\}$. Given ε such that*

$$0<\varepsilon\leqslant\frac{9\delta-\delta^2+2-\sqrt{65\delta^2-12\delta^3+44\delta+4}}{\delta^2(\delta-2)}, \tag{16}$$

let

$$d=\frac{1}{2\varepsilon\delta}\left(\delta(\varepsilon\delta-\varepsilon+1)-2-\sqrt{\delta^2(\delta-1)^2\varepsilon^2+2\delta(\delta^2-9\delta-2)\varepsilon+4(1-\delta)}\right). \tag{17}$$

Then G has a matching M such that $|M|>\varepsilon\cdot n$, and every edge $vw\in M$ has $\deg(v)$ *and* $\deg(w)$ *at most $\lceil d\rceil$.*

Proof. Let t be an integer with $t\geqslant\delta$. For a planar graph, $\alpha<6$. Thus, by Lemma 1, we have $|V_t|>(t-5)n/(t+1-\delta)$. Let I be a maximum independent set of $G[V_t]$. Then I is an independent set of G, and by Lemma 6, $|I|<2n/\delta$. Thus

$$|V_t\backslash I|>\left(\frac{t-5}{t+1-\delta}-\frac{2}{\delta}\right)n.$$

Let S be a set of edges in $G[V_t]$ such that every vertex in $V_t\backslash I$ has exactly one incident edge in S. Such a set exists as otherwise I would not be maximum. Observe that $|S|=|V_t\backslash I|$. Let M be a subset of S such that every vertex in I that is incident to an edge in S, is incident to precisely one edge in M. M is a matching, all of whose endpoints have degree at most t. Since every vertex $v\in I$ has $\deg(v)\leqslant t$,

$$|M|\geqslant\frac{|S|}{t}=\frac{|V_t\backslash I|}{t}>\frac{1}{t}\left(\frac{t-5}{t+1-\delta}-\frac{2}{\delta}\right)n.$$

For a given ε, we must choose t such that $|M|>\varepsilon\cdot n$; that is,

$$\frac{1}{t}\left(\frac{t-5}{t+1-\delta}-\frac{2}{\delta}\right)\geqslant\varepsilon. \tag{18}$$

Inequality (18) is equivalent to the following quadratic inequality in t.

$$(-\varepsilon\delta)t^2+(\delta-2-\delta(1-\delta)\varepsilon)t-(3\delta+2)\geqslant 0. \tag{19}$$

Solving (19) for t, we have that $d\leqslant t\leqslant t^+$, where

$$t^+=\frac{1}{2\varepsilon\delta}\left(\delta(\varepsilon\delta-\varepsilon+1)-2+\sqrt{\delta^2(\delta-1)^2\varepsilon^2+2\delta(\delta^2-9\delta-2)\varepsilon+4(1-\delta)}\right). \tag{20}$$

We now show that $\lceil d\rceil\leqslant t^+$. Now, $d+1\leqslant t^+$ if and only if

$$\delta^3(\delta-2)\varepsilon^2+2\delta(\delta^2-9\delta-2)\varepsilon+4(1-\delta)\geqslant 0. \tag{21}$$

By solving (21) for ε, we find that condition (16) guarantees that (21) is satisfied. Thus $d+1\leqslant t^+$ and hence $\lceil d\rceil\leqslant t^+$. Therefore, $t=\lceil d\rceil$ is an integer-valued solution to (19).

Note that, since (21) is satisfied, the content of the square-root in the definition of d is non-negative, and hence d is well-defined. (In fact (16) is a stronger condition that what is required here.) Since $3\leqslant\delta\leqslant 5$, the content of the square-root in (16) is non-negative, and the range for ε is well-defined.

It remains to verify that $\lceil d\rceil\geqslant\delta$. Now $d\geqslant\delta$ if $\varepsilon\geqslant 1-(5\delta+2)/\delta^2$, which is negative for $3\leqslant\delta\leqslant 5$. Since $\varepsilon>0$, we have $d\geqslant\delta$, and hence, $\lceil d\rceil\geqslant\delta$. This completes the proof. □

Fig. 2 illustrates the degree bound in Theorem 9.

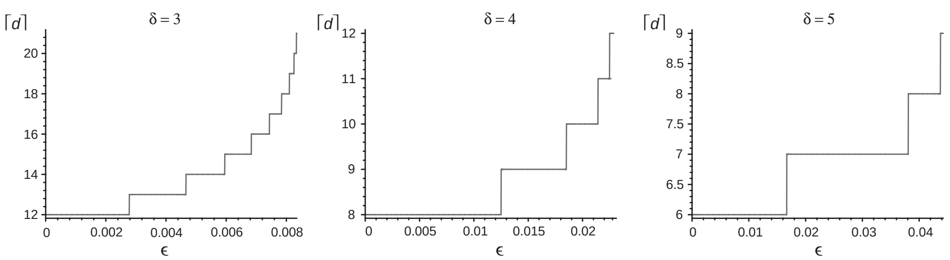

Fig. 2. The degree bound in a light matching of $\varepsilon\cdot n$ edges in a planar graph.

Finally, note that Lemma 6 can be generalised for graphs G with a 2-cell embedding in an arbitrary surface of Euler characteristic e. In particular, the number of edges in a bipartite subgraph of G is at most $2(n-e)$, and thus $|I| \leqslant 2(n-e)/\delta$. Therefore analogous results to Theorems 7–9 can easily be obtained.

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