Queue Layouts of Graphs with Bounded Degree and Bounded Genus

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Abstract

Motivated by the question of whether planar graphs have bounded queue-number, we prove that planar graphs with maximum degree Δ have queue-number $O(\Delta^2)$, which improves upon the best previous bound of $O(\Delta^6)$. More generally, we prove that graphs with bounded degree and bounded Euler genus have bounded queue-number. In particular graphs with Euler genus g and maximum degree Δ have queue-number $O(g + \Delta^2)$. As a byproduct we prove that if planar graphs have bounded queue-number, then graphs of Euler genus g have queue-number $O(g)$.

1 Introduction

Bekos, Förster, Gronemann, Mchedlidze, Montecchiani, Raftopoulou, and Ueckerdt [1] recently proved that planar graphs with bounded (maximum) degree have bounded queue-number. We improve their bound and more generally show that graphs with bounded degree and bounded genus have bounded queue-number.

First we introduce queue layouts and give the background to the above results. For a graph G and integer $k \geqslant 0$, a *k-queue layout* of G consists of a linear ordering $\preceq$ of $V(G)$ and a partition $E_1, E_2, \dots, E_k$ of $E(G)$, such that for $i \in \{1, \dots, k\}$, no two edges in E_i are nested with respect to $\preceq$. Here edges vw and xy are *nested* if $v \prec x \prec y \prec w$. The *queue-number* of a graph G, denoted by $\operatorname{qn}(G)$, is the minimum integer k such that G has a k-queue layout. These definitions were introduced by Heath et al. [12, 13] as a dual to stack layouts (also called book embeddings). In a stack layout, no two edges in E_i cross with respect to $\preceq$. Here edges vw and xy *cross* if $v \prec x \prec w \prec y$

Heath et al. [12] conjectured that every planar graph has bounded queue number. This conjecture has remained open despite much research on queue layouts [3, 5–8, 10–14, 16, 18, 19]. Dujmović

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and Wood [8] observed that every graph with m edges has a $O(\sqrt{m})$-queue layout using a random vertex ordering. Thus every planar graph with n vertices has queue-number $O(\sqrt{n})$. Di Battista, Frati, and Pach [2] proved the first breakthrough on this topic, by showing that every planar graph with n vertices has queue-number $O(\log^2 n)$. Dujmović [4] improved this bound to $O(\log n)$ with a simpler proof.

Dujmović et al. [6] established (poly-)logarithmic bounds for more general classes of graphs.[1] For example, they proved that every graph with n vertices and Euler genus g has queue-number $O(g + \log n)$, and that every graph with n vertices excluding a fixed minor has queue-number $\log^{O(1)} n$.

Recently, Bekos et al. [1] proved a second breakthrough result, by showing that planar graphs with bounded degree have bounded queue-number.

Theorem 1 ([1]). *Every planar graph with maximum degree* Δ *has queue-number at most* $32(2\Delta - 1)^6 - 1$.

Note that bounded degree alone is not enough to ensure bounded queue-number. In particular, Wood [20] proved that for every integer $\Delta \geqslant 3$ and all sufficiently large n, there are graphs with n vertices, maximum degree Δ, and queue-number $\Omega(\sqrt{\Delta} n^{1/2-1/\Delta})$.

The first contribution of this paper is to improve the bound of Bekos et al. [1] from $O(\Delta^6)$ to $O(\Delta^2)$.

Theorem 2. *Every planar graph with maximum degree* Δ *has queue-number at most* $12\Delta^2 + 16\Delta + 3$.

We extend this result by showing that graphs with bounded Euler genus and bounded degree have bounded queue-number.

Theorem 3. *Every graph with Euler genus* g *and maximum degree* Δ *has queue-number at most* $4g + 36\Delta^2 + 48\Delta + 9$.

We remark that using well-known constructions [5, 7, 9], Theorem 3 implies that graphs with bounded Euler genus and bounded degree have bounded track-number, which in turn can be used to prove linear volume bounds for three-dimensional straight-line grid drawings of the same class of graphs. These results can also be extended for graphs with bounded degree that can be drawn in a surface of bounded Euler genus with a bounded number of crossings per edge (using [10, Theorem 6]). We omit all these details.

The proof of Theorem 3 uses Theorem 2 as a 'black box'. Starting with a graph G of bounded Euler genus and bounded degree, we construct a planar subgraph G' of G. We then apply Theorem 2 to obtain a queue layout of G', from which we construct a queue layout of G. This approach suggests a direct connection between the queue-number of graphs with bounded Euler genus and planar graphs, regardless of degree considerations. The following theorem

[1] The *Euler genus* of a graph G is the minimum integer k such that G embeds in the orientable surface with $k/2$ handles (and k is even) or the non-orientable surface with k cross-caps. Of course, a graph is planar if and only if it has Euler genus 0; see [15] for more about graph embeddings in surfaces. A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges.

establishes this connection. A class of graphs is *hereditary* if it is closed under taking induced subgraphs.

Theorem 4. *Let $\mathcal{G}$ be a hereditary class of graphs, such that every planar graph in $\mathcal{G}$ has queue-number at most k. Then every graph in $\mathcal{G}$ with Euler genus g has queue-number at most $3k + 4g$.*

Theorem 3 is an immediate corollary of Theorems 2 and 4, where $\mathcal{G}$ is the class of graphs with maximum degree at most Δ. Theorem 4, where $\mathcal{G}$ is the class of all graphs, implies the following result of interest:

Corollary 5. *If every planar graph has queue-number at most k, then every graph with Euler genus g has queue-number at most $3k + 4g$.*

For a graph G and a set $A \subseteq V(G)$, let $G[A]$ be the subgraph of G induced by A, which has vertex set A and edge set $\{vw \in E(G) : v, w \in A\}$. For disjoint sets $A, B \subseteq V(G)$, let $G[A, B]$ be the bipartite graph with bipartition $\{A, B\}$ and edge set $\{vw \in E(G) : v \in A, w \in B\}$.

2 Planar Graphs of Bounded Degree

This section proves Theorem 2. The proof is inspired by the proof of Theorem 1 by Bekos et al. [1]. Here is high-level overview of their proof for a planar graph G with maximum degree Δ. First, Bekos et al. [1] construct a particular planar graph G_1 obtained from G by subdividing each edge at most three times. Then they construct a planar graph G_2 from G_1 by replacing certain edges by pairs of trees and a perfect matching between their leaves. G_2 is called a 'Δ-matched' graph. The heart of the proof of Bekos et al. [1] is to construct a $O(\Delta)$-queue layout of any Δ-matched graph, and thus of G_2. They then observe that the queue layout of G_2 also gives a $O(\Delta)$-queue layout of G_1. Finally, they use a generic lemma of Dujmović and Wood [8], which says that if some $(\leqslant c)$-subdivision of a graph has a k-queue layout, then the original graph has a $O(k^{2c})$-queue layout. Bekos et al. [1] apply this result with $k = O(\Delta)$ and $c = 3$, to obtain a $O(\Delta^6)$-queue layout of G.

It should be mentioned that there is a straightforward way to improve this $O(\Delta^6)$ bound. Lemma 11 in Appendix 1 shows that if some $(\leqslant c)$-subdivision of a graph has a k-queue layout for some fixed c, then the original graph has a $O(k^{c+1})$-queue layout. Moreover, in the proof of Bekos et al. [1], for every edge e of G that is subdivided three times, one of the edges in the subdivision of e is assigned to a single queue ($\mathcal{Q}_0$ in their notation). This observation, in conjunction with the proof of Lemma 11, leads to a $O(\Delta^3)$-queue layout of G.

Our proof of Theorem 2 initially follows a similar strategy. Starting with a planar graph G with maximum degree Δ, we consider the $(\leqslant 3)$-subdivision G_1 of G constructed by Bekos et al. [1]. Note that Bekos et al. [1] explain in Section 3.3 of their paper that one can work directly with G_1 instead of the Δ-matched graph G_2, and this is what we choose to do. The key properties of G_1 are summarised in the definition of 'well-layered' below. We then construct a partition of $V(G_1)$ with several desirable properties (see Lemma 7). This partition is implicit in the proof of Bekos et al. [1]—there is really nothing new in this part of our proof.

The main point of difference between our proof and that of Bekos et al. [1] is that we do not apply the generic 'unsubdividing' lemma of Dujmović and Wood [8]. Instead we refine the partition of $V(G_1)$ to obtain a similar partition of $V(G)$ (see Lemma 8). From this partition one can determine a $O(\Delta^2)$-queue layout of G. Note that in this $O(\Delta^2)$-queue layout, the vertex ordering is identical to that used by Bekos et al. [1], only the queue assignment is different. This fact shows the value in focusing on structural partitions rather than the final queue layout.

The following definitions are key concepts in our proofs (and that of several other papers on queue layouts [1, 5, 6, 8]). A *layering* of a graph G is a partition $(V_0, V_1, \dots, V_t)$ of $V(G)$ such that for every edge $vw \in E(G)$, if $v \in V_i$ and $w \in V_j$, then $|i - j| \leqslant 1$. If r is a vertex in a connected graph G and $V_i := \{v \in V(G) : \mathrm{dist}_G(r, v) = i\}$ for all $i \geqslant 0$, then $(V_0, V_1, \dots, V_t)$ is called a *BFS layering* of G, where $t := \max\{\mathrm{dist}_G(r, v) : v \in V(G)\}$. Associated with a bfs layering is a *bfs spanning tree* T obtained by choosing, for each non-root vertex $v \in V_i$ with $i \geqslant 1$, a neighbour w in V_{i-1}, and adding the edge vw to T. Thus $\mathrm{dist}_T(r, v) = \mathrm{dist}_G(r, v)$ for each vertex v of G. When the spanning tree T is obvious from the context, we call edges in T *tree edges* and edges not in T *non-tree edges*. An edge $vw \in E(G)$ with $v, w \in V_i$ for some $i \geqslant 0$ is called a *level* edge. An edge $vw \in E(G)$ with $v \in V_i$ and $w \in V_{i+1}$ for some $i \geqslant 0$ is called a *binding* edge. Every tree edge is *binding*.

The following lemma of Pupyrev [17] shows that every planar graph has a drawing that highlights particular aspects of a BFS layering, as illustrated in Figure 1.

Lemma 6 ([17]). *For every connected planar graph G and every vertex r of G, if T is the BFS tree and $(V_0, V_1, \dots, V_t)$ is the BFS layering of G rooted at r, then there is a drawing of G in $\mathbb{R}^2$ with the r at the origin and on the outer-face, such that for $i \in \{1, 2, \dots, t\}$,*

- *the vertices in V_i are drawn on a circle C_i of radius R_i centred at the origin, where $0 < R_1 < R_2 < \cdots < R_t$;*
- *each level edge $vw \in E(G)$ with $v, w \in V_i$ is drawn as an open curve between v and w strictly outside of C_i; and*
- *each binding edge vw with $v \in V_i$ and $w \in V_{i+1}$ is drawn either:*
  - *as an open curve from v to w strictly between C_i and C_{i+1} (called a* direct *edge), or*
  - *as an open curve starting at v that crosses C_{i+1} once at a point distinct from w, then stays outside of C_{i+1}, and ends at w (called a* hooked *edge).*
- *each tree edge $vw \in E(T)$ is direct and binding..*

2.1 Well-Layered Planar Graphs

A planar graph G is *well-layered* if there is a BFS spanning tree T of G rooted at a vertex r such that every non-tree edge $vw \in E(G) \setminus E(T)$ is a level edge in the corresponding BFS layering, and both v and w are leaves in T with degree 2 in G. This implies that the set of non-tree edges are a matching in G.

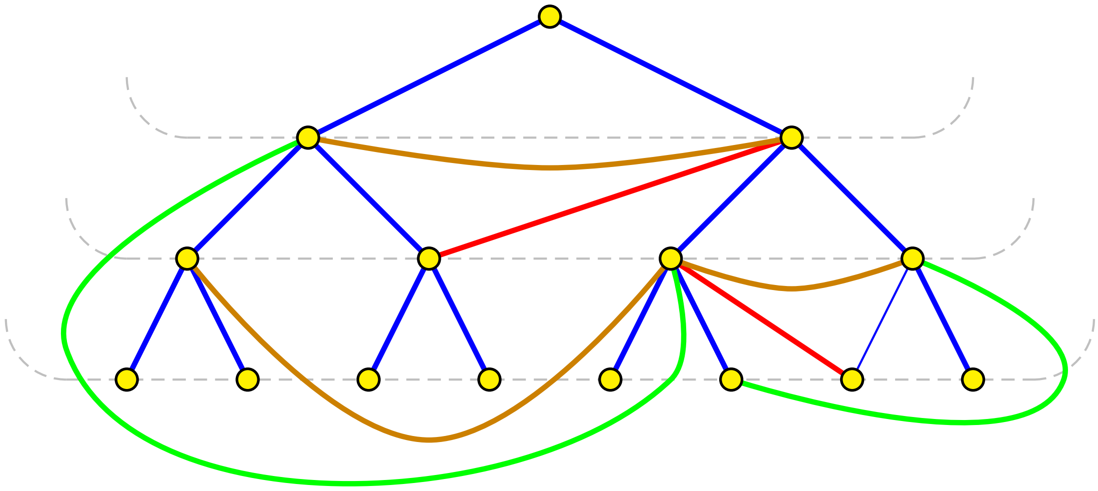

Figure 1: Drawing of planar graph on concentric circles: tree edges are blue, hooked edges are green, level edges are brown, direct non-tree edges are red.

Lemma 7. *Let G be well-layered planar graph with corresponding BFS spanning tree T and BFS layering $(V_0, V_1, \dots, V_t)$ rooted at a vertex r. Assume that every vertex in G has at most Δ children in T. Then for $i \in \{1, 2, \dots, t\}$, there is a partition $\{V_{i,a} : a \geqslant 0\}$ of V_i, and an ordering $\overrightarrow{V_{i,a}}$ of each set $V_{i,a}$, such that:*

(a) for each non-tree edge $vw \in E(G) \setminus E(T)$, both v and w are in $V_{i,a}$ for some $i, a \geqslant 0$,

(b) for each tree edge $vw \in E(T)$, if $v \in V_{i,a}$ and $w \in V_{i+1,b}$ for some $a, b \geqslant 0$, then $b - \Delta a \in \{0, 1, \dots, 2\Delta - 1\}$,

(c) for all $i, a \geqslant 0$, no two edges in $G[V_{i,a}]$ cross or nest with respect to the ordering $\overrightarrow{V_{i,a}}$, in particular, $\overrightarrow{V_{i,a}}$ defines a 1-queue layout of $G[V_{i,a}]$, and

(d) for all $i, a, b \geqslant 0$, the ordering $\overrightarrow{V_{i,a}}\overrightarrow{V_{i+1,b}}$ defines a 1-queue layout of $G[V_{i,a}, V_{i+1,b}]$.

Proof. Apply Lemma 6 to obtain a drawing of G on concentric circles $C_1, C_2, \dots, C_t$. For each vertex $v \in V(G)$, let $\ell(v) := \mathrm{dist}_G(v, r) - t$. Thus $\ell(v) \in \{0, 1, \dots, t\}$ for every vertex v of G, and $\ell(r) = t$. For each vertex v of G, let T_v be the subtree of T rooted at v.

For each non-tree edge $vw \in E(G) \setminus E(T)$, let D_{vw} be the cycle obtained from the vw-path in T by adding the edge vw. Note that if $v, w \in V_i$ then the vw-path in T is drawn within the interior of C_i (since every tree edge is binding and direct) and vw is drawn outside of C_i.

Let G^+ be the multigraph with vertex set $V(G)$, where each tree edge $vw \in E(T)$ has multiplicity 1 in G^+, and each non-tree edge $vw \in E(G) \setminus E(T)$ has multiplicity $\Delta^{\ell(v)}$ (which equals $\Delta^{\ell(w)}$ since every non-tree edge is a level edge). Note that T is a spanning tree of G^+.

A key property of this construction is that for each vertex v of G, the number of non-tree edges in G^+ with one endpoint in T_v is at most $\Delta^{\ell(v)}$. We prove this claim by induction on $\ell(v)$. First note that if v is a leaf in T, then T_v is simply the vertex v, and v is incident to at most one non-tree edge in G, and thus is incident to at most $\Delta^{\ell(v)}$ edges in G^+. So the claim holds for leaves. In particular, if $\ell(v)=0$ then v is a leaf in T, and the claim holds. Now consider a non-leaf vertex v with $\ell(v)\geqslant 1$. Let $v_1,\dots,v_d$ be the children of v, where $v_1,\dots,v_p$ are not leaves of T, and $v_{p+1},\dots,v_d$ are leaves of T. Note that $\ell(v_i)=\ell(v)-1$ for $i\in\{1,\dots,d\}$. By induction, for each $i\in\{1,\dots,p\}$, the number of non-tree edges in G^+ with one endpoint in T_{v_i} is at most $\Delta^{\ell(v_i)}=\Delta^{\ell(v)-1}$. For $i\in\{p+1,\dots,d\}$, we have already shown that the number of non-tree edges in G^+ incident to v_i is at most $\Delta^{\ell(v_i)}=\Delta^{\ell(v)-1}$ edges. In total, there are at most $d\Delta^{\ell(v)-1}\leqslant\Delta^{\ell(v)}$ non-tree edges in G^+ incident to T_v, as claimed.

Let G^* be the dual of G^+. Let T^* be the spanning subgraph of G^* consisting of those edges of G^* dual to edges of $E(G^+)\setminus E(T)$. It is well known (and easily follows from Euler's formula) that T^* is a spanning tree of G^*. (T^* is sometimes called a *co-tree*; note that T and T^* can be simultaneously drawn without crossing each other.) Let r^* be the vertex of T^* dual to the outer-face of G^+. Consider T^* to be rooted at r^*.

For each face f of G^+, let $d(f)$ be the distance in T^* between r^* and the vertex of T^* dual to f. For each vertex v of G^+, let $m(v)$ be the minimum of $d(f)$ taken over all faces f of G^+ incident with the subtree of T rooted at v, and let

$$g(v):=\left\lfloor\frac{m(v)}{\Delta^{\ell(v)}}\right\rfloor.$$

For $i,a\geqslant 0$, let

$$V_{i,a}:=\{v\in V_i:g(v)=a\}.$$

Let $\overrightarrow{V_{i,a}}$ be the ordering of $V_{i,a}$ along circle C_i, where the outer-face defines the start and end point. (In the language of Bekos et al. [1], $m(v)$ is analogous to the 'matching-value' of v, and $g(v)$ is the 'layer-group' of v.)

This concludes the description of the partition $\{V_{i,a}:i,a\geqslant 0\}$ and the orderings $\overrightarrow{V_{i,a}}$. We now show these satisfy the claims of the lemma.

We now prove (a). Consider a non-tree edge $vw\in E(G)\setminus E(T)$. By assumption, both v and w are in V_i for some $i\geqslant 0$, and both v and w are leaves in T. Thus $\ell(v)=\ell(w)$, and v is the only vertex in the subtree rooted at v, and similarly for w. Since $\deg_G(v)=\deg_G(w)=2$, the faces incident to v are exactly the same faces incident to w. Thus $m(v)=m(w)$, implying v and w are in $V_{i,a}$ where $a=\lfloor m(v)/\Delta^{\ell(v)}\rfloor$. This proves (a).

We now prove (b). Consider a non-leaf vertex v with $\ell(v)=\ell$. Then all the edges incident to v are in T. Let x and y be two children of v consecutive in the embedding of G. Observe that $m(y)-m(x)$ is maximised when all the non-tree edges incident to T_x go 'under' T_y. The number of such edges is at most $\Delta^{\ell(x)}=\Delta^{\ell-1}$. Thus $m(y)\leqslant m(x)+\Delta^{\ell-1}$. Since v has at most Δ children, $m(y)\leqslant m(x)+\Delta^{\ell}$ for all children x and y of v. Every face incident with T_v is incident to T_x for some child x of v. Thus $m(v)$ equals the minimum of $m(x)$ taken over all children x of v. Hence $m(y)\leqslant m(v)+\Delta^{\ell}$ for all children y of v, implying

$$g(y)=\left\lfloor\frac{m(y)}{\Delta^{\ell-1}}\right\rfloor\leqslant\frac{m(y)}{\Delta^{\ell-1}}\leqslant\Delta\frac{m(v)}{\Delta^{\ell}}+\Delta<\Delta\left(\left\lfloor\frac{m(v)}{\Delta^{\ell}}\right\rfloor+1\right)+\Delta=\Delta\,g(v)+2\Delta.$$

Since $g(v)$ and $g(y)$ are integers, $g(y)\leqslant\Delta\,g(v)+2\Delta-1$. Moreover, $m(v)\leqslant m(y)$, implying

$$\Delta\,g(v)=\Delta\left\lfloor\frac{m(v)}{\Delta^{\ell}}\right\rfloor\leqslant\Delta\,\frac{m(v)}{\Delta^{\ell}}\leqslant\frac{m(y)}{\Delta^{\ell-1}}<\left\lfloor\frac{m(y)}{\Delta^{\ell-1}}\right\rfloor+1=g(y)+1.$$

Since $g(v)$ and $g(y)$ are integers, $g(y) \geqslant \Delta\, g(v)$. Summarising, if y is a child of v in T then

$$g(y) - \Delta\, g(v) \in \{0, 1, \ldots, 2\Delta - 1\}.$$

This says that for each tree edge $vw \in E(T)$ where $v \in V_{i,a}$ and $w \in V_{i+1,b}$, we have $b - \Delta a \in \{0, 1, \ldots, 2\Delta - 1\}$, which proves (b).

We now prove (c), which claims that no two edges in $G[V_{i,a}]$ cross or nest with respect to the ordering $\overrightarrow{V_{i,a}}$. Consider edges $vw, pq \in E(G)$ with $v, w, p, q \in V_{i,a}$. Let $\ell := \ell(v) = \ell(w)$. Neither vw nor pq are tree edges. Suppose on the contrary that vw and pq cross with respect to $\overrightarrow{V_{i,a}}$. Without loss of generality, $v \prec p \prec w \prec q$ in $\overrightarrow{V_{i,a}}$. Then v, p, w, q appear in this order on the circle C_i. Since vw and pq are drawn outside C_i, these edges cross in the drawing of G, which is a contradiction. Thus no two edges in $G[V_{i,a}]$ cross with respect to $\overrightarrow{V_{i,a}}$. Now suppose that vw and pq nest with respect to $\overrightarrow{V_{i,a}}$. Without loss of generality, $v \prec p \prec q \prec w$ in $\overrightarrow{V_{i,a}}$. Thus v, p, q, w appear in this order on C_i. Hence both T_p and T_q are inside D_{vw} and the outer-face of G^+ is outside D_{vw}. Since vw is the only edge of D_{vw} not in T, every path in T^* from r^* to a vertex dual to a face incident with p or q must include the edges of T^* dual to vw. Let f be the face of G immediately below vw. Since vw has multiplicity Δ^ℓ in G^+, for every face f' incident with T_p or T_q, we have $d(f') \geqslant d(f) + \Delta^\ell$. Since w is incident with f, we have $m(w) \leqslant d(f)$. Thus $m(p) \geqslant d(f) + \Delta^\ell \geqslant m(w) + \Delta^\ell$. Hence

$$\frac{m(p)}{\Delta^\ell} \geqslant \frac{m(w)}{\Delta^\ell} + 1.$$

Therefore

$$g(p) = \left\lfloor \frac{m(p)}{\Delta^\ell} \right\rfloor > \left\lfloor \frac{m(w)}{\Delta^\ell} \right\rfloor = g(w),$$

which implies that p and w are not both in $V_{i,a}$. This contradiction shows that $\overrightarrow{V_{i,a}}$ defines a 1-queue layout of $G[V_{i,a}]$ for all $i, a \geqslant 0$. This proves (c).

We now prove (d). Suppose on the contrary that $v \prec x$ in $\overrightarrow{V_{i,a}}$ and $y \prec w$ in $\overrightarrow{V_{i+1,b}}$ for some edges $vw, xy \in E(G)$ for some $i, a, b \geqslant 0$. Thus v is to the left of x in C_i and y is to the left of w in C_{i+1}. Since every non-tree edge is a level edge, both vw and xy are tree edges, which are drawn direct between C_i and C_{i+1}. Thus vw and xy cross. This contradiction shows that no two edges of $G[V_{i,a}, V_{j,b}]$ are nested in the ordering $\overrightarrow{V_{i,a}}\overrightarrow{V_{i+1,b}}$, which thus defines a 1-queue layout of $G[V_{i,a}, V_{i+1,b}]$. This proves (d). $\square$

Note that Lemma 7 implies that every well-layered graph has a 2Δ-queue layout, as proved by Bekos et al. [1]. To see this, let $\overrightarrow{V_i}$ be the ordering $\overrightarrow{V_{i,0}}\overrightarrow{V_{i,1}}\ldots$ of V_i. Then take the ordering $\overrightarrow{V_0}\overrightarrow{V_1}\overrightarrow{V_2}\ldots$ of $V(G)$. By Lemma 7(c), every level edge can be assigned to a single queue Q^*. Assign each tree edge $vw \in E(T)$ where $v \in V_{i,a}$ and $w \in V_{i+1,b}$ to $Q_{b-\Delta a}$. By Lemma 7(b) this introduces 2Δ queues. Suppose that tree edges vw and pq in Q_j are nested for some $j \in \{0, 1, \ldots, 2\Delta - 1\}$, with $v \prec p \prec q \prec w$ in the ordering. Then $v \in V_{i,a}$, $p \in V_{i,b}$, $q \in V_{i+1,c}$ and $w \in V_{i+1,d}$ for some $i, a, b, c, d \geqslant 0$ with $j = d - \Delta a = c - \Delta b$. Thus $d - c = \Delta(a - b)$. Since $v \prec p \prec q \prec w$ in the ordering, $d - c \geqslant 0$ and $a - b \leqslant 0$. Thus $a = b$ and $c = d$, which contradicts Lemma 7(d).

2.2 General Planar Graphs

We now extend Lemma 7 for all planar graphs.

Lemma 8. *Let G be a planar graph with a BFS spanning tree T and BFS layering $(V_0, V_1, \ldots, V_t)$ rooted at a vertex r. Assume that every vertex in G has degree at most $\Delta + 1$ and has most Δ children in T. Then for $i \in \{1, 2, \ldots, t\}$, there is a partition $\{V_{i,a} : a \geqslant 0\}$ of V_i, and an ordering $\overrightarrow{V_{i,a}}$ of each set $V_{i,a}$, such that:*

(a) for each level edge $vw \in E(G)$, if $v \in V_{i,a}$ and $w \in V_{i,b}$ then $|a - b| \leqslant 1$;

(b) for each tree edge $vw \in E(T)$, if $v \in V_{i,a}$ and $w \in V_{i+1,b}$ then $b - a\Delta \in \{0, 1, \ldots, 2\Delta - 1\}$;

(c) for each non-tree binding edge $vw \in E(G) \setminus E(T)$, if $v \in V_{i,a}$ and $w \in V_{i+1,b}$ then $b - a\Delta \in \{-1, 0, \ldots, 2\Delta\}$;

(d) the ordering $\overrightarrow{V_{i,a}}\overrightarrow{V_{i,a+1}}$ defines a 1-queue layout of $G[V_{i,a}, V_{i,a+1}]$ for all $i, a \geqslant 0$.

(e) the ordering $\overrightarrow{V_{i,a}}\overrightarrow{V_{i+1,b}}$ defines a $(6\Delta + 1)$-queue layout of $G[V_{i,a}, V_{i+1,b}]$ for all $i, a, b \geqslant 0$.

(f) the ordering $\overrightarrow{V_{i,a}}$ defines a 2Δ-queue layout of $G[V_{i,a}]$ for all $i, a \geqslant 0$.

Proof. Apply Lemma 6 to obtain a drawing of G on concentric circles $C_1, C_2, \ldots, C_t$ rooted at r.

Let G' be obtained by subdividing edges of G as follows, as illustrated in Figure 2. Initialise $T' := T$ and $V'_i := V_i$ for each $i \geqslant 0$. For each level edge $vw \in E(G)$ with $v, w \in V_i$ for some $i \geqslant 0$:

- replace vw by a path $vxyw$ in G' (where x and y are new vertices);
- add the edges vx and wy to T' (so x and y are leaves in T' with degree 2 in G'); and
- add x and y to V'_{i+1}.

For each non-tree binding edge $vw \in E(G) \setminus E(T)$ with $v \in V_i$ and $w \in V_{i+1}$ for some $i \geqslant 0$:

- replace vw by a path $vxyzw$ in G' (where x, y, z are new vertices);
- add the edges vx, xy and wz to T' (so y and z are leaves in T' with degree 2 in G'); and
- add x to V'_{i+1}, and add y and z to V'_{i+2}.

Observe that T' is a bfs spanning tree of G', and G' is well-layered with respect to the layering $V'_0, V'_1, \ldots, V'_t$. For $i \geqslant 0$, let $\{V'_{i,a} : a \geqslant 0\}$ be the partition of V'_i from Lemma 7 applied to G'. Let $V_{i,a} := V'_{i,a} \cap V(G)$ for $i, a \geqslant 0$, where $\overrightarrow{V_{i,a}}$ inherits its order from $\overrightarrow{V'_{i,a}}$.

We now prove (a). Consider a level edge $vw \in E(G)$ with $v \in V_{i,a}$ and $w \in V_{i,b}$. Let $vxyw$ be the corresponding path in G'. Then xy is a level non-tree edge of G' with $v, w \in V'_{i+1}$. By

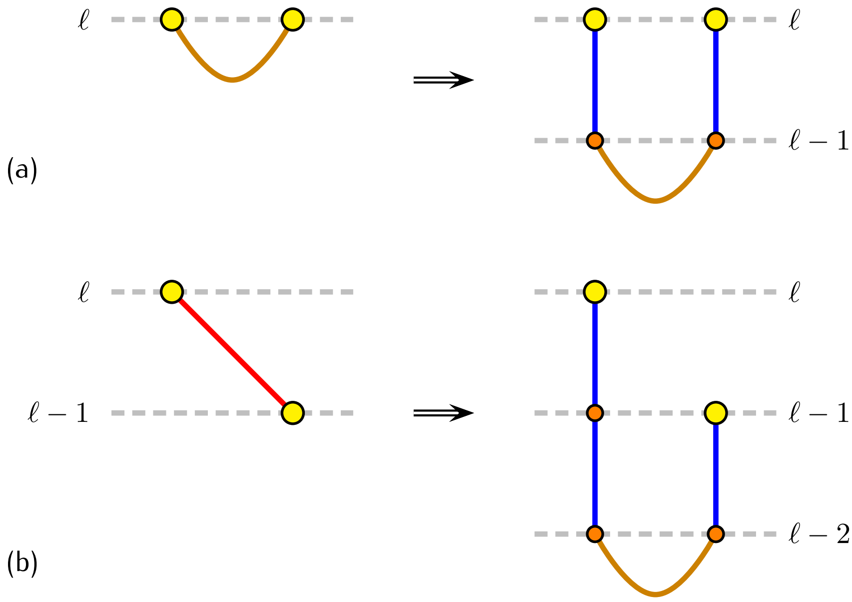

Figure 2: Creating the subdivision G': (a) level edge, (b) non-tree binding edge.

Lemma 7(a), both x and y are in $V'_{i+1,c}$ for some $c \geqslant 0$. Since vx and wy are tree edges in G' by Lemma 7(b), we have $c - \Delta a = \alpha$ and $c - \Delta b = \beta$ for some $\alpha, \beta \in \{0, 1, \dots, 2\Delta - 1\}$. Thus $c = \alpha + \Delta a = \beta + \Delta b$, implying $\Delta(a - b) = \beta - \alpha \leqslant 2\Delta - 1$ and $a - b \leqslant 1$. Similarly $b - a \leqslant 1$. Thus $|a - b| \leqslant 1$. This proves (a).

Property (b) follows immediately from Lemma 7(b) since a tree edge $vw \in E(T)$ with $v \in V_{i,a}$ and $w \in V_{i+1,b}$ is a tree edge in G' with $v \in V'_{i,a}$ and $w \in V'_{i+1,b'}$, in which case $b - \Delta a \in \{0, 1, \dots, 2\Delta - 1\}$.

We now prove (c). Consider a binding non-tree edge $vw \in E(G) \setminus E(T)$ with $v \in V_{i,a}$ and $w \in V_{i+1,b}$. Let $vxyzw$ be the corresponding path in G'. Then vx, xy and wz are tree edges in G', and yz is a level edge in G'. Moreover, $x \in V'_{i+1}$ and $y, z \in V'_{i+2}$. Then $x \in V'_{i+1,c}$ for some $c \geqslant 0$, and $y, z \in V'_{i+2,d}$ for some $d \geqslant 0$ by Lemma 7(a). Since vx, xy and wz are tree edges, by Lemma 7(b), $c - \Delta a = \alpha$ and $d - \Delta c = \beta$ and $d - \Delta b = \gamma$ for some $\alpha, \beta, \gamma \in \{0, 1, \dots, 2\Delta - 1\}$. Thus $d = \beta + \Delta c = \gamma + \Delta b$, implying $\Delta(c - b) = \gamma - \beta \leqslant 2\Delta - 1$ and $c - b \leqslant 1$. Similarly, $\Delta(b - c) = \beta - \gamma \leqslant 2\Delta - 1$ and $b - c \leqslant 1$. Now $b \leqslant c + 1 = \alpha + a\Delta + 1$, implying $b - a\Delta \leqslant \alpha + 1 \leqslant 2\Delta$. Similarly, $\Delta(c - b) = \gamma - \beta \leqslant 2\Delta - 1$ and $c - b \leqslant 1$. Thus $b \geqslant c - 1 = \Delta a + \alpha - 1 \geqslant \Delta a - 1$, implying $b - \Delta a \geqslant -1$. In summary, $b - a\Delta \in \{-1, 0, \dots, 2\Delta\}$. This proves (c).

We now prove (d). Suppose on the contrary that edges vw and pq in $G[V_{i,a}, V_{i,a+1}]$ are nested in $\overrightarrow{V_{i,a}}\overrightarrow{V_{i,a+1}}$. Without loss of generality, $v \prec p$ in $\overrightarrow{V_{i,a}}$ and $q \prec w$ in $\overrightarrow{V_{i,a+1}}$. Thus $v \prec p$ and $q \prec w$ in C_i. If $v \prec p \prec w$ in C_i, then $m(p) \geqslant m(w)$, which contradicts the fact that $g(w) > g(p)$. Thus $v \prec w \prec p$. If $v \prec q \prec w$ in C_i, then vw crosses pq in the drawing of G. Thus $q \prec v \prec w \prec p$ in C_i. Since $pq \in E(G)$, we have $m(v) \geqslant m(q)$, which contradicts the fact that $g(q) > g(v)$. Therefore the ordering $\overrightarrow{V_{i,a}}\overrightarrow{V_{i,a+1}}$ defines a 1-queue layout of $G[V_{i,a}, V_{i,a+1}]$.

This proves (d).

We now prove (e). That is, for $i, a, b \geqslant 0$, we show that the ordering $\overrightarrow{V_{i,a}}\overrightarrow{V_{i+1,b}}$ defines a $(6\Delta+1)$-queue layout of $G[V_{i,a}, V_{i+1,b}]$. Each edge in $G[V_{i,a}, V_{i+1,b}]$ is either direct or hooked. We first show that one queue suffices for direct edges in $G[V_{i,a}, V_{i+1,b}]$. Suppose on the contrary that there are two direct edges vw and xy in $G[V_{i,a}, V_{i+1,b}]$ with $v \prec x$ in $\overrightarrow{V_{i,a}}$ and $y \prec w$ in $\overrightarrow{V_{i+1,b}}$. Then vw and xy are drawn between C_i and C_{i+1} with $v \prec w$ in C_i and $x \prec y$ in C_{i+1}. Thus vw and xy cross. This contradiction shows that one queue suffices for direct edges in $G[V_{i,a}, V_{i+1,b}]$.

Now consider a hooked edge vw in $G[V_{i,a}, V_{i+1,b}]$ with $v \in V_{i,a}$ and $w \in V_{i+1,b}$. Let $vxyzw$ be the corresponding path in G'. Then vx, xy and wz are tree edges in G', and yz is a level edge in G'. Moreover, $x \in V'_{i+1,c}$ for some $c \geqslant 0$, and $y, z \in V'_{i+2,d}$ for some $d \geqslant 0$ by Lemma 7(a). Since vx, xy and wz are tree edges, by Lemma 7(b), $c - \Delta a = \alpha$ and $d - \Delta c = \beta$ and $d - \Delta b = \gamma$ for some $\alpha, \beta, \gamma \in \{0, 1, \ldots, 2\Delta - 1\}$. Thus $d = \beta + \Delta c = \gamma + \Delta b$, implying $\Delta(c - b) = \gamma - \beta \leqslant 2\Delta - 1$ and $c - b \leqslant 1$. Similarly, $\Delta(b - c) = \beta - \gamma \leqslant 2\Delta - 1$ and $b - c \leqslant 1$. Thus $c \in \{b-1, b, b+1\}$. Assign vw to queue Q^{η}_{γ} where $\eta = c - b$. This introduces 6Δ queues.

We claim that this is a valid queue assignment. Suppose on the contrary that there are hooked edges vw and pq in Q^{η}_{γ} with $v \prec p$ in $\overrightarrow{V_{i,a}}$ and $q \prec w$ in $\overrightarrow{V_{i+1,b}}$. Let $vxyzw$ be the path corresponding to vw in G'. Let $prstq$ be the path corresponding to pq in G'. Then x, y, z, r, s, t are distinct vertices, and $x, r \in V'_{i+1,b+\eta}$ and $y, z, s, t \in V'_{i+2,d}$ where $d = \Delta b + \gamma$. By Lemma 7(d) and since $v \prec p$ in $\overrightarrow{V_{i,a}}$, we have $x \prec r$ in $\overrightarrow{V_{i+1,c}}$. This in turn implies that $y \prec s$ in $\overrightarrow{V_{i+2,d}}$ by Lemma 7(d). Similarly, by Lemma 7(d) and since $q \prec w$ in $\overrightarrow{V_{i+1,b}}$, we have $t \prec z$ in $\overrightarrow{V_{i+2,d}}$. This implies that $y \prec t \prec s \prec z$ or $y \prec t \prec z \prec s$ or $y \prec s \prec t \prec z$ or $t \prec z \prec y \prec s$ or $t \prec y \prec s \prec z$ or $t \prec y \prec z \prec s$ in $\overrightarrow{V_{i+2,d}}$. Thus yz and st either nest or cross in $\overrightarrow{V_{i+2,d}}$, which contradicts Lemma 7(c). Hence no two edges in Q^{η}_{γ} nest. Therefore $(Q^{\eta}_{j} : \eta \in \{-1, 0, 1\}, j \in \{0, 1, \ldots, 2\Delta - 1\})$ is a 6Δ-queue layout of the hooked edges in $G[V_{i,a}, V_{i,a+1}]$ using the ordering $\overrightarrow{V_{i,a}}\overrightarrow{V_{i,a+1}}$. Including one queue for the direct edges, we obtain a $(6\Delta+1)$-queue layout of $G[V_{i,a}, V_{i,a+1}]$ using the ordering $\overrightarrow{V_{i,a}}\overrightarrow{V_{i,a+1}}$. This proves (e).

Finally we prove (f). Consider an edge vw with both end points v and w in $V_{i,a}$ for some $i, a \geqslant 0$. Then vw is a level edge. Let $vxyw$ be the corresponding path in G'. Then xy is a level non-tree edge of G' with $v, w \in V'_{i+1}$. By Lemma 7(a), both x and y are in $V'_{i+1,b}$ for some $b \geqslant 0$. Since vx and wy are tree edges in G' by Lemma 7(b), we have $b - \Delta a \in \{0, 1, \ldots, 2\Delta - 1\}$. Assign vw to queue $Q_{b-\Delta a}$. Suppose on the contrary that $v \prec p \prec q \prec w$ for two edges vw and pq in $Q_{b-\Delta a}$. Let $vxyw$ be the path in G' corresponding to vw. Let $pstq$ be the path in G' corresponding to pq. Then $x, y, s, t \in V'_{i+1,b}$. Note that vx, wy, ps and qt are tree edges in G', while xy and st are level edges in G'. Since $v \prec p$, we have $x \prec s$ in $\overrightarrow{V_{i+1,b}}$ by Lemma 7(d). Similarly, since $q \prec q$, we have $t \prec y$ in $\overrightarrow{V_{i+1,b}}$ by Lemma 7(d). Thus $x \prec s \prec t \prec y$ or $x \prec t \prec s \prec y$ or $x \prec t \prec y \prec s$ or $t \prec y \prec x \prec s$ or $t \prec x \prec s \prec y$ or $t \prec x \prec y \prec s$ in $\overrightarrow{V_{i+1,b}}$ In each case, xy and st either nest or cross, which contradicts Lemma 7(c). Thus no two edges in $Q_{b-\Delta a}$ are nested, and $(Q_j : j \in \{0, 1, \ldots, 2\Delta - 1\}$ is a 2Δ-queue layout of $G[V_{i,a}]$ using ordering $\overrightarrow{V_{i,a}}$. This proves (f). $\square$

We now show that Lemma 8 leads to a $O(\Delta^2)$-queue layout of an arbitrary planar graph.

Proof of Theorem 2. Let $\{V_{i,a} : i \in \{0,1,\ldots,t\}, a \in \{0,1,\ldots,n_i\}$ be the partition of of $V(G)$ from Lemma 8. Let $\overrightarrow{V_i}$ be the ordering $\overrightarrow{V_{i,0}}\overrightarrow{V_{i,1}}\ldots\overrightarrow{V_{i,n_i}}$ of V_i. Consider the ordering $\overrightarrow{V_0}\overrightarrow{V_1}\ldots\overrightarrow{V_t}$ of $V(G)$.

An edge with both endpoints in $V_{i,a}$ cannot nest an edge with both endpoints in $V_{j,b}$ for $(i,a) \neq (j,b)$, and 2Δ queues suffice for such edges by Lemma 8(f). An edge with endpoints in $V_{i,a}$ and $V_{i,a+1}$ cannot nest an edge with endpoints in $V_{j,b}$ and $V_{j,b+1}$ for $(i,a) \neq (j,b)$, and one queue suffice for such edges by Lemma 8(d). By Lemma 8(a) this accounts for all level edges. Thus $2\Delta + 1$ queues suffice for level edges.

For $i, a, b \geqslant 0$, by Lemma 8(e), there is a queue layout $(Q_j^{i,a,b} : j \in \{1, 2, \ldots, 6\Delta + 1\})$ of $G[V_{i,a}, V_{i+1,b}]$. For $j \in \{1, 2, \ldots, 6\Delta + 1\}$ and $\alpha \in \{-1, 0, \ldots, 2\Delta\}$, let

$$Q_j^\alpha := \bigcup \{Q_j^{i,a,b} : i, a, b \geqslant 0,\ b - \Delta a = \alpha\}.$$

By Lemma 8(b) and (c), this accounts for all binding edges.

Suppose that binding edges vw and pq in some Q_j^α are nested with $v \prec p \prec q \prec w$ in our ordering of $V(G)$. Then $v \in V_{i,a}$, $p \in V_{i,b}$, $q \in V_{i+1,c}$ and $w \in V_{i+1,d}$ for some $i, a, b, c, d \geqslant 0$ with $\alpha = d - \Delta a = c - \Delta b$. Thus $d - c = \Delta(a - b)$. Since $v \prec p \prec q \prec w$ in the ordering, $d - c \geqslant 0$ and $a - b \leqslant 0$. Thus $a = b$ and $c = d$, which contradicts Lemma 8(e).

Thus $(2\Delta + 2)(6\Delta + 1)$ queues suffice for binding edges. In total we use $(2\Delta + 2)(6\Delta + 1) + 2\Delta + 1 = 12\Delta^2 + 16\Delta + 3$ queues □

We emphasise that the vertex ordering used in the proof of Theorem 2 is identical to that used by Bekos et al. [1]. Our contribution is to show that $O(\Delta^2)$ queues suffice rather than the $O(\Delta^6)$ queues used by Bekos et al. [1]. On the other hand, we now show that up to a constant factor our analysis is tight. That is, the above ordering can produce $\Omega(\Delta^2)$ pairwise nested edges (a so-called 'rainbow'), which each must be assigned to a distinct queue. Start with a rooted binary tree with $2\Delta^2$ leaves. Label the leaves left-right

$$v_{1,1}, \ldots, v_{1,\Delta}; \ldots; v_{\Delta,1}, \ldots, v_{\Delta,\Delta}; w_{\Delta,\Delta}, \ldots, w_{\Delta,1}; \ldots; w_{1,\Delta}, \ldots, w_{1,1}.$$

Subdivide the edge incident to each leaf $v_{i,j}$. Let G be the graph obtained by adding the edge $v_{i,j}w_{i,j}$ for $i, j \in \{1, 2, \ldots, \Delta\}$, as illustrated in Figure 3. Let G' be the well-layered graph obtained by subdividing the edges of G as described above. Thus each edge $v_{i,j}w_{i,j}$ is replaced by a path $v_{i,j}x_{i,j}y_{i,j}z_{i,j}w_{i,j}$. Vertices $y_{i,j}$ and $z_{i,j}$, which are on level 0, are joined by a level edge. Edges $v_{i,j}x_{i,j}$, $x_{i,j}y_{i,j}$ and $z_{i,j}w_{i,j}$ are tree edges. The above algorithm does not introduce any parallel edges, since each level edge joins vertices on level 0. Vertices $v_{i,j}$ are on level 1, and vertices $w_{i,j}$ are on level 2. It follows that $g(w_{i,j}) = 0$ and $g(v_{i,j}) = i - 1$ for all i, j. Thus the vertex ordering of G produced by the above algorithm (after removing subdivision vertices) includes the sequence

$$w_{\Delta,\Delta}, \ldots, w_{\Delta,1}; \ldots; w_{1,\Delta}, \ldots, w_{1,1}, v_{1,1}, \ldots, v_{1,\Delta}; \ldots; v_{\Delta,1}, \ldots, v_{\Delta,\Delta};.$$

Here, $v_{i,j}w_{i,j}$ is nested with $v_{i',j'}w_{i',j'}$ for $(i,j) \neq (i',j')$. Thus Δ^2 queues are needed, as claimed. Curiously this example has maximum degree 3.

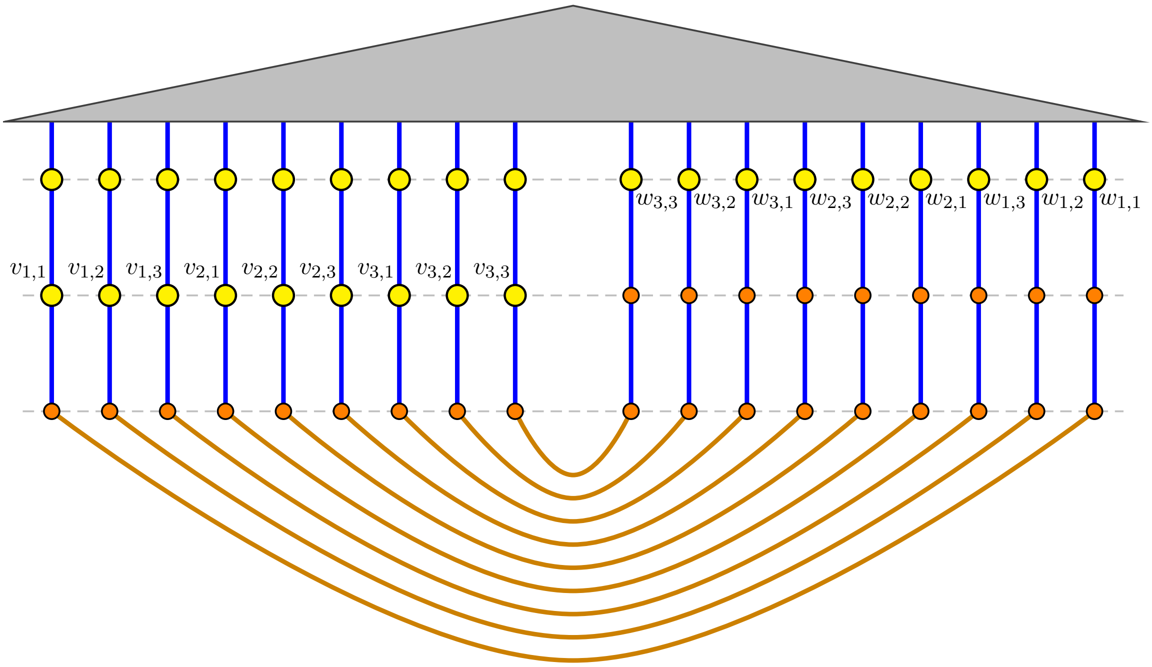

Figure 3: Example where the algorithm uses Δ^2 queues.

3 Graphs of Bounded Genus

This section proves our results for graphs of bounded Euler genus (Theorem 4 which implies Theorem 3). The next lemma is the key.

Lemma 9. *Let G be a connected graph G with Euler genus g. For every bfs layering $V_0, V_1, \dots, V_t$ of G, there is a set $Z \subseteq V(G)$ with at most $2g$ vertices in each layer V_i, such that $G - Z$ is planar.*

Proof. Fix an embedding of G in a surface of Euler genus g. Say G has n vertices, m edges, and f faces. By Euler's formula, $n - m + f = 2 - g$. Let $V_0, V_1, \dots, V_t$ be a bfs layering of G rooted at some vertex r. Let T be the corresponding bfs spanning tree. Let D be the graph with $V(D) = F(G)$, where for each edge e of $G - E(T)$, if f_1 and f_2 are the faces of G with e on their boundary, then there is an edge $f_1 f_2$ in D. (Think of D as the spanning subgraph of G^* consisting of those edges that do not cross edges in T.) Note that $|V(D)| = f = 2 - g - n + m$ and $|E(D)| = m - (n - 1) = |V(D)| - 1 + g$. Since T is a tree, D is connected; see [6, Lemma 11] for a proof. Let T^* be a spanning tree of D. Let $Q := E(D) \setminus E(T^*)$. Thus $|Q| = g$. Say $Q = \{v_1 w_1, v_2 w_2, \dots, v_g w_g\}$. For $i \in \{1, 2, \dots, g\}$, let Z_i be the union of the $v_i r$-path and the $w_i r$-path in T, plus the edge $v_i w_i$. Let Z be $Z_1 \cup Z_2 \cup \dots \cup Z_g$. Say Z has p vertices and q edges. Since Z consists of a subtree of T plus the g edges in Q, we have $q = p - 1 + g$.

We now describe how to 'cut' along the edges of Z to obtain a new graph G'; see Figure 4. First, each edge e of Z is replaced by two edges e' and e'' in G'. Each vertex of G incident with no edges in Z is untouched. Consider a vertex v of G incident with edges $e_1, e_2, \dots, e_d$ in Z in clockwise order. In G' replace v by new vertices $v_1, v_2, \dots, v_d$, where v_i is incident with e'_i, e''_{i+1} and all the edges incident with v clockwise from e_i to e_{i+1} (exclusive). Here

e_{d+1} means e_1 and e''_{d+1} means e''_1. This operation defines a cyclic ordering of the edges in G' incident with each vertex (where e''_{i+1} is followed by e'_i in the cyclic order at v_i). This in turn defines an embedding of G' in some orientable surface. (Note that if G is embedded in a non-orientable surface, then the edge signatures for G are ignored in the embedding of G'.)

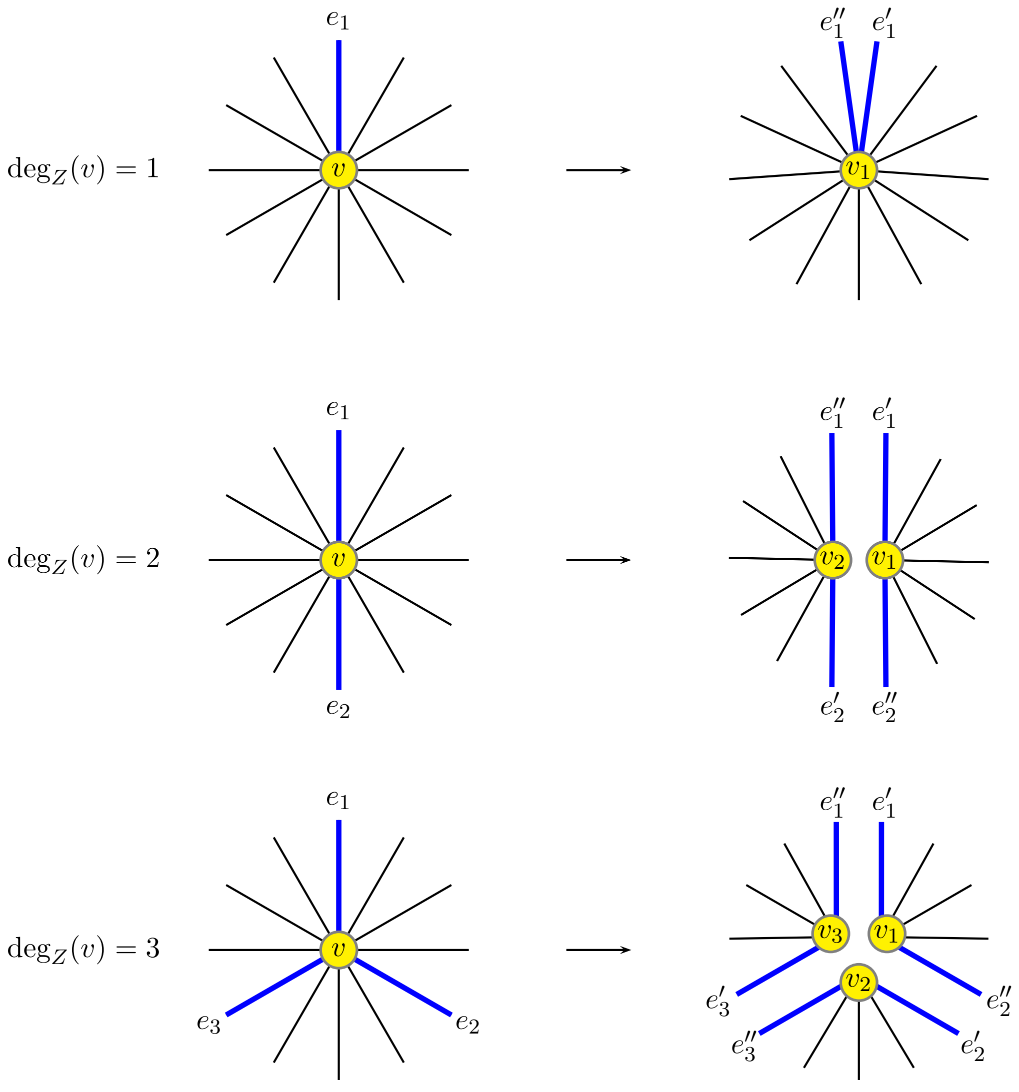

Figure 4: Cutting the blue edges in Z at each vertex.

Say G' has n' vertices and m' edges, and the embedding of G' has f' faces and Euler genus g'. Each vertex v in G with degree d in Z is replaced by d vertices in G'. Each edge in Z is replaced by two edges in G', while each edge of $G - E(Z)$ is maintained in G'. Thus

$$n' = n - p + \sum_{v \in V(G)} \deg_Z(v) = n + 2q - p = n + 2(p - 1 + g) - p = n + p - 2 + 2g$$

and $m' = m+q = m+p-1+g$. Each face of G is preserved in G'. Say r new faces are created by the cutting. Thus $f' = f + r$. Since D is connected, it follows that G' is connected. By Euler's formula, $n' - m' + f' = 2 - g'$. Thus $(n+p-2+2g) - (m+p-1+g) + (f+r) = 2 - g'$, implying $(n - m + f) - 1 + g + r = 2 - g'$. Hence $(2 - g) - 1 + g + r = 2 - g'$, implying $g' = 1 - r$. Since $r \geqslant 1$ and $g' \geqslant 0$, we have $g' = 0$ and $r = 1$. Therefore G' is planar.

Note that $G - V(Z)$ is a subgraph of G', and $G - V(Z)$ is planar. By construction, each path Z_i has at most two vertices in each layer V_j. Thus Z has at most $2g$ vertices in each V_j. $\square$

We need the following lemma of independent interest.

Lemma 10. *If a graph G has a k-queue layout, and $V_0, V_1, \ldots, V_t$ is a layering of G, then G has a $3k$-queue layout using ordering $V_0, V_1, \ldots, V_t$.*

Proof. Say $E_1, E_2, \ldots, E_k$ is the edge-partition and $\preceq$ is the ordering of $V(G)$ in a k-queue layout of G. For $a \in \{1, 2, \ldots, k\}$, let X_a be the set of edges $vw \in Q_a$ with $v, w \in V_i$ for some i; let Y_a be the set of edges $vw \in Q_a$ with $v \prec w$ and $v \in V_i$ and $w \in V_{i+1}$ for some i; and let Z_a be the set of edges $vw \in Q_a$ with $w \prec v$ and $v \in V_i$ and $w \in V_{i+1}$ for some i. Then $X_1, Y_1, Z_1, X_2, Y_2, Z_2, \ldots, X_k, Y_k, Z_k$ is a partition of $E(G)$.

Let $\preceq'$ be the ordering $V_0, V_1, \ldots, V_t$ of $V(G)$ where each V_i is ordered by $\preceq$. No two edges in some set X_a are nested in $\preceq'$, as otherwise the same two edges would be in Q_a and would be nested in $\preceq$. Suppose that $v \preceq' p \preceq' q \preceq' w$ for some edges $vw, pq \in Y_a$. So $v, p \in V_i$ and $w, q \in V_{i+1}$ for some i, and $v \prec p$ and $q \prec w$. Now $p \prec q$ by the definition of Y_a. Hence $v \prec p \prec q \prec w$, which is a contradiction since both vw and pq are in Q_a. Thus no two edges in Y_a are nested in $\preceq'$. By symmetry, no two edges in Z_a are nested in $\preceq'$. Hence $\preceq'$ is the ordering in a $3k$-queue layout of G. $\square$

We now prove Theorem 4, which says that if $\mathcal{G}$ is a hereditary class of graphs, such that every planar graph in $\mathcal{G}$ has queue-number at most k, then every graph in $\mathcal{G}$ with Euler genus g has queue-number at most $3k + 4g$.

Proof of Theorem 4. Let G be a graph in $\mathcal{G}$ with Euler genus g. Since the queue-number of G equals the maximum queue-number of the connected components of G, we may assume that G is connected. Let $V_0, V_1, \ldots, V_t$ be a bfs layering of G. By Lemma 9, there is a set $Z \subseteq V(G)$ with at most $2g$ vertices in each layer V_i, such that $G - Z$ is planar. Since $\mathcal{G}$ is hereditary, $G - Z \in \mathcal{G}$, and by assumption $G - Z$ has a k-queue layout. Note that $V_0 \setminus Z, V_1 \setminus Z, \ldots, V_t \setminus Z$ is a layering of $G - Z$. By Lemma 10, $G - Z$ has a $3k$-queue layout using ordering $V_0 \setminus Z, V_1 \setminus Z, \ldots, V_t \setminus Z$. Recall that $|V_j \cap Z| \leqslant 2g$ for all $j \in \{0, 1, \ldots, t\}$. Let $\preceq$ be the ordering

$$V_0 \cap Z, V_0 \setminus Z,\ V_1 \cap Z, V_1 \setminus Z,\ \ldots, V_t \cap Z, V_t \setminus Z$$

of $V(G)$. where each set $V_j \cap Z$ is ordered arbitrarily, and each set $V_j \setminus Z$ is ordered according to the above $3k$-queue layout of $G - Z$. Edges of $G - Z$ inherit their queue assignment. We now assign edges incident with vertices in Z to queues. For $i \in \{1, \ldots, 2g\}$ and odd $j \geqslant 1$, put each edge incident with the i-th vertex in $V_j \cap Z$ in a new queue S_i. For $i \in \{1, \ldots, 2g\}$ and even $j \geqslant 0$, put each edge incident with the i-th vertex in $V_j \cap Z$ (not already assigned to a queue) in a new queue T_i. Suppose that two edges vw and pq in S_i are nested, where $v \prec p \prec q \prec w$. Say $v \in V_a$ and $p \in V_b$ and $q \in V_c$ and $w \in V_d$. By construction, $a \leqslant b \leqslant c \leqslant d$. Since vw is an edge, $d \leqslant a + 1$. At least one endpoint of vw is in $V_j \cap Z$ for some odd j, and one endpoint of pq is in $V_\ell \cap Z$ for some odd ℓ. Since v, w, p, q are distinct, $j \neq \ell$. Thus $|i - j| \geqslant 2$. This is a contradiction since $a \leqslant b \leqslant c \leqslant d \leqslant a + 1$. Thus S_i is a queue. Similarly T_i is a queue. Hence this step introduces $4g$ new queues. We obtain a $(3k + 4g)$-queue layout of G. $\square$

4 Excluded Minors

Whether the result of Bekos et al. [1] can be generalised for arbitrary excluded minors is an interesting question. That is, do graphs excluding a fixed minor and with bounded degree have bounded queue-number? It might even be true that graphs excluding a fixed minor have bounded queue-number.

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A Unsubdividing

Dujmović and Wood [8] proved that if some $(\leqslant c)$-subdivision of a graph G has a k-queue layout, then G has a $O(k^{2c})$-queue layout. Here we improve this bound to $O(k^{c+1})$.

Lemma 11. *For every $(\leqslant c)$-subdivision G' of a graph G, if G' has a k-queue layout using vertex ordering $\preceq$, then G has a $\frac{2k}{2k-1}((2k)^{c+1}-1)$-queue layout using $\preceq$ restricted to $V(G)$.*

Proof. Let $E_1,\dots,E_k$ be the partition of $E(G')$ into queues. For each edge $xy \in E_i$, let $q(xy) := i$. For distinct vertices $a,b \in V(G')$, let $f(a,b) := 1$ if $a \prec b$ and let $f(a,b) := -1$ if $b \prec a$. For $\ell \in \{0,1,\dots,c\}$, let X_ℓ be the set of edges in G that are subdivided exactly ℓ times in G'. We will use distinct sets of queues for the X_ℓ. Consider an edge vw in X_ℓ with $v \prec w$. Say $v = x_0, x_1, \dots, x_\ell, x_{\ell+1} = w$ is the corresponding path in G'. Let $f(vw) := (f(x_0,x_1),\dots,f(x_\ell,x_{\ell+1}))$ and $q(vw) := (q(x_0,x_1),\dots,q(x_\ell,x_{\ell+1}))$. Consider edges $vw, pq \in X_\ell$ with v,w,p,q distinct and $f(vw) = f(pq)$ and $g(vw) = g(pq)$. Assume $v \prec p$. Say $v = x_0, x_1, \dots, x_\ell, x_{\ell+1} = w$ and $p = y_0, y_1, \dots, x_\ell, x_{\ell+1} = q$ are the paths respectively corresponding to vw and pq in G'. Thus $f(x_i,x_{i+1}) = f(y_i,y_{i+1})$ and $q(x_ix_{i+1}) = q(y_iy_{i+1})$ for $i \in \{0,1,\dots,\ell\}$. Thus x_ix_{i+1} and y_iy_{i+1} are not nested. Since $v = x_0 \prec y_0 = p$, it follows by induction that $x_i \prec y_i$ for $i \in \{0,1,\dots,\ell+1\}$. In particular, $w = x_{\ell+1} \prec y_{\ell+1} = q$. Thus vw and pq are not nested. There are $2^{\ell+1}$ values for f, and $k^{\ell+1}$ values for q. Thus $(2k)^{\ell+1}$ queues suffice for X_ℓ. In total, $\sum_{\ell=0}^{c}(2k)^{\ell+1} = \frac{2k}{2k-1}((2k)^{c+1}-1)$ queues suffice for G. □