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Drawings of planar graphs with few slopes and segments ☆

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Abstract

We study straight-line drawings of planar graphs with few segments and few slopes. Optimal results are obtained for all trees. Tight bounds are obtained for outerplanar graphs, 2-trees, and planar 3-trees. We prove that every 3-connected plane graph on n vertices has a plane drawing with at most $\frac{5}{2}n$ segments and at most $2n$ slopes. We prove that every cubic 3-connected plane graph has a plane drawing with three slopes (and three bends on the outerface). In a companion paper, drawings of non-planar graphs with few slopes are also considered.

Keywords: Graph drawing; Planar graph; Slope; Segment; Canonical decomposition

1. Introduction

A common requirement for an aesthetically pleasing drawing of a graph is that the edges are straight. This paper studies the following additional requirements of straight-line graph drawings:

(1) minimise the number of segments in the drawing, and
(2) minimise the number of distinct edge slopes in the drawing.

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Table 1
Summary of results (ignoring additive constants). Here n is the number of vertices, η is the number of vertices of odd degree, and Δ is the maximum degree. The lower bounds are existential, except for trees, for which the lower bounds are universal

Graph family	# segments		# slopes	
	$\geqslant$	$\leqslant$	$\geqslant$	$\leqslant$
trees	$\frac{\eta}{2}$	$\frac{\eta}{2}$	$\lceil\frac{\Delta}{2}\rceil$	$\lceil\frac{\Delta}{2}\rceil$
maximal outerplanar	n	n	–	n
plane 2-trees	$2n$	$2n$	$2n$	$2n$
plane 3-trees	$2n$	$2n$	$2n$	$2n$
plane 2-connected	$\frac{5}{2}n$	–	$2n$	–
planar 2-connected	$2n$	–	n	–
plane 3-connected	$2n$	$\frac{5}{2}n$	$2n$	$2n$
planar 3-connected	$2n$	$\frac{5}{2}n$	n	$2n$
plane 3-connected cubic	–	$n+2$	3	3

First we formalise these notions. Consider a mapping of the vertices of a graph to distinct points in the plane. Now represent each edge by the closed line segment between its endpoints. Such a mapping is a *straight-line drawing* if the only vertices that each edge intersects are its own endpoints. For the sake of brevity, we refer to a straight-line drawing simply as a *drawing*.

By a *segment* in a drawing, we mean a maximal set of edges that form a line segment. The *slope* of a line L is the angle swept from the X-axis in an anticlockwise direction to L (and is thus in $[0,\pi)$). The *slope* of an edge or segment is the slope of the line that extends it. Of course two edges have the same slope if and only if they are parallel.

A *crossing* in a drawing is a pair of edges that intersect at some point other than a common endpoint. A drawing is *plane* if it has no crossings. A *plane graph* is a planar graph with a fixed combinatorial embedding and a specified outerface. We emphasise that a plane drawing of a plane graph must preserve the embedding and outerface. That every plane graph has a plane drawing is a famous result independently due to Wagner [26] and Fáry [12].

In this paper we prove lower and upper bounds on the minimum number of segments and slopes in (plane) drawings of graphs. In a companion paper [10], we consider drawings of non-planar graphs with few slopes. A summary of our results is given in Table 1. A number of comments are in order when considering these results:

- The minimum number of slopes in a drawing of (plane) graph G is at most the minimum number of segments in a drawing of G.
- Upper bounds for plane graphs are stronger than for planar graphs, since for planar graphs one has the freedom to choose the embedding and outerface. On the other hand, lower bounds for planar graphs are stronger than for plane graphs.
- Deleting an edge in a drawing cannot increase the number of slopes, whereas it can increase the number of segments. Thus, the upper bounds for slopes are applicable to all subgraphs of the mentioned graph families, unlike the upper bounds for segments.

The paper is organised as follows. In Section 3 we consider drawings with two or three slopes, and conclude that it is $\mathcal{NP}$-complete to determine whether a graph has a plane drawing on two slopes.

Section 4 studies plane drawings of graphs with small treewidth. In particular, we consider trees, outerplanar graphs, 2-trees, and planar 3-trees. For any tree, we construct a plane drawing with the minimum number of segments and the minimum number of slopes. For outerplanar graphs, 2-trees, and planar 3-trees, we determine bounds on the minimum number of segments and slopes that are tight in the worst-case.

Section 5 studies plane drawings of 3-connected plane and planar graphs. In the case of slope-minimisation for plane graphs we obtain a bound that is tight in the worst case. However, our lower bound examples have linear maximum degree. We drastically improve the upper bound in the case of cubic graphs. We prove that every 3-connected plane cubic graph has a plane drawing with three slopes, except for three edges on the outerface that have their own slope. As a corollary we prove that every 3-connected plane cubic graph has a plane 'drawing' with three slopes and three bends on the outerface.

We now review some related work from the literature.

Plane orthogonal drawings with two slopes (and few bends) have been extensively studied [2,3,19–25]. For example, Ungar [25] proved that every cyclically 4-edge-connected plane cubic graph has a plane drawing with two slopes and four bends on the outerface. Thus our result for 3-connected plane cubic graphs (Corollary 25) nicely complements this theorem of Ungar.

Contact and intersection graphs of segments in the plane with few slopes is an interesting line of research. The *intersection graph* of a set of segments has one vertex for each segment, and two vertices are adjacent if and only if the corresponding segments have a non-empty intersection. Hartman et al. [14] proved that every bipartite planar graph is the intersection graph of some set of horizontal and vertical segments. A *contact graph* is an intersection graph of segments for which no two segments have an interior point in common. Strengthening the above result, Fraysseix et al. [7] (and later, Czyzowicz et al. [4]) proved that every bipartite planar graph is a contact graph of some set of horizontal and vertical segments. Similarly, Castron et al. [5] proved that every triangle-free planar graph is a contact graph of some set of segments with only three distinct slopes. It is an open problem whether every planar graph is the intersection graph of a set of segments in the plane; see [6,17] for the most recent results. It is even possible that every k-colourable planar graph ($k \leqslant 4$) is the intersection graph of some set of segments using only k distinct slopes.

1.1. Definitions

We consider undirected, finite, and simple graphs G with vertex set $V(G)$ and edge set $E(G)$. The number of vertices and edges of G are respectively denoted by $n = |V(G)|$ and $m = |E(G)|$. The maximum degree of G is denoted by $\Delta(G)$.

For all $S \subseteq V(G)$, the (*vertex-*) *induced* subgraph $G[S]$ has vertex set S and edge set $\{vw \in E(G)\colon v, w \in S\}$. For all $S \subseteq V(G)$, let $G \setminus S$ be the subgraph $G[V(G) \setminus S]$. For all $v \in V(G)$, let $G \setminus v = G \setminus \{v\}$. For all $A, B \subseteq V(G)$, let $G[A, B]$ be the bipartite subgraph of G with vertex set $A \cup B$ and edge set $\{vw \in E(G)\colon v \in A \setminus B, w \in B \setminus A\}$.

For all $S \subseteq E(G)$, the (*edge-*) *induced* subgraph $G[S]$ has vertex set $\{v \in V(G)\colon \exists\, vw \in S\}$ and edge set S. For all pairs of vertices $v, w \in V(G)$, let $G \cup vw$ be the graph with vertex set $V(G)$ and edge set $E(G) \cup \{vw\}$.

A vertex v is *k-simplicial* if the neighbours of v form a k-clique. For each integer $k \geqslant 1$, k-trees are the class of graphs defined recursively as follows. The complete graph K_{k+1} is a *k-tree*, and the graph obtained from a k-tree by adding a new k-simplicial vertex adjacent to each vertex of an existing k-clique is also a *k-tree*. The *treewidth* of a graph G is the minimum k such that G is a spanning subgraph of a k-tree. For example, the graphs of treewidth one are the forests. Graphs of treewidth two, called *series-parallel*, are planar since in the construction of a 2-tree, each new vertex can be drawn close to the midpoint of the edge that it is added onto. Maximal outerplanar graphs are examples of 2-trees.

2. Some special plane graphs

As illustrated in Fig. 1, we have the following characterisation of plane drawings with a segment between every pair of vertices. In this sense, these are the plane drawings with the least number of segments.

Theorem 1. *In a plane drawing of a planar graph G, every pair of vertices of G is connected by a segment if and only if at least one of the following conditions hold*:

(a) *all the vertices of G are collinear,*
(b) *all the vertices of G, except for one, are collinear,*
(c) *all the vertices of G, except for two vertices v and w, are collinear, such that the line-segment $\overline{vw}$ passes through one of the collinear vertices,*
(d) *all the vertices of G, except for two vertices v and w, are collinear, such that the line-segment $\overline{vw}$ does not intersect the line-segment containing $V(G) \setminus \{v, w\}$,*
(e) *G is the* 6*-vertex octahedron graph* (*say* $V(G) = \{1, 2, \dots, 6\}$ *and* $E(G) = \{12, 13, 23, 45, 46, 56, 14, 15, 25, 26, 34, 36\}$) *with the triangle* $\{4, 5, 6\}$ *inside the triangle* $\{1, 2, 3\}$*, and each of the triples* $\{1, 4, 6\}$*,* $\{2, 5, 4\}$*,* $\{3, 6, 5\}$ *are collinear.*

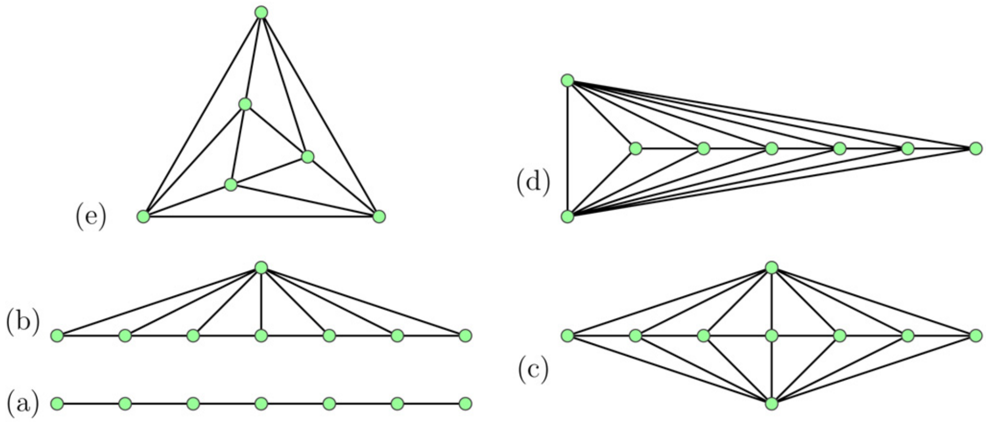

Fig. 1. The plane graph drawings with a segment between every pair of vertices.

Proof. As illustrated in Fig. 1, in a plane graph that satisfies one of (a)–(e), every pair of vertices is connected by a segment. For the converse, consider a plane graph G in which every pair of vertices is connected by a segment. Let L be a maximum set of collinear vertices. Let $\hat{L}$ be the line containing L. Then $|L| \geqslant 2$. If $|L| = 2$, then $G = K_n$ for some $n \leqslant 4$, which is included in case (a), (b), or (d). Now suppose that $|L| \geqslant 3$.

Without loss of generality, $\hat{L}$ is horizontal. Let S and T be the sets of vertices respectively above and below $\hat{L}$. Assume $|S| \geqslant |T|$.

If $|S| \leqslant 1$, then it is easily seen that G is in case (a), (b), (c), or (d). Otherwise $|S| \geqslant 2$. Choose $v \in S$ to be the closest vertex to $\hat{L}$ (in terms of perpendicular distance), and choose $w \in S \setminus \{v\}$ to be the next closest vertex. This is possible since G is finite. Let p be the point of intersection between $\hat{L}$ and the line through v and w.

Suppose on the contrary that there are at least two vertices $x, y \in L$ on one side of p. Say x is between p and y. Then the segments vy and wx cross at a point closer to $\hat{L}$ than v. Since G is plane, there is a vertex in S at this point, contradicting our choice of v. Hence there is at most one vertex in L on each side of p. Since $|L| \geqslant 3$, p is a vertex in L, and $|L| = 3$. Thus there is exactly one vertex in L on each side of p. Let these vertices be x and y.

Suppose on the contrary that there is a vertex $u \in S \setminus \{v, w\}$. Then u is above w, and u is not on the line containing v, w, p (as otherwise L is not a maximum set of collinear points). Thus the segment uv crosses either wx or wy at a point closer to $\hat{L}$ than w. Since G is plane, there is a vertex in S at this point, contradicting our choice of w. Thus $|S| = 2$, which implies $|T| \leqslant 2$.

Now $V(G) = \{v, w, p, x, y\} \cup T$. We have $|V(G)| \geqslant 6$, as otherwise G is in case (a), (b), (c) or (d). Hence $T \neq \emptyset$. Consider a vertex $q \in T$. The segment qv crosses $\hat{L}$ at some vertex in L. It cannot cross at p (as otherwise L would not be a maximum set of collinear vertices). Thus every vertex $q \in T$ is collinear with vx or vy. Suppose there are two vertices $q_1, q_2 \in T$ with q_1 collinear with vx and q_2 collinear with vy. Then the segments q_1y and q_2x would cross at a point below $\hat{L}$ but not collinear with vx or vy, which is a contradiction. Suppose there are two vertices $q_1, q_2 \in T$ both collinear with vx; say q_1 is closer to $\hat{L}$ than q_2. Then the segments q_1y and q_2p would cross at a point below $\hat{L}$ but not collinear with vx or vy, which is a contradiction. We obtain a similar contradiction if there are two vertices $q_1, q_2 \in T$ both collinear with vy. Thus there is exactly one vertex $q \in T$. Without loss of generality, q is collinear with vx. Then $\{v, w, x, y, p, q\}$ induce the octahedron in case (e) where $1 = q$, $2 = y$, $3 = w$, $4 = x$, $5 = p$, and $6 = v$. □

3. Drawings on two or three slopes

For drawings on two or three slopes the choice of slopes is not important.

Lemma 2. *A graph has a* (*plane*) *drawing on three slopes if and only if it has a* (*plane*) *drawing on any three slopes.*

Proof. Let D be a drawing of a graph G on slopes s_1, s_2, s_3. Let t_1, t_2, t_3 be three given slopes. Let T be a triangle with slopes s_1, s_2, s_3. Let T' be a triangle with slopes t_1, t_2, t_3. It is well known that there is an affine transformation

α to transform T into T'. Let D' be the result of applying α to D. Since parallel lines are preserved under α, every edge in D' has slope in $\{t_1, t_2, t_3\}$. Since sets of collinear points are preserved under α, no edge passes through another vertex in D'. Thus D' is a drawing of G with slopes t_1, t_2, t_3. Moreover, two edges cross in D if and only if they cross in D'. Thus D' is plane whenever D is plane. □

Corollary 3. *A graph has a (plane) drawing on two slopes if and only if it has a (plane) drawing on any two slopes.*

Garg and Tamassia [13] proved that it is $\mathcal{NP}$-complete to decide whether a graph has a rectilinear planar drawing (that is, with vertical and horizontal edges). Thus Corollary 3 implies:

Corollary 4. *It is $\mathcal{NP}$-complete to decide whether a graph has a plane drawing with two slopes.*

Note that it is easily seen that K_4 has a drawing on four slopes, but does not have a drawing on slopes $\{0, \epsilon, \frac{\pi}{2}, \frac{\pi}{2} + \epsilon\}$ for small enough ϵ.

4. Planar graphs with small treewidth

4.1. Trees

In this section we study drawings of trees with few slope and few segments. We start with the following universal lower bounds.

Lemma 5. *The number of slopes in a drawing of a graph is at least half the maximum degree, and at least the minimum degree. The number of segments in a drawing of a graph is at least half the number of odd degree vertices.*

Proof. At most two edges incident to a vertex v can have the same slope. Thus the edges incident to v use at least $\frac{1}{2}\deg(v)$ slopes. Hence the number of slopes is at least half the maximum degree. For some vertex v on the convex hull, every edge incident to v has a distinct slope. Thus the number of slopes is at least the minimum degree.

If a vertex is internal on every segment then it has even degree. Thus each vertex of odd degree is an endpoint of some segment. Thus the number of vertices of odd degree is at most twice the number of segments. (The number of odd degree vertices is always even.) □

We now show that the lower bounds in Lemma 5 are tight for trees. In fact, they can be simultaneously attained by the same drawing.

Theorem 6. *Let T be a tree with maximum degree Δ, and with η vertices of odd degree. The minimum number of segments in a drawing of T is $\frac{\eta}{2}$. The minimum number of slopes in a drawing of T is $\lceil \frac{\Delta}{2} \rceil$. Moreover, T has a plane drawing with $\frac{\eta}{2}$ segments and $\lceil \frac{\Delta}{2} \rceil$ slopes.*

Proof. The lower bounds are Lemma 5. The upper bound will follow from the following hypothesis, which we prove by induction on the number of vertices: "Every tree T with maximum degree Δ has a plane drawing with $\lceil \frac{\Delta}{2} \rceil$ slopes, in which every odd degree vertex is an endpoint of exactly one segment, and no even degree vertex is an endpoint of a segment." The hypothesis is trivially true for a single vertex. Let x be a leaf of T incident to the edge xy. Let $T' = T \setminus x$. Suppose T' has maximum degree Δ'.

First suppose that y has even degree in T, as illustrated in Fig. 2(a). Thus y has odd degree in T'. By induction, T' has a plane drawing with $\lceil \frac{\Delta'}{2} \rceil \leqslant \lceil \frac{\Delta}{2} \rceil$ slopes, in which y is an endpoint of exactly one segment. That segment contains some edge e incident to y. Draw x on the extension of e so that there are no crossings. In the obtained drawing D, the number of slopes is unchanged, x is an endpoint of one segment, and y is not an endpoint of any segment. Thus D satisfies the hypothesis.

Now suppose that y has odd degree in T, as illustrated in Fig. 2(b). Thus y has even degree in T'. By induction, T' has a plane drawing with $\lceil \frac{\Delta'}{2} \rceil$ slopes, in which y is not an endpoint of any segment. Thus the edges incident to y

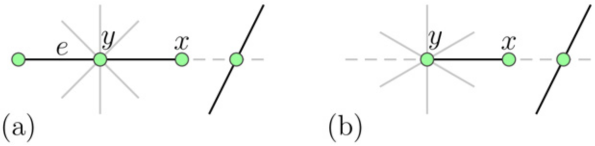

Fig. 2. Adding a leaf x to a drawing of a tree: (a) $\deg(y)$ even and (b) $\deg(y)$ odd.

use $\frac{1}{2}\deg_{T'}(y) \leqslant \lceil\frac{\Delta}{2}\rceil - 1$ slopes. If the drawing of T' has any other slopes, let s be one of these slopes, otherwise let s be an unused slope. Add edge xy to the drawing of T' with slope s so that there are no crossings. In the obtained drawing D, there is a new segment with endpoints x and y. Since both x and y have odd degree in T, and since x and y were not endpoints of any segment in the drawing of T', the induction hypothesis is maintained. The number of slopes in D is $\max\{\lceil\frac{\Delta'}{2}\rceil, \frac{1}{2}\deg_{T'}(y) + 1\} \leqslant \lceil\frac{\Delta}{2}\rceil$. □

4.2. Outerplanar graphs

A planar graph G is *outerplanar* if G admits a combinatorial embedding with all the vertices on the boundary of a single face. An outerplanar graph G is *maximal* if $G \cup vw$ is not outerplanar for any pair of non-adjacent vertices $v, w \in V(G)$. A plane graph is *outerplanar* if all the vertices are on the boundary of the outerface. A maximal outerplanar graph has a unique outerplanar embedding.

Theorem 7. *Every n-vertex maximal outerplanar graph G has an outerplanar drawing with at most n segments. For all $n \geqslant 3$, there is an n-vertex maximal outerplanar graph that has at least n segments in any drawing.*

Proof. We prove the upper bound by induction on n with the additional invariant that the drawing is *star-shaped*. That is, there is a point p in (the interior of) some internal face of D, and every ray from p intersects the boundary of the outerface in exactly one point.

For $n = 3$, G is a triangle, and the invariant holds by taking p to be any point in the internal face. Now suppose $n > 3$. It is well known that G has a degree-2 vertex v whose neighbours x and y are adjacent, and $G' = G \setminus v$ is maximal outerplanar. By induction, G' has a drawing D' with at most $n - 1$ segments, and there is a point p in some internal face of D', such that every ray from p intersects the boundary of D' in exactly one point. The edge xy lies on the boundary of the outerface and of some internal face F. Without loss of generality, xy is horizontal in D', and F is below xy. Since G' is maximal outerplanar, F is bounded by a triangle rxy.

For three non-collinear points a, b and c in the plane, define the *wedge* (a, b, c) to be the infinite region that contains the interior of the triangle abc, and is enclosed on two sides by the ray from b through a and the ray from b through c. By induction, p is in the wedge (y, x, r) or in the wedge (x, y, r). By symmetry we can assume that p is in (y, x, r).

Let R be the region strictly above xy that is contained in the wedge (x, p, y). The line extending the edge xr intersects R. As illustrated in Fig. 3, place v on any point in R that is on the line extending xr. Draw the two incident edges vx and vy straight. This defines our drawing D of G. By induction, $R \cap D' = \emptyset$. Thus vx and vy do not create crossings in D. Every ray from p that intersects R, intersects the boundary of D in exactly one point. All other rays from p intersect the same part of the boundary of D as in D'. Since p remains in some internal face, D is star-shaped. By induction, D' has $n-1$ segments. Since vx and rx are in the same segment, there is at most one segment in $D \setminus D'$. Thus D is a star-shaped outerplanar drawing of G with n segments. This concludes the proof of the upper bound.

For the lower bound, let G_n be the maximal outerplanar graph on $n \geqslant 3$ vertices whose weak dual (that is, dual graph disregarding the outerface) is a path and the maximum degree of G_n is at most four, as illustrated in Fig. 4.

We claim that every drawing of G_n has at least n segments (even if crossings are allowed). We proceed by induction on n. The result is trivial for $n = 3$. Suppose that every drawing of G_{n-1} has at least $n - 1$ segments, but there exists a drawing D of G_n with at most $n - 1$ segments. Let v be a degree-2 vertex in G_n adjacent to x and y. By the definition of G_n, one of x and y, say x, has degree three in G_n. Observe that $G_n \setminus v$ is isomorphic to G_{n-1}. Thus we have a drawing of G_n with exactly $n - 1$ segments, which contains a drawing of $G_n \setminus v$ with $n - 1$ segments. Thus the edge vx shares a segment with some other edge xr, and the edge vy shares a segment with some other edge ys. Since vxy

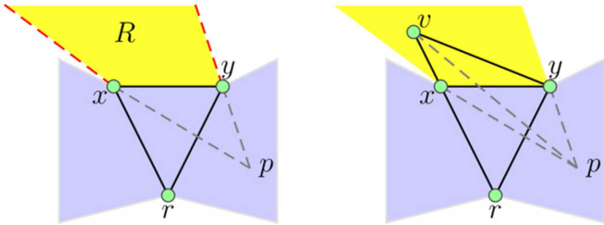

Fig. 3. Construction of a star-shaped drawing of an outerplanar graph.

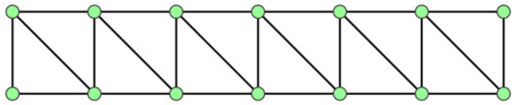

Fig. 4. The graph G_{14}.

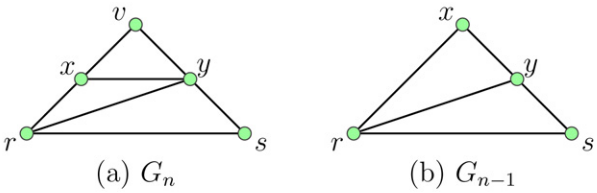

Fig. 5. Construction of a drawing of G_{n-1} from a drawing of G_n.

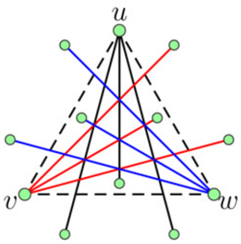

Fig. 6. A triangle forces many different slopes.

is a triangle, $r \neq y$, $s \neq x$ and $r \neq s$. Since x has degree three, y is adjacent to r, as illustrated in Fig. 5. That accounts for all edges incident to y and x. Thus xy is a segment in D.

Now construct a drawing D' of G_{n-1} with x moved to the position of v in the drawing of G_n. The drawing D consists of D' plus the edge xy. Since xy is a segment in D, D' has one less segment than D. Thus D' is a drawing of G_{n-1} with at most $n-2$ segments, which is the desired contradiction. □

Open Problem 8. Is there a polynomial time algorithm to compute an outerplanar drawing of a given outerplanar graph with the minimum number of segments?

4.3. 2-trees

In this section we study drawings of 2-trees with few slopes and segments. The following lower bound on the number of slopes is immediate, as illustrated in Fig. 6.

Observation 9. Let u, v and w be three non-collinear vertices in a drawing D of a graph G. Let $d(u)$ denote the number of edges incident to u that intersect the interior of the triangle uvw, and similarly for v and w. Then D has at least $d(u) + d(v) + d(w) + |E(G) \cap \{uv, vw, uw\}|$ slopes.

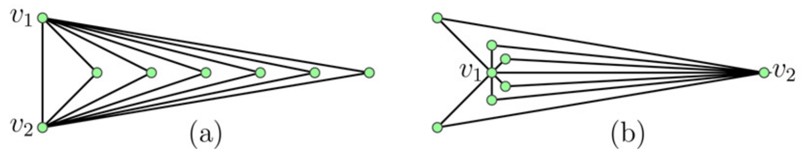

Fig. 7. The graph G_8 in Lemma 10.

Lemma 10. *Every n-vertex 2-tree has a plane drawing with at most $2n-3$ segments (and thus at most $2n-3$ slopes). For all $n \geqslant 3$, there is an n-vertex plane 2-tree that has at least $2n-3$ slopes (and thus at least $2n-3$ segments) in every plane drawing.*

Proof. The upper bound follows from the Fáry–Wagner theorem since every 2-tree is planar and has $2n-3$ edges. Consider the 2-tree G_n with vertex set $\{v_1, v_2, \dots, v_n\}$ and edge set $\{v_1v_2, v_1v_i, v_2v_i\colon 3 \leqslant i \leqslant n\}$. Fix a plane embedding of G_n with the edge v_1v_2 on the triangular outerface, as illustrated in Fig. 7(a). The number of slopes is at least $(n-3)+(n-3)+0+3 = 2n-3$ by Observation 9.

In Lemma 10 the embedding is fixed. A better bound can be obtained if we do not fix the embedding. For example, the graph G_n from Lemma 10 has a plane drawing with $\frac{3n}{2}-2$ segments, as illustrated in Fig. 7(b).

Theorem 11. *Every n-vertex 2-tree G has a plane drawing with at most $\frac{3}{2}n$ segments*

The key idea in the proof of Theorem 11 is to position a set of vertices at each step, rather than a single vertex. The next lemma says how to partition a 2-tree appropriately. It has subsequently been generalised for k-trees by Dujmović and Wood [11].

Lemma 12. *Let G be a 2-tree. Then for some $k \geqslant 1$, $V(G)$ can be partitioned $(S_0, S_1, S_2, \dots, S_k)$ such that the following properties hold, where for $0 \leqslant i \leqslant k$, G_i is defined to be the induced subgraph $G[\bigcup_{j=0}^{i} S_j]$:*

(a) *for $1 \leqslant i \leqslant k$, G_i is a 2-tree,*
(b) *S_0 consists of two adjacent vertices,*
(c) *for $1 \leqslant i \leqslant k$, S_i is an independent set of G,*
(d) *for $1 \leqslant i \leqslant k$, each vertex in S_i has exactly two neighbours in G_{i-1}, and they are adjacent,*
(e) *for $2 \leqslant i \leqslant k$, the vertices in S_i have a common neighbour v in G_{i-1}, and v has degree two in G_{i-1}.*

Proof. We proceed by induction on $|V(G)|$. By definition, $|V(G)| \geqslant 3$.

First suppose that $|V(G)| = 3$. Then $G = K_3$. Say $V(G) = \{u, v, w\}$. Define $S_0 = \{u, v\}$ and $S_1 = \{w\}$. Then (S_0, S_1) is the desired partition of G with $k = 1$ since: (a) $G_1 = G$ is a 2-tree; (b) S_0 consists of two adjacent vertices u and v; (c) $S_1 = \{w\}$ is an independent set; (d) each vertex in S_1, namely w, has exactly two neighbours in G_0, namely u and v; and condition (e) is vacuous with $k = 1$.

Now assume that $|V(G)| > 3$. Let L be the set of vertices of degree two in G. By the definition of 2-tree, L is a nonempty independent set, and the two neighbours of each vertex in L are adjacent.

Suppose that $G \setminus L$ is a single edge vw. Define $S_0 = \{v, w\}$ and $S_1 = L$. Then (S_0, S_1) is the desired partition of G with $k = 1$ since: (a) $G_1 = G$ is a 2-tree; (b) S_0 consists of two adjacent vertices v and w; (c) $S_1 = L$ is an independent set of G; (d) each vertex in S_1 has exactly two neighbours in G_0, namely v and w; and condition (e) is vacuous with $k = 1$.

Otherwise $G \setminus L$ is not a single edge, in which case, $G \setminus L$ is a 2-tree. Thus $G \setminus L$ has a vertex v of degree 2. Let S be the set of neighbours of v in L. Now $S \neq \emptyset$, as otherwise $v \in L$. By induction, for some $k \geqslant 2$, there is a partition $(S_0, S_1, S_2, \dots, S_{k-1})$ of $V(G \setminus S)$ that satisfies conditions (a)–(e). Define $S_k = S$. To prove that $(S_0, S_1, S_2, \dots, S_k)$ is the desired partition of G, we need only consider the $i = k$ case, since for $i < k$, the claims hold by induction. We have: (a) $G_k = G$ is a 2-tree; (b) S_0 consists of two adjacent vertices; (c) $S_k = S$ is an independent set by construction; (d) each vertex in S_k has degree two in G, and thus has exactly two neighbours in G_{k-1} since S_k is an independent set; (e) by construction, the vertices in S_k have a common neighbour v in G_{k-1}, and v has degree two in G_{k-1}. □

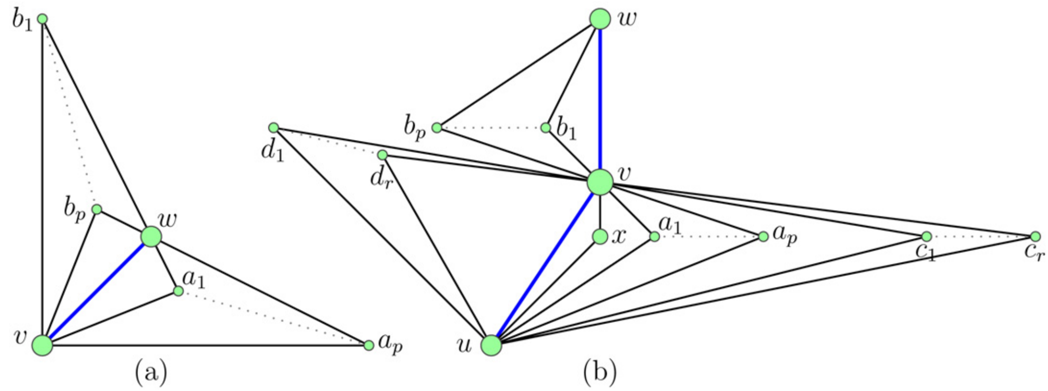

Fig. 8. (a) Drawing S_1, (b) drawing S_k.

Proof of Theorem 11. Let $(S_0, S_1, S_2, \dots, S_k)$ be the partition of $V(G)$ from Lemma 12. First suppose that $k = 1$. By Lemma 12(b) and (d), $S_0 = \{v, w\}$ and S_1 is an independent set of vertices, each adjacent to both v and w. Let $S_1 = \{a_1, a_2, \dots, a_p\} \cup \{b_1, b_2, \dots, b_q\}$, where $q \leqslant p \leqslant q + 1$. As illustrated in Fig. 8(a), G can be drawn such that $a_i v$ and $b_i v$ form a single segment, for all $1 \leqslant i \leqslant q$. The number of segments is at most $1 + |S_1| + \lceil \frac{1}{2}|S_1| \rceil \leqslant \frac{1}{2}(3n - 3)$.

Now suppose that $k \geqslant 2$. By Lemma 12(a), G_{k-1} is a 2-tree. Thus by induction, G_{k-1} has a plane drawing with at most $\frac{3}{2}(n - |S_k|)$ segments. By Lemma 12(d) and (e), the vertices in S_k have degree two in G, and have a common neighbour v in G_{k-1} with degree two in G_{k-1}. Let u and w be the neighbours of v in G_{k-1}. Then the neighbourhood of each vertex in S_k is either $\{v, u\}$ or $\{v, w\}$. Let S_k^u and S_k^w be the sets of vertices in S_k whose neighbourhood respectively is $\{v, u\}$ and $\{v, w\}$. Without loss of generality, $|S_k^u| \geqslant |S_k^w|$. Let $S_k^w = \{b_1, \dots, b_p\}$. For the time being assume that $|S_k^u| - p$ is even. Let $r = \frac{1}{2}(|S_k^u| - p)$. Thus r is a nonnegative integer, and S_k^u can be partitioned

$$S_k^u = \{a_1, \dots, a_p\} \cup \{c_1, \dots, c_r\} \cup \{d_1, \dots, d_r\}.$$

As illustrated in Fig. 8(b), G can be drawn such that $a_i v$ and $b_i v$ form a single segment for all $1 \leqslant i \leqslant p$, and $c_i v$ and $d_i v$ form a single segment for all $1 \leqslant i \leqslant r$. Clearly the vertices can be placed to avoid crossings with the existing drawing of G_{k-1}. In particular, vertices $\{b_1, \dots, b_p, d_1, \dots, d_r\}$ are drawn inside the triangle (u, v, w). The number of new segments in the drawing is $3p + 3r = \frac{3}{2}|S_k|$.

In the case that $|S_k^u| - p$ is odd, a vertex x from S_k^u can be drawn so that xv and xw form a single segment; then apply the above algorithm to $S_k \setminus \{x\}$. The number of new segments is then $3p + 3r + 1$, where $|S_k| = 2p + 2r + 1$. It follows that the number of new segments is at most $\frac{1}{2}(3|S_k| - 1)$.

In both cases, the total number of segments is at most $\frac{3}{2}(n - |S_k|) + \frac{3}{2}|S_k| = \frac{3}{2}n$. □

4.4. Planar 3-trees

We now turn our attention to drawings of planar 3-trees.

Theorem 13. *Every n-vertex plane 3-tree has a plane drawing with at most $2n - 2$ segments (and thus at most $2n - 2$ slopes). For all $n \geqslant 4$, there is an n-vertex plane 3-tree with at least $2n - 2$ slopes (and thus at least $2n - 2$ segments) in every drawing.*

Proof. We prove the upper bound by induction on n with the hypothesis that "every plane 3-tree with $n \geqslant 4$ vertices has a plane drawing with at most $2n - 2$ segments, such that for every internal face F there is an edge e incident to exactly one vertex of F, and the extension of e intersects the interior of F". The base case is trivial since K_4 is the only 3-tree on four vertices, and any plane drawing of K_4 satisfies the hypothesis.

Suppose that the claim holds for plane 3-trees on $n - 1$ vertices. Let G be a plane 3-tree on n vertices. Every k-tree on at least $k + 2$ vertices has two non-adjacent simplicial vertices of degree exactly k [9]. In particular, G has two non-adjacent simplicial degree-3 vertices, one of which, say v, is not on the outerface. Thus G can be obtained from

$G \setminus v$ by adding v inside some internal face (p, q, r) of $G \setminus v$, adjacent to p, q and r.[5] By induction, $G \setminus v$ has a drawing with $2n - 4$ segments in which there is an edge e incident to exactly one of $\{p, q, r\}$, and the extension of e intersects the interior of the face. Position v in the interior of the face anywhere on the extension of e, and draw segments from v to each of p, q and r. We obtain a plane drawing of G with $2n - 2$ segments. The extension of vp intersects the interior of (v, q, r); the extension of vq intersects the interior of (v, p, r); and the extension of vr intersects the interior of (v, p, q). All other faces of G are faces of $G \setminus v$. Thus the inductive hypothesis holds for G, and the proof of the upper bound is complete.

The proof of the lower bound is given in Lemma 18 below. □

5. 3-connected plane graphs

The following is the main result of this section.

Theorem 14. *Every 3-connected plane graph with n vertices has a plane drawing with at most $\frac{5}{2}n - 3$ segments and at most $2n - 10$ slopes.*

The proof of Theorem 14 is based on the canonical decomposition of Kant [16], which is a generalisation of a similar structure for plane triangulations introduced by de Fraysseix et al. [8]. Let G be a 3-connected plane graph. Kant [16] proved that G has a canonical decomposition defined as follows. Let $\sigma = (V_1, V_2, \dots, V_K)$ be an ordered partition of $V(G)$. That is, $V_1 \cup V_2 \cup \dots \cup V_K = V(G)$ and $V_i \cap V_j = \emptyset$ for all $i \neq j$. Define G_i to be the plane subgraph of G induced by $V_1 \cup V_2 \cup \dots \cup V_i$. Let C_i be the subgraph of G induced by the edges on the boundary of the outerface of G_i. As illustrated in Fig. 9, σ is a *canonical decomposition* of G (also called a *canonical decomposition*) if:

- $V_1 = \{v_1, v_2\}$, where v_1 and v_2 lie on the outerface and $v_1 v_2 \in E(G)$.
- $V_K = \{v_n\}$, where v_n lies on the outerface, $v_1 v_n \in E(G)$, and $v_n \neq v_2$.
- Each C_i $(i > 1)$ is a cycle containing $v_1 v_2$.
- Each G_i is biconnected and internally 3-connected; that is, removing any two interior vertices of G_i does not disconnect it.
- For each $i \in \{2, 3, \dots, K - 1\}$, one of the following conditions holds:
  (1) $V_i = \{v_i\}$ where v_i is a vertex of C_i with at least three neighbours in C_{i-1}, and v_i has at least one neighbour in $G \setminus G_i$.
  (2) $V_i = (s_1, s_2, \dots, s_\ell, v_i)$, $\ell \geqslant 0$, is a path in C_i, where each vertex in V_i has at least one neighbour in $G \setminus G_i$. Furthermore, the first and the last vertex in V_i have one neighbour in C_{i-1}, and these are the only two edges between V_i and G_{i-1}.

The vertex v_i is called the *representative* vertex of the set V_i, $2 \leqslant i \leqslant K$. The vertices $\{s_1, s_2, \dots, s_\ell\} \subseteq V_i$ are called *division* vertices. Let $S \subset V(G)$ be the set of all division vertices. A vertex u is a *successor* of a vertex $w \in V_i$ if uw is an edge and $u \in G \setminus G_i$. A vertex u is a *predecessor* of a vertex $w \in V_i$ if uw is an edge and $u \in V_j$ for some $j < i$. We also say that u is a predecessor of V_i. Let $P(V_i) = (p_1, p_2, \dots, p_q)$ denote the set of predecessors of V_i ordered by the path from v_1 to v_2 in $C_{i-1} \setminus v_1 v_2$. Vertex p_1 and p_q are the *left* and *right predecessors* of V_i respectively, and vertices $p_2, p_3, \dots p_{q-1}$ are called *middle predecessors* of V_i.

Theorem 15. *Let G be an n-vertex m-edge plane 3-connected graph with a canonical decomposition σ. Define S as above (with respect to σ). Then G has a plane drawing D with at most*

$$m - \max\{\lceil n/2 \rceil - |S| - 3,\ |S|\}$$

segments, and at most

$$m - \max\{n - |S| - 4,\ |S|\}$$

slopes.

[5] Note that this implies that the planar 3-trees are precisely those graphs that are produced by the LEDA 'random' maximal planar graph generator. This algorithm, starting from K_3, repeatedly adds a new vertex adjacent to the three vertices of a randomly selected internal face.

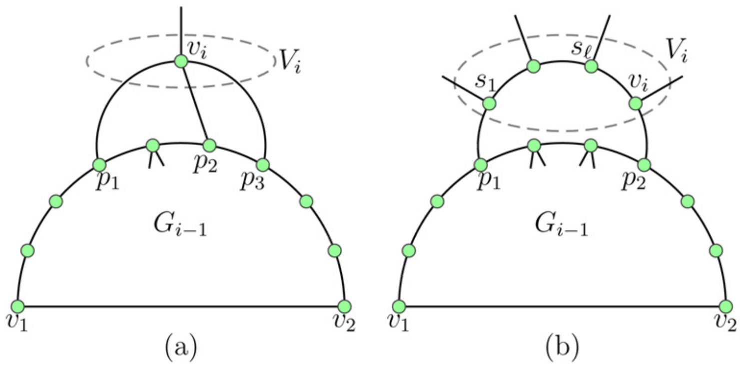

Fig. 9. The canonical decomposition of a 3-connected plane graph.

Proof. We first define D and then determine the upper bounds on the number of segments and slopes in D. For every vertex v, let $X(v)$ and $Y(v)$ denote the x and y coordinates of v, respectively. If a vertex v has a neighbour w, such that $X(w) < X(v)$ and $Y(w) < Y(v)$, then we say vw is a *left edge* of v. Similarly, if v has a neighbour w, such that $X(w) > X(v)$ and $Y(w) < Y(v)$, then we say vw is a *right edge* of v. If vw is an edge such that $X(v) = X(w)$ and $Y(v) < Y(w)$, than we say vw is a *vertical edge above* v and *below* w.

We define D inductively on $\sigma = (V_1, V_2, \dots, V_K)$ as follows. Let D_i denote a drawing of G_i. A vertex v is a *peak in* D_i, if each neighbour w of v has $Y(w) \leqslant Y(v)$ in D_i. We say that a point p in the plane is *visible in* D_i from vertex $v \in D_i$, if the segment $\overline{pv}$ does not intersect D_i except at v. At the ith induction step, $2 \leqslant i \leqslant K$, D_i will satisfy the following invariants:

Invariant 1: $C_i \setminus v_1v_2$ is *strictly X-monotone*; that is, the path from v_1 to v_2 in $C_i \setminus v_1v_2$ has increasing X-coordinates.
Invariant 2: Every peak in D_i, $i < K$, has a successor.
Invariant 3: Every representative vertex $v_j \in V_j$, $2 \leqslant j \leqslant i$ has a left and a right edge. Moreover, if $|P(V_j)| \geqslant 3$ then there is a vertical edge below v_j.
Invariant 4: D_i has no edge crossings.

For the base case $i = 2$, position the vertices v_1, v_2 and v_3 at the corners of an equilateral triangle so that $X(v_1) < X(v_3) < X(v_2)$ and $Y(v_1) < Y(v_2) < Y(v_3)$. Draw the division vertices of V_2 on the segment v_1v_3. This drawing of D_2 satisfies all four invariants. Now suppose that we have a drawing of D_{i-1} that satisfies the invariants. There are two cases to consider in the construction of D_i, corresponding to the two cases in the definition of the canonical decomposition.

Case 1. $|P(V_i)| \geqslant 3$: If v_i has a middle predecessor v_j with $|P(V_j)| \geqslant 3$, let $w = v_j$. Otherwise let w be any middle predecessor of v_i. Let L be the open ray $\{(X(w), y)\colon\ y > Y(w)\}$. By invariant 1 for D_{i-1}, there is a point in L that is visible in D_{i-1} from every predecessor of v_i. Represent v_i by such a point, and draw segments between v_i and each of its predecessors. That the resulting drawing D_i satisfies the four invariants can be immediately verified.

Case 2. $|P(V_i)| = 2$: Suppose that $P(V_i) = \{w, u\}$, where w and u are the left and the right predecessors of V_i, respectively. Suppose $Y(w) \geqslant Y(u)$. (The other case is symmetric.) Let P be the path between w and u on $C_{i-1} \setminus v_1v_2$. As illustrated in Fig. 10, let A_i be the region $\{(x, y)\colon\ y > Y(w)$ and $X(w) \leqslant x \leqslant X(u)\}$.

Assume, for the sake of contradiction, that $D_{i-1} \cap A_i \neq \emptyset$. By the monotonicity of D_{i-1}, $P \cap A_i \neq \emptyset$. Let $p \in P \cap A_i$. Since $Y(p) > Y(w) \geqslant Y(u)$, P is X-monotone and thus has a vertex between w and u that is a peak. By the definition of the canonical decomposition σ, the addition of V_i creates a face of G, since V_i is added in the outerface of G_{i-1}. Therefore, each vertex between w and u on P has no successor, and is thus not a peak in D_{i-1} by invariant 2, which is the desired contradiction. Therefore $D_{i-1} \cap A_i = \emptyset$.

Let L be the open ray $\{(X(u), y)\colon\ y > Y(u)\}$. If $w \notin S$, then by invariant 3, w has a left and a right edge in D_{i-1}. Let c be the point of intersection between L and the line extending the left edge at w. If $w \in S$, then let c be any point in A_i on L. By invariant 1, there is a point $c' \notin \{c, w\}$ on $\overline{wc}$ such that c' is visible in D_{i-1} from u. Represent v_i by c', and draw two segments $\overline{v_iu}$ and $\overline{v_iw}$. These two segments do not intersect any part of D_{i-1} (and neither is horizontal).

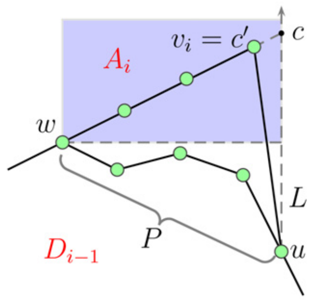

Fig. 10. Illustration for Case 2.

Represent any division vertices in V_i by arbitrary points on the open segment $\overline{wv_i} \cap A_i$. Therefore, in the resulting drawing D_i, there are no crossings and the remaining three invariants are maintained.

This completes the construction of D. The following claim will be used to bound the number of segments and slopes in D. It basically says that a division vertex (and v_2) can be the higher predecessor for at most one set V_i with $|P(V_i)| = 2$.

Claim 16. *Let $V_i, V_j \in \sigma$ with $i < j$ and $|P(V_i)| = |P(V_j)| = 2$. Let w_i be the higher of the two predecessors of V_i in D_{i-1}, and let w_j be the higher of the two predecessors of V_j in D_{j-1}. If $w_i \in S$ or $w_i = v_2$, then $w_i \neq w_j$.*

Proof. Suppose that $w_i \in V_k$, $k < i$. First assume that $w_i \in S$. Then each division vertex lies on some non-horizontal segment and it is not an endpoint of that segment. Thus w_i is not a peak in D_k, and therefore it is not a peak in every D_ℓ, $\ell \geqslant k$. For all $\epsilon > 0$, let

$$A'_\epsilon = \{(x, y)\colon\ y > Y(w_i),\ X(w_i) - \epsilon \leqslant x < X(w_i)\}, \quad \text{and}$$
$$A''_\epsilon = \{(x, y)\colon\ y > Y(w_i),\ X(w_i) < x \leqslant X(w_i) + \epsilon\}.$$

Then for all small enough ϵ, either $A'_\epsilon \cap D_k \neq \emptyset$ or $A''_\epsilon \cap D_k \neq \emptyset$. Without loss of generality, $A'_\epsilon \cap D_k = \emptyset$ and $A''_\epsilon \cap D_k \neq \emptyset$. Then at iteration $i > k$, the region A_i, as defined in Case 2 of the construction of D_i, contains A'_ϵ for all small enough ϵ. Thus, $A'_\epsilon \cap D_i \neq \emptyset$ for all small enough ϵ. Since $j \geqslant i + 1$, $A'_\epsilon \cap D_{j-1} \neq \emptyset$ or $A''_\epsilon \cap D_{j-1} \neq \emptyset$ for all small enough ϵ. Therefore, $w_i \neq w_j$ (since V_j is drawn by Case 2 of the construction of D_j, where it is known that $A_j \cap D_{j-1} = \emptyset$). The case $w_i = v_2$ is the same, since the region $A''_\epsilon \cap D_i = \emptyset$, for every ϵ and every $1 \leqslant i \leqslant K$, so only region A'_ϵ is used, and thus the above argument applies. □

For the purpose of counting the number of segments and slopes in D assume that we draw edge v_1v_2 at iteration step $i = 1$ and $G_2 \setminus v_1v_2$ at iteration $i = 2$. In every iteration i of the construction, $2 \leqslant i \leqslant K$, at most $|P(V_i)|$ new segments and slopes are created. We call an iteration i of the construction *segment-heavy* if the difference between the number of segments in D_i and D_{i-1} is exactly $|P(V_i)|$, and *slope-heavy* if the difference between the number of slopes in D_i and D_{i-1} is exactly $|P(V_i)|$. Let h_s and h_ℓ denote the total number of segment-heavy and slope-heavy iterations, respectively. Then D uses at most

$$1 + \sum_{i=2}^{K}\left(\left|P(V_i)\right| - 1\right) + h_s \tag{1}$$

segments, and at most

$$1 + \sum_{i=2}^{K}\left(\left|P(V_i)\right| - 1\right) + h_\ell \tag{2}$$

slopes.

We first express $\sum_{i=2}^{K} |P(V_i)|$ in terms of m and $|S|$, and then establish an upper bound on h_s and h_ℓ. For $i \geqslant 2$, let E_i denote the set of edges of G_i with at least one endpoint in V_i, and let ℓ_i denote the number of division vertices in V_i. Then $m = 1 + \sum_{i=2}^{K} |E_i| = 1 + \sum_{i=2}^{K} (\ell_i + |P(V_i)|) = 1 + |S| + \sum_{i=2}^{K} |P(V_i)|$. Thus $\sum_{i=2}^{K} |P(V_i)| = m - |S| - 1$. Since the trivial upper bound for h_s and h_ℓ is $K-1$, and by (1) and (2), we have that D uses at most $1 + \sum_{i=2}^{K} |P(V_i)| = 1 + m - |S| - 1 = m - |S|$ segments and slopes.

We now prove a tighter bound on h_s. Let R denote the set of representative vertices of segment-heavy steps i with $|P(V_i)| \geqslant 3$. Consider a step i such that $|P(V_i)| \geqslant 3$. If v_i has at least one predecessor v_j with $|P(V_j)| \geqslant 3$, then v_i is drawn on the line that extends the vertical edge below v_j, and thus step i introduces at most $|P(V_i)| - 1$ new segments and is not segment-heavy. Therefore, step i is segment-heavy only if no middle predecessor w of v_i is in R. Thus for each segment-heavy step i with $|P(V_i)| \geqslant 3$, there is a unique vertex $w \notin R$. In other words, for each vertex in R, there is a unique vertex in $V(G) \setminus R$. Thus $|R| \leqslant \lfloor \frac{n}{2} \rfloor$. Since the number of segment-heavy steps i with $|P(V_i)| \geqslant 3$ is equal to $|R|$, there is at most $\lfloor \frac{n}{2} \rfloor$ such steps.

The remaining steps, those with $|P(V_i)| = 2$, introduce $|P(V_i)|$ segments only if the higher of the two predecessors of V_i is in S or is v_2. (It cannot be v_1, since $Y(v_1) < Y(v)$ for every vertex $v \neq v_1$.) By the above claim, there may be at most $|S| + 1$ such segment-heavy steps. Therefore, $h_s \leqslant \lfloor \frac{n}{2} \rfloor + |S| + 1$. By (1) and since $K = n - 1 - |S|$, D has at most $m - \lceil \frac{n}{2} \rceil + |S| + 3$ segments.

Finally, we bound h_ℓ. There may be at most one slope-heavy step i with $|P(v_i)| \geqslant 3$, since there is a vertical edge below every such vertex v_i by invariant 3. As in the above case for segments, there may be at most $|S| + 1$ slope-heavy steps i with $|P(v_i)| = 2$. Therefore, $h_\ell \leqslant |S| + 2$. By (2) and since $K = n - 1 - |S|$, we have that D has at most $m - n + |S| + 4$ slopes. □

Proof of Theorem 14. Whenever a set V_i is added to G_{i-1}, at least $|V_i| - 1$ edges that are not in G can be added so that the resulting graph is planar. Thus $|S| = \sum_i (|V_i| - 1) \leqslant 3n - 6 - m$. Hence Theorem 15 implies that G has a plane drawing with at most $m - \frac{n}{2} + |S| + 3 \leqslant \frac{5}{2}n - 3$ segments, and at most $m - n + |S| - 4 \leqslant 2n - 10$ slopes. □

We now prove that the bound on the number of segments in Theorem 14 is tight.

Lemma 17. *For all $n \equiv 0 \pmod 3$, there is an n-vertex planar triangulation with maximum degree six that has at least $2n - 6$ segments in every plane drawing, regardless of the choice of outerface.*

Proof. Consider the planar triangulation G_k with vertex set $\{x_i, y_i, z_i\colon 1 \leqslant i \leqslant k\}$ and edge set $\{x_i y_i, y_i z_i, z_i x_i\colon 1 \leqslant i \leqslant k\} \cup \{x_i x_{i+1}, y_i y_{i+1}, z_i z_{i+1}\colon 1 \leqslant i \leqslant k-1\} \cup \{x_i y_{i+1}, y_i z_{i+1}, z_i x_{i+1}\colon 1 \leqslant i \leqslant k-1\}$. G_k has $n = 3k$ vertices. G_k is the famous 'nested-triangles' graph. We say $\{(x_i, y_i, z_i)\colon 1 \leqslant i \leqslant k\}$ are *the triangles* of G_k. This graph has a natural plane embedding with the triangle $x_i y_i z_i$ nested inside the triangle $(x_{i+1}, y_{i+1}, z_{i+1})$ for all $1 \leqslant i \leqslant k-1$, as illustrated in Fig. 11.

We first prove that if (x_k, y_k, z_k) is the outerface then G_k has at least $6k$ segments in any plane drawing. First observe that no two edges in the triangles can share a segment. Thus they contribute $3k$ segments.

We claim that the six edges between triangles (x_i, y_i, z_i) and $(x_{i+1}, y_{i+1}, z_{i+1})$ contribute a further three segments. Consider the two edges $x_i x_{i+1}$ and $z_i x_{i+1}$ incident on x_{i+1}. We will show that at least one of them contributes a

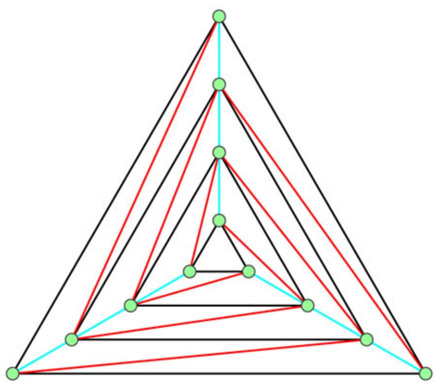

Fig. 11. The graph G_4 in Lemma 17.

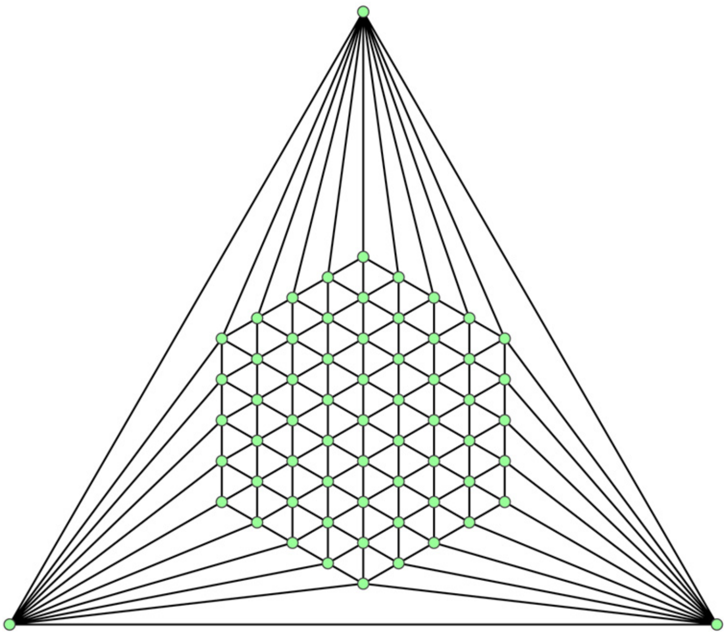

Fig. 12. A plane triangulation with only $\mathcal{O}(\sqrt{n})$ segments.

new segment. Let R_x be the region bounded by the lines containing $x_i y_i$ and $x_i z_i$ that shares only x_i with triangle (x_i, y_i, z_i). Similarly, let R_z be the region bounded by the lines containing $x_i z_i$ and $y_i z_i$ that shares only z_i with the same triangle. We note that these two regions are disjoint. Furthermore, if edge $x_i x_{i+1}$ belongs to a segment including edges contained in triangle (x_i, y_i, z_i), then x_{i+1} lies in region R_x. Similarly, if $z_i x_{i+1}$ belongs to a segment including edges contained in triangle (x_i, y_i, z_i), then x_{i+1} lies in region R_z. Both cases cannot be true simultaneously so either edge $x_i x_{i+1}$ or edge $z_i x_{i+1}$ contributes a new segment to the drawing. Symmetric arguments apply to the edges incident on y_{i+1} and z_{i+1} so the edges between triangles contribute at least three segments.

Thus in total we have at least $3k + 3(k-1) = 2n - 3$ segments. Now suppose that some face, other than (x_k, y_k, z_k), is the outerface. Thus the triangles are split into two nested sets. Say there are p triangles in one set and q in the other. By the above argument, any drawing has at least $(2p-3) + (2q-3) = 2n - 6$ segments. □

Lemma 17 gives a tight lower bound of $2n - c$ on the number of segments in drawings of maximal planar graphs. However, there are plane drawings with as little as $\mathcal{O}(\sqrt{n})$ segments, as illustrated in Fig. 12. Note that for planar graphs without degree two vertices, if there are k segments in some drawing, then the corresponding arrangement has at most $\binom{k}{2}$ vertices. Thus $n \leqslant \binom{k}{2}$ and $k > \sqrt{2n}$.

We now prove that the bound on the number of slopes in Theorem 14 is tight up to an additive constant.

Lemma 18. *For all $n \geqslant 4$, there is an n-vertex planar (3-tree) triangulation G_n that has at least $n + 2$ slopes in every plane drawing. For a particular choice of outerface, there are at least $2n - 2$ slopes in every plane drawing of G_n.*

Proof. Let G_n be the graph with vertex set $\{v_1, v_2, \dots, v_n\}$ and edge set $\{v_1 v_i, v_2 v_i\colon 3 \leqslant i \leqslant n\} \cup \{v_i v_{i+1}\colon 1 \leqslant i \leqslant n-1\}$. Observe that $G_4 = K_4$, and G_n is obtained from G_{n-1} by adding a 3-simplicial vertex v_n adjacent to v_1, v_2 and v_{n-1}. Thus G_n is a 3-tree and a planar triangulation. Every 3-cycle in G_n contains v_1 or v_2. Thus v_1 or v_2 is in the boundary of the outerface in every plane drawing of G_n. By Observation 9, the number of slopes in any plane drawing of G_n is at least $(n-3) + 1 + 1 + 3 = n + 2$. As illustrated in Fig. 13(a), if we fix the outerface of G_n to be (v_1, v_2, v_n), then the number of slopes is at least $(n-3) + (n-3) + 1 + 3 = 2n - 2$ slopes by Observation 9 □

As illustrated in Fig. 13(b), the graph G_n in Lemma 18 has a plane drawing (using a different embedding) with only $\lceil \frac{3n}{2} \rceil$ slopes.

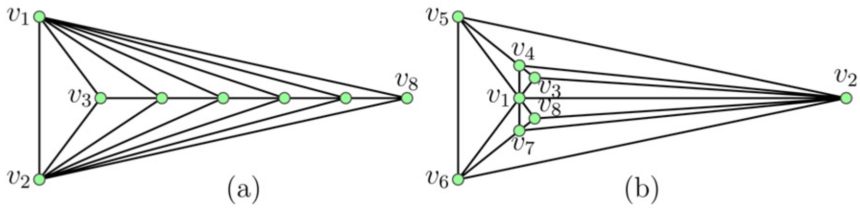

Fig. 13. The graph G_8 from Lemma 18.

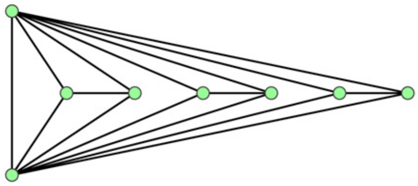

Fig. 14. The graph G_8 in Lemma 21.

Since deleting an edge from a drawing cannot increase the number of slopes, and every plane graph can be triangulated to a 3-connected plane graph, Theorem 14 implies:

Corollary 19. *Every n-vertex plane graph has a plane drawing with at most $2n - 10$ slopes.*

Open Problem 20. Is there some $\epsilon > 0$, such that every n-vertex planar triangulation has a plane drawing with $(2 - \epsilon)n + \mathcal{O}(1)$ slopes?

On the other hand, Theorem 14 does not imply any upper bound on the number of segments for all planar graphs. A natural question to ask is whether Theorem 14 can be extended to plane graphs that are not 3-connected. We have the following lower bound.

Lemma 21. *For all even $n \geqslant 4$, there is a 2-connected plane graph with n vertices (and $\frac{5}{2}n - 4$ edges) that has as many segments as edges in every drawing.*

Proof. Let G_n be the graph with vertex set $\{v, w, x_i, y_i\colon 1 \leqslant i \leqslant \frac{1}{2}(n-2)\}$ and edge set $\{vw, x_iy_i, vx_i, vy_i, wx_i, wy_i\colon 1 \leqslant i \leqslant \frac{1}{2}(n-2)\}$. Consider the plane embedding of G_n with the cycle (v, w, y_n) as the outerface, as illustrated in Fig. 14. Since the outerface is a triangle, no two edges incident to v can share a segment, and no two edges incident to w can share a segment. Consider two edges e and f both incident to a vertex x_i or y_i. The endpoints of e and f induce a triangle. Thus e and f cannot share a segment. Therefore no two edges in G_n share a segment. □

Note that the drawing technique from Fig. 7 can be used to draw the graph G_n in Lemma 21 with only $2n + \mathcal{O}(1)$ segments.

Open Problem 22. What is the minimum c such that every n-vertex plane (or planar) graph has a plane drawing with at most $cn + \mathcal{O}(1)$ segments?

5.1. Cubic 3-connected plane graphs

A graph in which every vertex has degree three is *cubic*. It is easily seen that Theorem 15 implies that every cubic plane 3-connected graph on n vertices has a plane drawing with at most $\frac{5}{4}n + \mathcal{O}(1)$ segments. This result can be improved as follows.

Lemma 23. *Every cubic plane 3-connected graph G on n vertices has a plane drawing with at most $n + 2$ segments.*

Proof. Let D be the plane drawing of G from Theorem 15. Recall the definitions and the arguments for counting segments in Theorem 15. By (1), the number of segments is at most

$$1 + h_s + \sum_{i=2}^{K} \left(|P(V_i)| - 1 \right).$$

By the properties of the canonical decomposition for plane cubic graphs, $|P(V_i)| = 2$ for all $2 \leqslant i \leqslant K - 1$, and $|P(V_K)| = 3$. Thus $|R| \leqslant 1$. As in Theorem 15, the number of segment-heavy steps with $|P(V_i)| = 2$ is at most $|S| + 1$. Thus $h_s \leqslant |S| + 2$. Therefore the number of segments in D is at most

$$1 + \left(|S| + 2\right) + (K - 2) + 2 = |S| + 3 + K = |S| + 3 + n - 1 - |S| = n + 2,$$

as claimed. □

Our bound on the number of slopes in a drawing of a 3-connected plane graph (Theorem 14) can be drastically improved when the graph is cubic.

Theorem 24. *Every cubic 3-connected plane graph has a plane drawing in which every edge has slope in* $\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}$, *except for three edges on the outerface.*

Proof. Let $\sigma = (V_1, V_2, \ldots, V_K)$ be a canonical decomposition of G. We re-use the notation from Theorem 15, except that a representative vertex of V_i may be the first or last vertex in V_i. Since G is cubic, $|P(V_i)| = 2$ for all $1 < i < K$, and every vertex not in $\{v_1, v_2, v_n\}$ has exactly one successor. We proceed by induction on i with the hypothesis that G_i has a plane drawing D_i that satisfies the following invariants.

Invariant 1: $C_i \setminus v_1v_2$ is *X-monotone*; that is, the path from v_1 to v_2 in $C_i \setminus v_1v_2$ has non-decreasing X-coordinates.
Invariant 2: Every peak in D_i, $i < K$, has a successor.
Invariant 3: If there is a vertical edge above v in D_i, then all the edges of G that are incident to v are in G_i.
Invariant 4: D_i has no edge crossings.

Let D_2 be the drawing of G_2 constructed as follows. Draw v_1v_2 horizontally with $X(v_1) < X(v_2)$. This accounts for one edge whose slope is not in $\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}$. Now draw v_1v_3 with slope $\frac{\pi}{4}$, and draw v_2v_3 with slope $\frac{3\pi}{4}$. Add any division vertices on the segment v_1v_3. Now v_3 is the only peak in D_2, and it has a successor by the definition of the canonical decomposition. Thus all the invariants are satisfied for the base case D_2.

Now suppose that $2 < i < K$ and we have a drawing of D_{i-1} that satisfies the invariants. Suppose that $P(V_i) = \{u, w\}$, where u and w are the left and the right predecessors of V_i, respectively. Without loss of generality, $Y(w) \leqslant Y(u)$. Let the representative vertex v_i be last vertex in V_i. Position v_i at the intersection of a vertical segment above w, and a segment of slope $\frac{\pi}{4}$ from u, and add any division vertices on $\overline{uv_i}$, as illustrated in Fig. 15(a). Note that there is no vertical edge above w by invariant 3 for D_{i-1}. (For the case in which $Y(u) < Y(w)$, we take the representative vertex v_i to be the first vertex in V_i, and the edge wv_i has slope $\frac{3\pi}{4}$, as illustrated in Fig. 15(b).)

Clearly the resulting drawing D_i is X-monotone. Thus invariant 1 is maintained. The vertex v_i is the only peak in D_i that is not a peak in D_{i-1}. Since v_i has a successor by the definition of the canonical decomposition, invariant 2

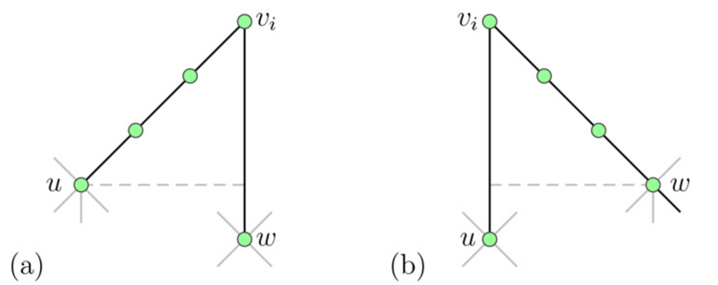

Fig. 15. Construction of a 3-slope drawing of a cubic 3-connected plane graph.

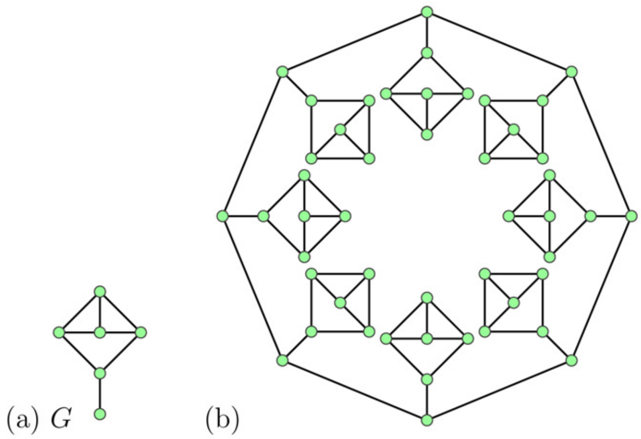

Fig. 16. Counterexample to the claim of Kant [15].

is maintained. The vertical edge wv_i satisfies invariant 3, since v_i is the sole successor of w. Thus invariant 3 is maintained. No vertex between u and w (on the path from u to w in $C_{i-1} \setminus v_1v_2$) is higher than the higher of u and w. Otherwise there would be a peak, not equal to v_n, with no successor, and thus violating invariant 2 for D_{i-1}. Thus the edges in $D_i \setminus D_{i-1}$ do not cross any edges in D_i. In particular, there is no edge ux in D_{i-1} with slope $\frac{\pi}{4}$ and $Y(x) > Y(u)$.

It remains to draw the vertex v_n. Suppose v_n is adjacent to v_1, u, and w, where $X(v_1) < X(u) < X(w)$. By invariants 1 and 3 applied to v_1, u and w, there is point p vertically above u that is visible from v_1 and w. Position v_n at p and draw its incident edges. We obtain the desired drawing of G. The edge v_nu has slope $\frac{\pi}{2}$, while v_nv_1 and v_nw are the remaining two edges whose slope is not in $\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}$. □

A number of notes regarding Theorem 24 are in order:

- By Lemma 2 we could have used any set of three slopes instead of $\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}$ in Theorem 24.
- By Observation 9, the bound of six on the number of slopes in Theorem 24 is optimal for any 3-connected cubic plane graph whose outerface is a triangle. It is easily seen that there is such a graph on n vertices for all even $n \geqslant 4$.
- Theorem 24 was independently obtained by Kant [15]. We believe that our proof is much simpler. Kant [15] also claimed to prove that every plane graph with maximum degree three has a 3-slope drawing (except for one bent edge). This claim is false. Consider the plane graph G illustrated in Fig. 16(a). It is easily seen that G has no 3-slope plane (straight-line) drawing. Thus the cubic plane graph illustrated in Fig. 16(b), which contains a linear number of copies of G, must have a linear number of bends in any plane drawing on three slopes.
  Kant [15] also claimed to prove that every planar graph with maximum degree three (except K_4) has a drawing in which every angle (between consecutive edges incident to a vertex) is at least $\frac{\pi}{3}$, except for at most four angles. The example in Fig. 16(b) is a counterexample to this claim as well. It is easily seen that every drawing of G has an angle less than $\frac{\pi}{3}$. (Assume otherwise, and start from back-to-back drawings of two equilateral triangles.) Thus the cubic plane graph illustrated in Fig. 16(b) has a linear number of angles less than $\frac{\pi}{3}$.

Corollary 25. *Every cubic* 3*-connected plane graph has a plane 'drawing' with three slopes and three bends on the outerface.*

Proof. Apply the proof of Theorem 24 with two exceptions. First the edge v_1v_2 is drawn with one bend. The segment incident to v_1 has slope $\frac{3\pi}{4}$, and the segment incident to v_2 has slope $\frac{\pi}{4}$. The second exception regards how to draw the edges incident to v_n. Suppose v_n is adjacent to v_1, u, and w, where $X(v_1) < X(u) < X(w)$. There is a point s above v_1, a point p above u, and a point t above w, so that the slope of sp is $\frac{\pi}{4}$ and the slope of tp is $\frac{3\pi}{4}$. Place v_n at

p, draw the edge $v_n u$ vertical, draw the edge $v_1 v_n$ with one bend through s (with slopes $\{\frac{\pi}{2}, \frac{\pi}{4}\}$), and draw the edge wv_n with one bend through t (with slopes $\{\frac{\pi}{2}, \frac{3\pi}{4}\}$). □

Open Problem 26. Does there exist a function f such that every plane graph with maximum degree Δ has a plane drawing with $f(\Delta)$ slopes? This is open even for maximal outerplanar graphs. Note that there exist bounded degree (non-planar) graphs for which the number of slopes is unbounded in every drawing [1,10,18]. The best bounds are in our companion paper [10], in which we prove that there exists Δ-regular n-vertex graphs with at least $n^{1-\frac{8+\epsilon}{\Delta+4}}$ slopes in every drawing.

Open Problem 27. In all our results, we have not studied other aesthetic criteria such as symmetry and small area (with the vertices at grid points). Many open problems remain when combining "few slopes or segments" with other aesthetic criteria. For example, can Theorem 24 be generalised to prove that every cubic 3-connected plane graph on n vertices has a plane grid drawing with polynomial (in n) area, such that every edge has one of three slopes (except for three edges on the outerface)?

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