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Three-Dimensional Graph Drawing

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Years and Authors of Summarized Original Work

2005; Dujmović, Morin, Wood

Problem Definition

A *three-dimensional straight-line grid drawing* of a graph, henceforth called a *3D drawing*, represents the vertices by distinct grid-points in $\mathbb{Z}^3$ and represents each edge by the line segment between its end vertices, such that no two edges cross. In contrast to the case in the plane, it is folklore that every graph has a 3D drawing. For example, the "moment curve" algorithm places the ith vertex at (i, i^2, i^3). It is easily seen that no four vertices are coplanar, and thus no two edges cross. Since every graph has a 3D drawing, we are interested in optimizing certain measures of their aesthetic quality. If a 3D drawing is contained in an axis-aligned box with side lengths $X-1$, $Y-1$, and $Z-1$, then we speak of an $X \times Y \times Z$ drawing with *volume* $X \cdot Y \cdot Z$. This entry considers the problem of producing a 3D drawing of a given graph with small volume.

Key Results

Observe that the drawings produced by the moment curve algorithm have $\mathcal{O}(n^6)$ volume, where n is the number of vertices. Cohen et al. [2] improved this bound, by proving that if p is a prime with $n < p \leq 2n$, and the ith vertex is at $(i, i^2 \bmod p, i^3 \bmod p)$, then there is still no crossing. The resulting $\mathcal{O}(n^3)$ volume bound is optimal for the complete graph K_n since each grid plane may contain at most four vertices. It is therefore of interest to identify fixed graph parameters that allow for 3D drawings with small volume, as summarized in the following table.

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Graph family	Min. volume	Reference
Arbitrary	$\Theta(n^3)$	[2]
Bounded chromatic number	$\Theta(n^2)$	[19]
Bounded maximum degree	$\mathcal{O}(n^{3/2})$	[7]
Bounded degeneracy	$\mathcal{O}(n^{3/2})$	[9]
H-minor-free (H fixed)	$n \log^{\mathcal{O}(1)} n$	[12]
Bounded genus	$\mathcal{O}(n \log n)$	[12]
Apex-minor-free	$\mathcal{O}(n \log n)$	[12]
Planar	$\mathcal{O}(n \log n)$	[6]
Bounded treewidth	$\Theta(n)$	[11]

The first such parameter to be studied was the chromatic number. Pach et al. [19] proved that graphs of bounded chromatic number have 3D drawings with $\mathcal{O}(n^2)$ volume. If p is a suitably chosen prime, the main step of their algorithm represents the vertices in the ith color class by grid-points in the set $\{(i, t, it) : t \equiv i^2 \pmod{p}\}$. It follows that the volume bound is $\mathcal{O}(k^2 n^2)$ for k-colorable graphs.

Pach et al. [19] also proved an $\Omega(n^2)$ lower bound for the volume of 3D drawings of the complete bipartite graph $K_{n,n}$. This lower bound was generalized for all graphs by Bose et al. [1], who proved that every 3D drawing of an n-vertex m-edge graph has volume at least $\frac{1}{8}(n+m)$. In particular, the maximum number of edges in an $X \times Y \times Z$ drawing is exactly $(2X-1)(2Y-1)(2Z-1) - XYZ$.

Graphs with bounded maximum degree have bounded chromatic number and, thus, by the result of Pach et al. [19], have 3D drawings with $\mathcal{O}(n^2)$ volume. Pach et al. [19] conjectured that such graphs have 3D drawings with $o(n^2)$ volume, which was verified by Dujmović and Wood [7], who proved a $\mathcal{O}(n^{3/2})$ bound. The best lower bound is $\Omega(n)$. Determining the optimal volume for 3D drawings of bounded degree graphs is a challenging open problem; see [13]. The $\mathcal{O}(n^{3/2})$ upper bound for bounded degree graphs was generalized for graphs with bounded degeneracy [9].

The first nontrivial $\mathcal{O}(n)$ volume bound was established by Felsner et al. [15] for outerplanar graphs. Their elegant algorithm "wraps" a 2D drawing around a triangular prism to obtain a 3D drawing. This result naturally led to the following open problem due to Felsner et al. [15], which motivated much subsequent research: does every planar graph have a 3D drawing with $\mathcal{O}(n)$ volume?

For some time, the $\mathcal{O}(n^2)$ bound for 2D drawings was the best known bound in 3D. Then Dujmović and Wood [7] proved that every planar graph has a 3D drawing with $\mathcal{O}(n^{3/2})$ volume. A breakthrough came with the $\mathcal{O}(n \log^8 n)$ bound of Di Battista et al. [4], which was improved to $\mathcal{O}(n \log n)$ by Dujmović [6] (with a much simpler proof). The most recent work in this direction, by Dujmović et al. [12], extended this $\mathcal{O}(n \log n)$ bound to all graphs of bounded Euler genus and more generally proved that every graph excluding a fixed minor has a 3D drawing with $n \log^{\mathcal{O}(1)} n$ volume.

The $\mathcal{O}(n)$ volume bound for outerplanar graphs mentioned above was generalized by Dujmović et al. [11] as follows:

Theorem 1 ([11]). *Graphs with bounded treewidth have 3D drawings with $\mathcal{O}(n)$ volume.*

This result is the focus of the remainder of this entry. Treewidth is a measure of the similarity of a graph to a tree. It can be defined as follows. A graph is *chordal* if every induced cycle is a

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triangle. The *treewidth* of a graph G is the minimum integer k such that G is a spanning subgraph of a chordal graph with no $(k+2)$-clique. Many graphs arising in applications of graph drawing have small treewidth. Trees have treewidth 1, while outerplanar and series-parallel graphs have treewidth 2. Another example arises in software engineering applications. Thorup [20] proved that the control-flow graphs of go-to free programs in many programming languages have treewidth bounded by a small constant, in particular, 3 for Pascal and 6 for C.

Reference [11] is also important because it discovered the connection between 3D drawings, track layouts, and queue layouts; also see [10, 16].

Track Layouts: Track layouts are a combinatorial tool that effectively eliminates the geometry from 3D drawings and exposes the underlying combinatorial structure. They were introduced in [11] although they are implicit in some previous work [15, 16].

Let $V_1, \dots, V_t$ be the color classes in a (proper) vertex t-coloring of a graph G. Suppose that each color class V_i is equipped with a total order, denoted by $\preceq$. Call V_i a *track* and $V_1, \dots, V_t$ a *t-track assignment*. An *X-crossing* in $V_1, \dots, V_t$ consists of two edges vw and xy such that $v \prec x$ in some track V_i and $y \prec w$ in some other track V_j. A t-track assignment with no X-crossing is called a *t-track layout*.

One can produce a track layout from an $A \times B \times C$ drawing of a graph G as follows. Let $V_{x,y}$ be the set of vertices of G with an X-coordinate of x and a Y-coordinate of y. Order each set $V_{x,y}$ by the corresponding Z-coordinates. We obtain an AB-track layout of G, except that consecutive vertices in each track might be adjacent. Doubling each track and putting alternate vertices in $V_{x,y}$ on distinct tracks gives a $2AB$-track layout of G. Most interestingly, a converse result is also true.

Theorem 2 ([11]). *If an n-vertex graph has a t-track layout, then G has a $\mathcal{O}(t) \times \mathcal{O}(t) \times \mathcal{O}(n)$ drawing with $\mathcal{O}(t^2 n)$ volume.*

The proof of Theorem 2 is inspired by the generalizations of the moment curve algorithm by Cohen et al. [2] and Pach et al. [19]. Loosely speaking, Cohen et al. [2] allow three "free" dimensions, whereas Pach et al. [19] use a coloring to "fix" one dimension with two dimensions free. Theorem 2 uses a track layout to fix two dimensions with one dimension free; see Fig. 1. In particular, say $(V_1, \dots, V_t)$ is the given t-track layout. Let p be the smallest prime such that $p > k$. Then $p \leq 2k$ by Bertrand's postulate. For $1 \leq i \leq k$, represent the vertices in V_i by the grid-points $\{(i, i^2 \bmod p, t) : 1 \leq t \leq p \cdot |V_i|,\ t \equiv i^3 \pmod{p}\}$, such that the Z-coordinates respect the given total order of V_i.

Note that Dujmović and Wood [7] combined the method of Pach et al. [19] with the proof of Theorem 2 to conclude a $\mathcal{O}(tn)$ volume bound of 3D drawings of t-track graphs with bounded chromatic number.

As an example of how to construct a track layout, we now show that every tree T has a 3-track layout (which is implicitly proved in [15]). Let r be a vertex of T. Let V_i be the vertices at distance i from r. Note that $(V_0, V_1, \dots)$ is a coloring of T. Clearly, each color class V_i can be ordered so that there is no X-crossing; see Fig. 2a. Hence $(V_0, V_1, \dots)$ is a track layout. Note that, working from the root down, the child nodes of each node can be ordered arbitrarily. This will be important later. Now, imagine wrapping this track layout around a prism; see Fig. 2b. That is, for $0 \leq i \leq 2$, group tracks $V_i \prec V_{3+i} \prec V_{6+i} \prec \dots$ to obtain a 3-track layout of T.

An Algorithm for Graphs of Bounded Treewidth: Theorem 1 is an immediate consequence of Theorem 2 and the following claim, which we prove by induction on $k \geq 0$: for each integer $k \geq 0$,

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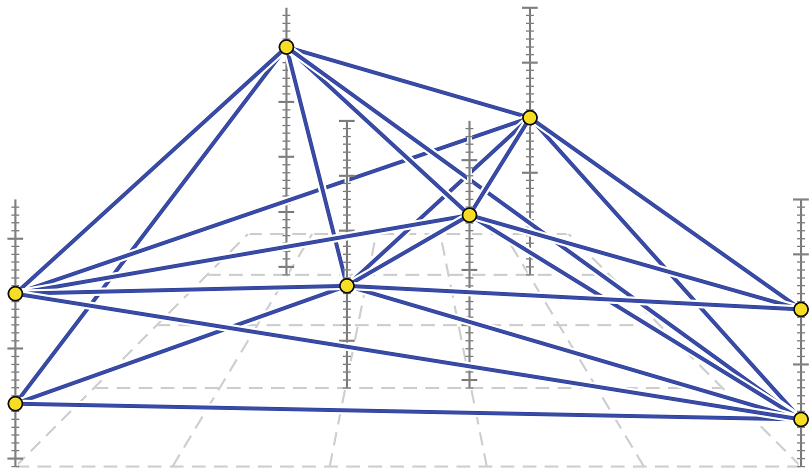

Fig. 1 A 3D drawing produced from a track layout

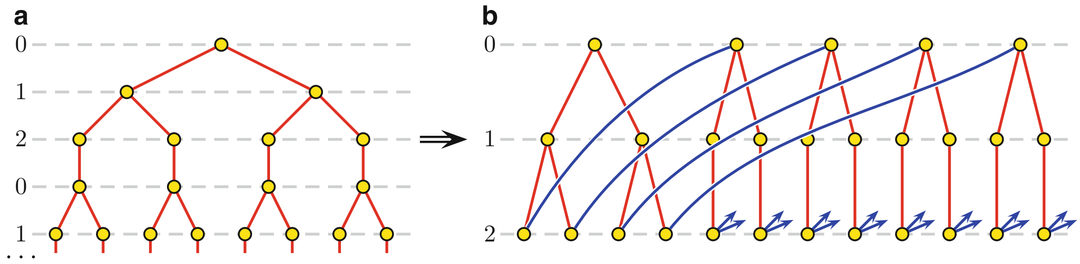

Fig. 2 A 3-track layout of a tree

there is an integer t_k such that every k-tree has a t_k-track layout. A 0-tree has no edges and thus has a 1-track layout. A 1-tree is a tree which has a 3-track layout. Thus the result holds with $t_0 = 1$ and $t_1 = 3$. Let G be a k-tree. Various authors have proved that G can be decomposed as follows [11, 18]. There is a tree T rooted at some node r and a partition $\{B_x : x \in V(T)\}$ of $V(G)$ indexed by the nodes of T with the following properties:

- For each edge vw of G, there is a node x of T such that $v, w \in B_x$, or there is an edge xy of T such that $v \in B_x$ and $w \in B_y$.
- For each node x of T, the induced subgraph $G[B_x]$ is a $(k-1)$-tree.
- For each non-root node y of T, if x is the parent node of y, and C_y is the set of vertices in B_x adjacent to some vertex in B_y, then C_y is a clique in G called the *parent clique* of y.

By induction, for each node x of T, there is a t_{k-1}-track layout of $G[B_x]$. Each clique C in $G[B_x]$ has size at most k. Define the *signature* of C to be the set of (at most k) tracks that contain C. Since there is no X-crossing, the set of cliques of $G[B_x]$ with the same signature can be linearly ordered $C_1 \prec \cdots \prec C_p$, such that if v and w are vertices in the same track, and in distinct cliques C_i and C_j with $i < j$, then $v \prec w$ in that track. Call this a *clique ordering*.

Let T_0, T_1, T_2 be a 3-track layout of T described above. Replace each track T_i by t_{k-1} subtracks, and replace each node $x \in T_i$ by the t_{k-1}-track layout of $G[B_x]$. This defines a $3 \cdot t_{k-1}$ track assignment for G. Clearly an edge in some $G[B_x]$ is in no X-crossing with any other edge.

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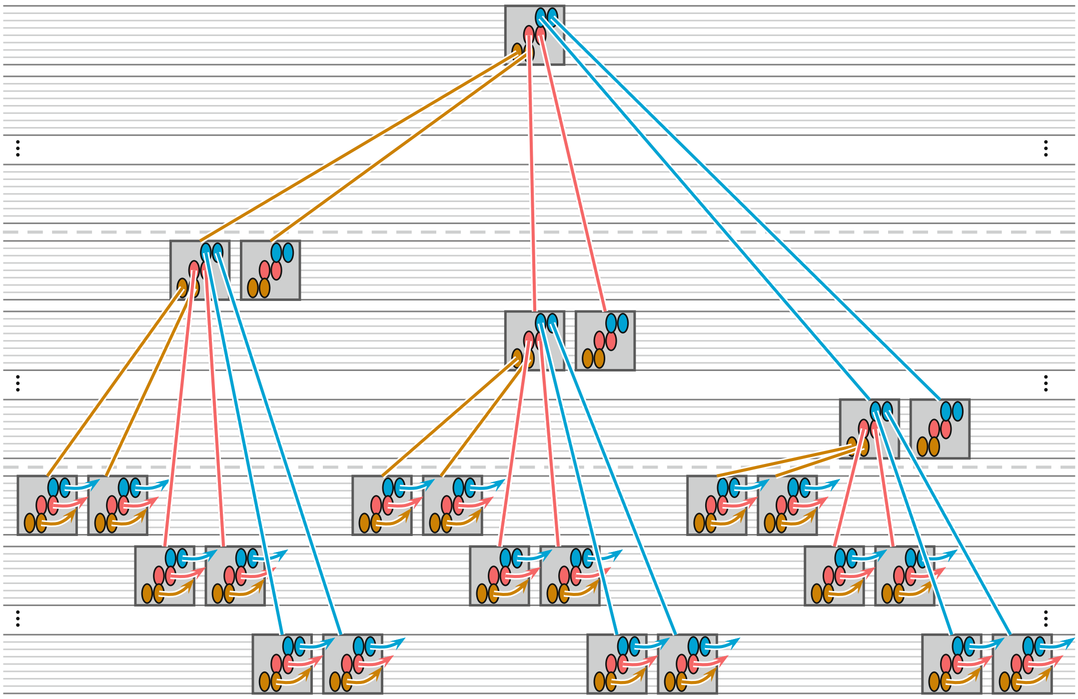

Fig. 3 Final track layout with $3(t_{k-1})^k$ groups of t_{k-1} tracks

There is no X-crossing between two edges between a parent bag B_x and some same child bag B_y, since the end points in B_x of such edges form a clique (the parent clique of y) and therefore are in distinct tracks. The only possible X-crossing is between edges ab and cd, where a and c are in some parent bag B_x and b and d are in distinct child bags B_y and B_z, respectively.

To solve this problem, when determining the 3-track layout of T, the child nodes of each node x are ordered in their track so that $y \prec z$ whenever the parent cliques C_y and C_z have the same signature and $C_y \prec C_z$ in the clique ordering. Then group the child nodes of x according to the signatures of their parent cliques, and for each signature σ, use a distinct set of t_{k-1} tracks for the child bags whose parent cliques have signature σ. Now the ordering of the child bags with the same signature agrees with the clique ordering of their parent cliques and therefore agrees with the ordering of any neighbors in the parent bag. It follows that there is no X-crossing, as illustrated in Fig. 3. The number of tracks is at most $3t_{k-1}$ times the number of signatures, which is at most $\sum_{i=1}^{k} \binom{t_{k-1}}{i} \leq (t_{k-1})^k$. This completes the proof with $t_k := 3(t_{k-1})^{k+1}$.

This proof makes no effort to reduce the bound on t_k. The recurrence roughly solves to $3^{(k+2)!}$. The original proof by Dujmović et al. [11] reduces this bound to a doubly exponential function in k. Further improvements were made by Di Giacomo et al. [5], but the bound is still doubly exponential. The best lower bound, due to Dujmović et al. [11], is $\Omega(k^2)$. For $k = 2$, the best upper bound is 15, due to Di Giacomo et al. [5].

Other Models for 3D Graph Drawing:

- Polyline grid drawings, where bends in the edges are allowed (at grid-points) [3, 8]
- Orthogonal 3D drawings, where the edges are routed along the grid-lines [14, 21]
- Upward 3D drawings of directed acyclic graphs [5, 9]
- Symmetrical 3D drawings with vertices in $\mathbb{R}^3$ [17]

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Cross-References

▸ Planar Graph Grid Drawing

Recommended Reading

1. Bose P, Czyzowicz J, Morin P, Wood DR (2004) The maximum number of edges in a three-dimensional grid-drawing. J Graph Algorithms Appl 8(1):21–26
2. Cohen RF, Eades P, Lin T, Ruskey F (1996) Three-dimensional graph drawing. Algorithmica 17(2):199–208
3. Devillers O, Everett H, Lazard S, Pentcheva M, Wismath S (2006) Drawing K_n in three dimensions with one bend per edge. J Graph Algorithms Appl 10(2):287–295
4. Di Battista G, Frati F, Pách J (2013) On the queue number of planar graphs. SIAM J Comput 42(6):2243–2285
5. Di Giacomo E, Liotta G, Meijer H, Wismath SK (2009) Volume requirements of 3D upward drawings. Discret Math 309(7):1824–1837
6. Dujmović V (2015) Graph layouts via layered separators. J Comb Theory Ser B 110:79–89
7. Dujmović V, Wood DR (2004) Three-dimensional grid drawings with sub-quadratic volume. In: Pach J (ed) Towards a theory of geometric graphs. Contemporary mathematics, vol 342. American Mathematical Society, Providence, pp 55–66
8. Dujmović V, Wood DR (2005) Stacks, queues and tracks: layouts of graph subdivisions. Discret Math Theor Comput Sci 7:155–202
9. Dujmović V, Wood DR (2006) Upward three-dimensional grid drawings of graphs. Order 23(1):1–20
10. Dujmović V, Pór A, Wood DR (2004) Track layouts of graphs. Discret Math Theor Comput Sci 6(2):497–522
11. Dujmović V, Morin P, Wood DR (2005) Layout of graphs with bounded tree-width. SIAM J Comput 34(3):553–579
12. Dujmović V, Morin P, Wood DR (2013) Layered separators for queue layouts, 3D graph drawing and nonrepetitive coloring. In: Proceedings of 54th Annual Symposium on Foundations of Computer Science (FOCS ’13), Berkeley. IEEE, pp 280–289. http://arxiv.org/abs/1306.1595
13. Dujmović V, Morin P, Sheffer A (2014) Crossings in grid drawings. Electron J Combin 21(1):#P1.41
14. Eades P, Symvonis A, Whitesides S (2000) Three dimensional orthogonal graph drawing algorithms. Discret Appl Math 103:55–87
15. Felsner S, Liotta G, Wismath SK (2003) Straight-line drawings on restricted integer grids in two and three dimensions. J Graph Algorithms Appl 7(4):363–398
16. Heath LS, Leighton FT, Rosenberg AL (1992) Comparing queues and stacks as mechanisms for laying out graphs. SIAM J Discret Math 5(3):398–412
17. Hong SH, Eades P (2003) Drawing trees symmetrically in three dimensions. Algorithmica 36(2):153–178
18. Kündgen A, Pelsmajer MJ (2008) Nonrepetitive colorings of graphs of bounded tree-width. Discret Math 308(19):4473–4478
19. Pach J, Thiele T, Tóth G (1999) Three-dimensional grid drawings of graphs. In: Chazelle B, Goodman JE, Pollack R (eds) Advances in discrete and computational geometry. Contemporary mathematics, vol 223. American Mathematical Society, Providence, pp 251–255

DOI 10.1007/978-3-642-27848-8_656-1

20. Thorup M (1998) All structured programs have small tree-width and good register allocation. Inf Comput 142(2):159–181
21. Wood DR (2003) Optimal three-dimensional orthogonal graph drawing in the general position model. Theor Comput Sci 299(1–3):151–178