SUBGRAPH DENSITIES IN A SURFACE

TONY HUYNH, GWENAËL JORET, AND DAVID R. WOOD

Abstract. Given a fixed graph H that embeds in a surface Σ, what is the maximum number of copies of H in an n-vertex graph G that embeds in Σ? We show that the answer is $\Theta(n^{f(H)})$, where $f(H)$ is a graph invariant called the 'flap-number' of H, which is independent of Σ. This simultaneously answers two open problems posed by Eppstein (1993). The same proof also answers the question for minor-closed classes. That is, if H is a $K_{3,t}$ minor-free graph, then the maximum number of copies of H in an n-vertex $K_{3,t}$ minor-free graph G is $\Theta(n^{f'(H)})$, where $f'(H)$ is a graph invariant closely related to the flap-number of H. Finally, when H is a complete graph we give more precise answers.

1. Introduction

All graphs in this paper are undirected, finite, and simple, unless stated otherwise. Many classical theorems in extremal graph theory concern the maximum number of copies of a fixed graph H in an n-vertex graph in some class $\mathcal{G}$. Here, a *copy* means a subgraph isomorphic to H. For example, Turán's Theorem determines the maximum number of copies of K_2 (that is, edges) in an n-vertex K_t-free graph [63]. More generally, Zykov's Theorem determines the maximum number of copies of a given complete graph K_s in an n-vertex K_t-free graph [67]. The excluded graph need not be complete. The Erdős–Stone Theorem [19] determines, for every non-bipartite graph X, the asymptotic maximum number of copies of K_2 in an n-vertex graph with no X-subgraph. Analogues of the Erdős–Stone Theorem for copies of K_s have recently been studied by Alon and Shikhelman [4, 5]. See [3, 20, 23–26, 32, 45, 46, 51, 62] for recent related results.

This paper studies similar questions when the class $\mathcal{G}$ consists of the graphs that embed[1] in a given surface Σ (rather than being defined by an excluded subgraph). For graphs H and G, let $C(H,G)$ be the number of copies of H in G. For a surface Σ, let $C(H,\Sigma,n)$ be the maximum of $C(H,G)$, where the maximum is taken over all n-vertex graphs G

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[1]See [49] for background about graphs embedded in surfaces. For $h \geqslant 0$, let $\mathbb{S}_h$ be the sphere with h handles. For $c \geqslant 0$, let $\mathbb{N}_c$ be the sphere with c cross-caps. Every surface is homeomorphic to $\mathbb{S}_h$ or $\mathbb{N}_c$. The *Euler genus* of $\mathbb{S}_h$ is $2h$. The *Euler genus* of $\mathbb{N}_c$ is c. A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. If G embeds in a surface Σ, then every minor of G also embeds in Σ.

that embeds in Σ. This paper determines the asymptotic behaviour of $C(H, \Sigma, n)$ as $n \to \infty$ for any fixed surface Σ and any fixed graph H.

Before stating our theorem, we mention some related results that determine $C(H, \mathbb{S}_0, n)$ for specific planar graphs H where the surface is the sphere $\mathbb{S}_0$. Alon and Caro [2] determined $C(H, \mathbb{S}_0, n)$ precisely if H is either a complete bipartite graph or a triangulation without non-facial triangles. Hakimi and Schmeichel [33] studied $C(C_k, \mathbb{S}_0, n)$ where C_k is the k-vertex cycle; they proved that $C(C_3, \mathbb{S}_0, n) = 3n-8$ and $C(C_4, \mathbb{S}_0, n) = \frac{1}{2}(n^2+3n-22)$. See [34, 35] for more results on $C(C_3, \mathbb{S}_0, n)$ and see [1] for more results on $C(C_4, \mathbb{S}_0, n)$. Győri et al. [29] proved that $C(C_5, \mathbb{S}_0, n) = 2n^2 - 10n + 12$ (except for $n \in \{5, 7\}$). Győri et al. [30] determined $C(P_4, \mathbb{S}_0, n)$ precisely, where P_k is the k-vertex path. Alon and Caro [2] and independently Wood [64] proved that $C(K_4, \mathbb{S}_0, n) = n-3$. More generally, Perles (see [2]) conjectured that if H is a fixed 3-connected planar graph, then $C(H, \mathbb{S}_0, n) = O(n)$. Perles noted the converse: If H is planar, not 3-connected and $|V(H)| \geqslant 4$, then $C(H, \mathbb{S}_0, n) \geqslant \Omega(n^2)$. Perles' conjecture was proved by Wormald [66] and independently by Eppstein [17], who asked the following two open problems:

- Characterise the subgraphs occurring $O(n)$ times in graphs of given genus.
- Characterise the subgraphs occurring a number of times which is a nonlinear function of n.

This paper answers both these questions (and more).

We start with the following natural question: when is $C(H, \Sigma, n)$ bounded by a constant depending only on H and Σ (and independent of n)? We prove that H being 3-connected and non-planar is a sufficient condition. In fact we prove a stronger result that completely answers the question. We need the following standard definitions. A *k-separation* of a graph H is a pair (H_1, H_2) of edge-disjoint subgraphs of H such that $H_1 \cup H_2 = H$, $V(H_1) \setminus V(H_2) \neq \emptyset$, $V(H_2) \setminus V(H_1) \neq \emptyset$, and $|V(H_1 \cap H_2)| = k$. A k'-separation for some $k' \leqslant k$ is called a *$(\leqslant k)$-separation*. If (H_1, H_2) is a separation of H with $X = V(H_1) \cap V(H_2)$, then let H_i^- and H_i^+ be the simple graphs obtained from H_i by removing and adding all edges between vertices in X, respectively.

A graph H is *strongly non-planar* if H is non-planar and for every $(\leqslant 2)$-separation (H_1, H_2) of H, both H_1^+ and H_2^+ are non-planar. Note that every 3-connected non-planar graph is strongly non-planar. The following is our first contribution. It says that $C(H, \Sigma, n)$ is bounded if and only if H is strongly non-planar.

Theorem 1.1. *There exists a function $c_{1.1}(h, g)$ such that for every strongly non-planar graph H with h vertices and every surface Σ of Euler genus g,*

$$C(H, \Sigma, n) \leqslant c_{1.1}(h, g).$$

Conversely, for every graph H that is not strongly non-planar and for every surface Σ in which H embeds, there is a constant $c > 0$ such that for all $n \geqslant 4|V(H)|$, there *is an n-vertex graph that embeds in Σ and contains at least cn copies of H; that is, $C(H, \Sigma, n) \geqslant cn$.*

There are two important observations about Theorem 1.1. First, the characterisation of graphs H does not depend on the surface Σ. Indeed, the only dependence on Σ is in the constants. Second, Theorem 1.1 shows that $C(H, \Sigma, n)$ is either bounded or $\Omega(n)$.

Theorem 1.1 is in fact a special case of the following more general theorem. The next definition is a key to describing our results. A *flap* in a graph H is a ($\leqslant 2$)-separation (A, B) such that A^+ is planar. Separations (A, B) and (C, D) of H are *independent* if $E(A^-) \cap E(C^-) = \emptyset$ and $(V(A) \setminus V(B)) \cap (V(C) \setminus V(D)) = \emptyset$.[2] If H is planar and with no ($\leqslant 2$)-separation, then the *flap-number* of H is defined to be 1. Otherwise, the *flap-number* of H is defined to be the maximum number of pairwise independent flaps in H. Let $f(H)$ denote the flap-number of H.

The following is our main theorem.

Theorem 1.2. *For every graph H and every surface Σ in which H embeds,*

$$C(H, \Sigma, n) = \Theta(n^{f(H)}).$$

It is immediate from the definitions that $f(H) = 0$ if and only if H is strongly non-planar. So Theorem 1.1 follows from the $f(H) \leqslant 1$ cases of Theorem 1.2.

As an aside, note that Theorem 1.2 can be restated as follows: for every graph H and every surface Σ in which H embeds,

$$\lim_{n \to \infty} \frac{\log C(H, \Sigma, n)}{\log n} = f(H).$$

The above limit is sometimes referred to as the *asymptotic logarithmic density* of H in Σ. A related result of Nešetřil and Ossona de Mendez [51], shows that the asymptotic logarithmic density of the number of *induced* copies of a fixed graph H in an infinite nowhere dense hereditary graph class $\mathcal{G}$ is an integer that is at most $\alpha(H)$ (the size of a maximum stable set in H). Our results (in the case that $\mathcal{G}$ is the class of graphs embeddable in a fixed surface) imply this result (since the number of induced copies of H in G is at most $C(H, G)$). Moreover, our bounds are often more precise since $f(H)$ can be significantly less than $\alpha(H)$. For example, for all $n \geqslant 3$, $f(K_{n,n}) = 0$, but $\alpha(K_{n,n}) = n$.

The lower bound in Theorem 1.2 is proved in Section 2. Section 3 introduces some tools from the literature that are used in the proof of the upper bound. Theorem 1.1

[2]It is worth noticing that neither condition implies the other. If G is a 4-cycle $abcd$, then the two 2-separations (A, B) and (C, D) obtained by considering respectively the cutsets $\{b, d\}$ and $\{a, c\}$ satisfy the second condition but not the first. If G consists of 5 non-adjacent vertices a, b, c, d, e then the two 2-separations (A, B) and (C, D) with $V(A) = \{a, b, c, e\}$, $V(B) = \{b, c, d\}$, $V(C) = \{b, c, d, e\}$, $V(D) = \{a, b, c\}$ satisfy the first condition but not the second.

is proved in Section 4. The upper bound in Theorem 1.2 is then proved in Section 5. Section 6 presents more precise bounds on $C(H, \Sigma, n)$ when H is a complete graph K_s. Section 7 considers the maximum number of copies of a graph H in an n-vertex graph in a given minor-closed class. Section 8 reinterprets our results in terms of homomorphism inequalities, and presents some open problems that arise from this viewpoint.

Before continuing, to give the reader some more intuition about Theorem 1.2, we now asymptotically determine $C(T, \Sigma, n)$ for a tree T.

Corollary 1.3. *For every fixed tree T, let $\beta(T)$ be the size of a maximum stable set in the subforest F of T induced by the vertices with degree at most 2. Then for every fixed surface Σ,*

$$C(T, \Sigma, n) = \Theta(n^{\beta(T)}).$$

Proof. By Theorem 1.2, it suffices to show that $\beta(T) = f(T)$.

Let $I = \{v_1, \dots, v_{\beta(T)}\}$ be a maximum stable set in F. Let x_i (and possibly y_i) be the neighbours of v_i. Let $A_i := T[\{v_i, x_i, y_i\}]$ and $B_i := T - v_i$. Then (A_i, B_i) is a flap of T. Since I is a stable set, for each $v_i \in I$ neither x_i nor y_i are in I, implying that $E(A_i^-) \cap E(A_j^-) = \emptyset$ for distinct $i, j \in [\beta(T)]$. Moreover, $V(A_i) \setminus V(B_i) = \{v_i\}$, so $(V(A_i) \setminus V(B_i)) \cap (V(A_j) \setminus V(B_j)) = \emptyset$ for all distinct i, j. Hence $(A_1, B_1), \dots, (A_{\beta(T)}, B_{\beta(T)})$ are pairwise independent flaps in T. Thus $\beta(T) \leqslant f(T)$. Theorem 1.2 then implies that $C(T, \Sigma, n) = \Omega(n^{\beta(T)})$. This lower bound is particularly easy to see when T is a tree. Let G be the graph obtained from T by replacing each vertex $v_i \in I$ by $\lfloor \frac{n-|V(T)|}{\beta(T)} \rfloor$ vertices with the same neighbourhood as v_i, as illustrated in Corollary 1.3. Then G is planar with at most n vertices and at least $(\frac{n-|V(T)|}{\beta(T)})^{\beta(T)}$ copies of T. Thus $C(T, \Sigma, n) \geqslant C(T, \mathbb{S}_0, n) = \Omega(n^{\beta(T)})$ for fixed T.

For the converse, let $(A_1, B_1), \dots, (A_{f(T)}, B_{f(T)})$ be pairwise independent flaps in T. Choose $(A_1, B_1), \dots, (A_{f(T)}, B_{f(T)})$ to minimise $\sum_{i=1}^{f(T)} |V(A_i)|$. A simple case-analysis shows that $|V(A_i) \setminus V(B_i)| = 1$, and if v_i is the vertex in $V(A_i) \setminus V(B_i)$, then $N(v_i) = V(A_i) \cap V(B_i)$, implying v_i has degree 1 or 2 in T. Moreover, $v_i v_j \notin E(T)$ for distinct $i, j \in [f(T)]$ as otherwise $E(A_i^-) \cap E(A_j^-) \neq \emptyset$. Hence $\{v_1, \dots, v_{f(T)}\}$ is a stable set of vertices in T all with degree at most 2. Hence $\beta(T) \geqslant f(T)$. □

2. Lower Bound

Now we prove the lower bound in Theorem 1.2. Let H be an h-vertex graph with flap-number k. Let Σ be a surface in which H embeds. Our goal is to show that $C(H, \Sigma, n) = \Omega(n^k)$ for all $n \geqslant 4|V(H)|$. We may assume that $k \geqslant 2$ and H is connected. Let $(A_1, B_1), \dots, (A_k, B_k)$ be pairwise independent flaps in H. If (A_i, B_i) is a 1-separation, then let v_i be the vertex in $A_i \cap B_i$. If (A_i, B_i) is a 2-separation, then let v_i and w_i be the two vertices in $A_i \cap B_i$. Let H' be obtained from H as follows: if (A_i, B_i) is a 2-separation,

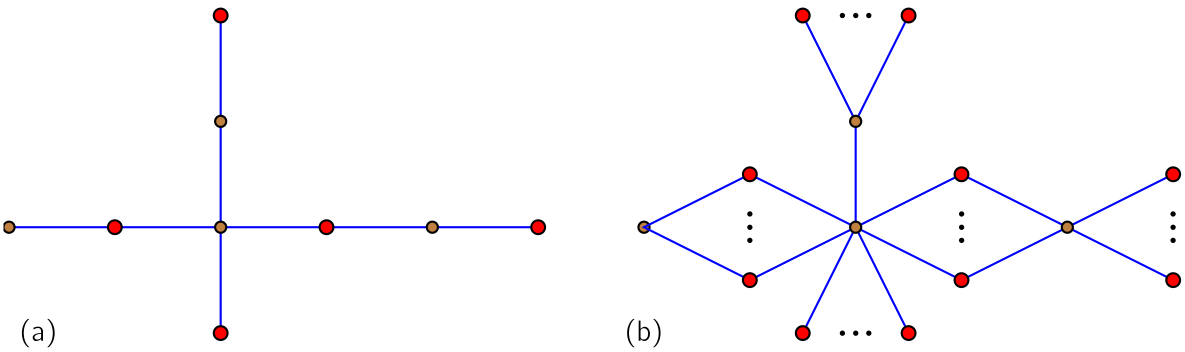

Figure 1. (a) A tree T with $\beta(T) = 5$. (b) A planar graph with $\Omega(n^5)$ copies of T.

then delete $A_i - V(B_i)$ from H, and add the edge $v_i w_i$ (if it does not already exist). Note that H' is a minor of H, since we may assume that whenever (A_i, B_i) is a 2-separation, there is a $v_i w_i$-path in A_i (otherwise (A_i, B_i) can be replaced by a $(\leqslant 1)$-separation). Since H embeds in Σ, so does H'. By assumption, A_i^+ is planar for each i. Fix an embedding of A_i^+ with v_i and w_i (if it exists) on the outerface (which exists since $v_i w_i$ is an edge of A_i^+ in the case of a 2-separation). Let G be the graph obtained from an embedding of H' in Σ by pasting $q := \lfloor \frac{n}{|V(H)|} - 1 \rfloor$ copies of A_i^+ onto v_i (if (A_i, B_i) is a 1-separation) and onto $v_i w_i$ (if (A_i, B_i) is a 2-separation). These copies of A_i^+ can be embedded into a face of H', as illustrated in Figure 2.

Since $(V(A_i) \setminus V(B_i)) \cap (V(A_j) \setminus V(B_j)) = \emptyset$ for distinct $i, j \in [k]$,

$$|V(G)| = |V(H)| + q \sum_i |V(A_i) \setminus V(B_i)| \leqslant (q+1)|V(H)| \leqslant n.$$

By construction, G has at least $q^k \geqslant (\frac{n}{|V(H)|} - 2)^k$ copies of H. Hence $C(H, \Sigma, n) = \Omega(n^k)$.

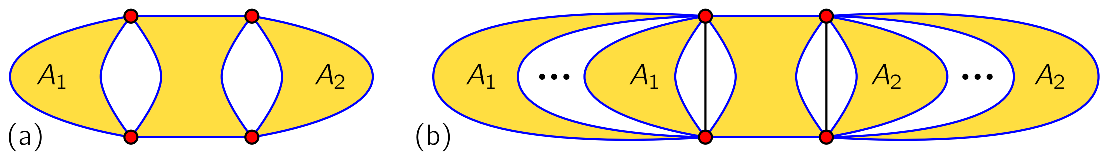

Figure 2. (a) A graph H with flap-number 2. (b) A graph with $\Omega(n^2)$ copies of H.

3. Tools

In Sections 3–5 of this paper we work in the following setting. For graphs G and H, an *image* of H in G is an injection $\phi : V(H) \to V(G)$ such that $\phi(u)\phi(v) \in E(G)$ for all $uv \in E(H)$. Let $I(H, G)$ be the number of images of H in G, and let $I(H, \Sigma, n)$ be the maximum of $I(H,G)$ taken over all n-vertex graphs G that embed in Σ. If H is fixed then $C(H,G)$ and $I(H,G)$ differ by a constant factor. In particular, if $|V(H)|=h$ then

$$C(H,G) \leqslant I(H,G) \leqslant h!\,C(H,G).$$
$$C(H,\Sigma,n) \leqslant I(H,\Sigma,n) \leqslant h!\,C(H,\Sigma,n).$$

So to prove our main theorems, it suffices to work with images rather than copies.

To prove the upper bound in Theorem 1.2 we need several tools from the literature. The first two were proved by Eppstein [17]. To state the first result we need the following definition. A collection $\mathcal{H}$ of images of H in G is *coherent* if for all images $\phi_1,\phi_2 \in \mathcal{H}$ and for all distinct vertices $x,y \in V(H)$, we have $\phi_1(x) \neq \phi_2(y)$.

Lemma 3.1 ([17]). *Let H be a graph with h vertices and G be a graph. Every collection of at least $c_{3.1}(h,t) := h!^2 t^h$ images of H in G contains a coherent subcollection of size at least t.*

Theorem 3.2 ([17]). *There exists a function $c_{3.2}(h,g)$ such that for every planar graph H with h vertices and no ($\leqslant 2$)-separation, and every surface Σ of Euler genus g,*

$$I(H,\Sigma,n) \leqslant c_{3.2}(h,g)n.$$

The next key tool is the following result by Miller [48] and Archdeacon [7].

Theorem 3.3 (Additivity of Euler genus [7, 48]). *For all graphs G_1 and G_2, if $|V(G_1) \cap V(G_2)| \leqslant 2$ then the Euler genus of $G_1 \cup G_2$ is at least the Euler genus of G_1 plus the Euler genus of G_2.*

We also use the following result of Erdős and Rado [18]; see [6] for a recent quantitative improvement. A *t-sunflower* is a collection $\mathcal{S}$ of t sets for which there exists a set R such that $X \cap Y = R$ for all distinct $X,Y \in \mathcal{S}$. The set R is called the *kernel* of $\mathcal{S}$.

Lemma 3.4 (Sunflower Lemma [18]). *There exists a function $c_{3.4}(h,t)$ such that every collection of $c_{3.4}(h,t)$ many h-subsets of a set contains a t-sunflower.*

Finally, we mention some well-known corollaries of Euler's Formula that we use implicitly. Every graph with $n \geqslant 3$ vertices and Euler genus g has at most $3(n+g-2)$ edges. Moreover, for bipartite graphs the above bound is $2(n+g-2)$. For example, this implies that the complete bipartite graph $K_{3,2g+3}$ has Euler genus greater than g.

4. Strongly Non-Planar Graphs

We begin by proving a quantitative version of the upper bound in Theorem 1.1. In fact, we will prove that Theorem 1.1 holds more generally for what we call 'partially subdivided graphs'. A *partially subdivided graph* is a pair $(H,\mathcal{P})$, where H is a graph and $\mathcal{P}$ is a collection of internally disjoint paths in H such that the two ends of each path in $\mathcal{P}$ are not adjacent in H, and every internal vertex of each path in $\mathcal{P}$ has degree 2 in H. Let

$H - \mathcal{P}$ be the graph obtained from H by deleting every internal vertex of each path in $\mathcal{P}$. Finally, let $H/\mathcal{P}$ be the minor of H obtained by contracting all but one edge from each path in $\mathcal{P}$.

Theorem 4.1. *Let $c_{4.1}(h, g) := c_{3.1}(h, c_{3.4}(h, 2g+3))$. Then for every partially subdivided graph $(H, \mathcal{P})$ such that $H/\mathcal{P}$ is strongly non-planar and $|V(H)| = h$, for every surface Σ with Euler genus g, and for every graph G embedded in Σ, there are at most $c_{4.1}(h, g)$ images of $H - \mathcal{P}$ in G that extend to an image of H in G.*

Proof. Assume for the sake of contradiction, that there is a collection $\mathcal{H}$ of more than $c_{4.1}(h, g)$ images of H in G, such that all restrictions of these images to $H - \mathcal{P}$ are distinct. By Lemma 3.1, $\mathcal{H}$ contains a coherent subfamily $\mathcal{H}_0$ of size at least $c_{3.4}(h, 2g+3)$. Let $\mathcal{V}$ be the collection of vertex sets of the images of H in $\mathcal{H}_0$. By coherence, $|\mathcal{V}| = |\mathcal{H}_0| \geqslant c_{3.4}(h, 2g+3)$.

By the Sunflower Lemma, $\mathcal{V}$ contains a $(2g+3)$-sunflower $\mathcal{F}$. (Abusing notations slightly, we equate sets in $\mathcal{F}$ with the corresponding images of H in $\mathcal{H}_0$.) Let Z be the kernel of $\mathcal{F}$. Thus $F \cap F' = Z$ for all $F, F' \in \mathcal{F}$. Let I be the set of internal vertices of all $P \in \mathcal{P}$. Since the restrictions of the images of H in $\mathcal{F}$ to $H - \mathcal{P}$ are all distinct, for each $F \in \mathcal{F}$ there exists a vertex $w_F \in F \setminus Z$ such that w_F is the image of a vertex in $V(H) \setminus I$. By coherence, we may assume that every w_F is the image of the same vertex of $V(H) \setminus I$. For each $F \in \mathcal{F}$, let C_F be the component of $F - Z$ which contains w_F and let N_F be the vertices of Z with at least one neighbour in $V(C_F)$. By coherence, N_F is the same for all $F \in \mathcal{F}$. Therefore, we obtain a $K_{|N_F|,|\mathcal{F}|}$ minor in G by contracting each C_F to a vertex. Since $|\mathcal{F}| = 2g+3$ and $K_{3,2g+3}$ does not embed in Σ, we must have $|N_F| \leqslant 2$.

For each $F \in \mathcal{F}$, consider the pair of subgraphs $(F_1, F_2) := (F[Z], F[(V(F) \setminus Z) \cup N_F])$ of F. Since $|N_F| \leqslant 2$, either (F_1, F_2) is a $(\leqslant 2)$-separation of F or $F_2 = F$. Let $F/\mathcal{P}$ and $(F_1/\mathcal{P}, F_2/\mathcal{P})$ be obtained from F and (F_1, F_2) respectively, by contracting all but one edge from the image of each $P \in \mathcal{P}$. When performing these contractions, we may assume that no edge of F incident to w_F is contracted. Therefore, since $w_F \in V(F_2) \setminus V(F_1)$, it follows that either $(F_1/\mathcal{P}, F_2/\mathcal{P})$ is a $(\leqslant 2)$-separation of $F/\mathcal{P}$ or $F_2/\mathcal{P} = F/\mathcal{P}$. Since $F/\mathcal{P} \cong H/\mathcal{P}$ and $H/\mathcal{P}$ is strongly non-planar, either $(F_2/\mathcal{P})^+$ is non-planar or $F_2/\mathcal{P} = F/\mathcal{P}$. In the first case, F_2^+ is also non-planar since $(F_2/\mathcal{P})^+$ is a minor of F_2^+. In the second case, $F_2 = F$ is also non-planar. Let $F_2' := F_2^+$ if the first case holds, and $F_2' = F$ if the second case holds. If $N_F := \{x, y\}$ and $xy \notin E(F)$, let Σ' be obtained from Σ by adding a handle and using the handle to draw the edge xy, and let $G' := G \cup \{xy\}$. Otherwise, let $\Sigma' := \Sigma$ and $G' := G$. We conclude by noting that the family of subgraphs $\{F_2' \mid F \in \mathcal{F}\}$ of G' contradicts the Additivity of Euler genus (Theorem 3.3), since they each have Euler genus at least 1, they pairwise intersect only in N_F, are drawn on a surface Σ' of Euler genus at most $g+2$, and $|\mathcal{F}| = 2g+3 > g+2$. □

Note that if H is a strongly non-planar graph, then we recover Theorem 1.1 by applying Theorem 4.1 to the partially subdivided graph $(H, \emptyset)$. We need the stronger statement in Theorem 4.1 for the proof of Theorem 5.9 to come.

5. Proof of Main Theorem

The proof of our main theorem uses a variant of the SPQR tree, which we now introduce.

5.1. SPQRK Trees. The *SPQR tree* of a 2-connected graph G is a tree that displays all the 2-separations of G. Since we need to consider graphs which are not necessarily 2-connected, we use a variant of the SPQR tree which we call the *SPQRK tree*.

Let G be a connected graph. The *SPQRK tree* T_G of G is a tree, where each node $a \in V(T_G)$ is associated with a multigraph H_a which is a minor of G. Each vertex $x \in V(H_a)$ is a vertex of G, that is, $V(H_a) \subseteq V(G)$. Each edge $e \in E(H_a)$ is classified either as a *real* or *virtual* edge. By the construction of an SPQRK tree each edge $e \in E(G)$ appears in exactly one minor H_a as a real edge, and each edge $e \in E(H_a)$ which is classified real is an edge of G. The SPQRK tree T_G is defined recursively as follows.

(1) If G is 3-connected, then T_G consists of a single *R-node* a with $H_a := G$. All edges of H_a are real in this case.
(2) If G is a cycle, then T_G consists of a single *S-node* a with $H_a := G$. Again, all edges of H_a are real in this case.
(3) If G is isomorphic to K_1 or K_2, then T_G consists of a single *K-node* a with $H_a := G$. Again, all edges of H_a are real in this case.
(4) If G is 2-connected and has a cutset $\{x, y\}$ such that the vertices x and y have degree at least 3, we construct T_G inductively as follows. Let $C_1, \dots, C_r$ ($r \geqslant 2$) be the connected components of $G - \{x, y\}$. First add a *P-node* a to T_G, for which H_a is the graph with $V(H_a) := \{x, y\}$ consisting of r parallel virtual edges and one additional real edge if xy is an edge of G.

    Next let G_i be the graph $G[V(C_i) \cup \{x, y\}]$ with the additional edge xy if it is not already there. Since we include the edge xy, each G_i is 2-connected and we can construct the corresponding SPQRK tree T_{G_i} by induction. Let a_i be the (unique) node in T_{G_i} for which xy is a real edge in H_{a_i}. In order to construct T_G, we make xy a virtual edge in the node a_i, and connect a_i to a in T_G.
(5) If G has a cut-vertex x and $C_1, \dots, C_s$ ($s \geqslant 2$) are the connected components of $G - x$, then construct T_G inductively as follows. First, add a *Q-node* a to T_G, for which H_a is the graph consisting of the single vertex x. For each $i \in [s]$, let $G_i := G[V(C_i) \cup \{x\}]$. Since G_i is connected, we can construct the corresponding SPQRK tree T_{G_i} by induction. If there is a unique node $b_i \in V(T_{G_i})$ such that $x \in V(H_{b_i})$, then make a adjacent to b_i in T_G. If x is in at least two nodes of $V(T_{G_i})$, then $x \in V(C) \cap V(D)$ for some ($\leqslant 2$)-separation (C, D) of G_i. Since

$G_i - x$ is connected, there must be a P-node b_i in T_{G_i} such that $x \in V(H_{b_i})$. Note that b_i is not necessarily unique. Choose one such b_i and make a adjacent to b_i in T_G.

As a side remark, note that the SPQRK tree T_G of G is in fact not unique—there is some freedom in choosing b_i in the last point in the definition above—however, for our purposes we do not need uniqueness, we only need that T_G displays all the ($\leqslant 2$)-separations of G.

The next lemma is the crux of the proof. Let J and G be graphs and X and Y be cliques in J and G respectively, with $|X| = |Y|$. Let $\phi : V(J) \to V(G)$ be an image of J in G. We say that ϕ *fixes* X *at* Y if $\phi(X) = Y$. Let $(J', \mathcal{P})$ be a partially subdivided graph such that $J = J'/\mathcal{P}$. We call $uv \in E(J)$ a *fake* edge if u and v are the set of ends of some $P \in \mathcal{P}$. Otherwise, uv is a *true edge*.

Lemma 5.1. *Let* $c_{5.1}(j, g) := 12(g+1)c_{3.1}(j, c_{3.4}(j, 2g+3))$. *Let* Σ *be a surface of Euler genus* g. *Let* $(J', \mathcal{P})$ *be a connected, partially subdivided planar graph with* $|V(J')| = j$ *and let* $J := J'/\mathcal{P}$.

Let X *be a clique in* J *such that:*

(1) *there do not exist independent flaps* (A, B) *and* (C, D) *of* J *with* $X \subseteq V(B \cap D)$,
(2) $|X| \in \{1, 2\}$, *and if* $|X| = 2$, *then* X *is a true edge,*
(3) *if* $|X| = 1$ *and* $J \cong P_3$, *then* e *is a true edge, where* e *is the unique edge of* J *not incident to* X,
(4) *if* $|X| = 1$ *and* $J \cong C_3$, *then* e *is a true edge, where* e *is the unique edge of* J *not incident to* X,
(5) *if* $X = 2$ *and* $J \cong C_4$, *then* e *is a true edge, where* e *is the unique edge of* J *not incident to a vertex of* X,
(6) *if* J *is* 3*-connected, then all edges of* J *with neither end in* X *are true,*
(7) *if* (A, B) *is a flap of* J, *with* $X \subseteq V(B)$, $|V(A \cap B)| = 1$, *and* $A \cong P_3$, *then* e *is a true edge, where* e *is the unique edge of* A *not incident to* $V(A \cap B)$,
(8) *if* (A, B) *is a flap of* J, *with* $X \subseteq V(B)$, $|V(A \cap B)| = 1$, *and* $A \cong C_3$, *then* e *is a true edge, where* e *is the unique edge of* A *not incident to* $V(A \cap B)$,
(9) *if* (A, B) *is a flap of* J, *with* $X \subseteq V(B)$, $|V(A \cap B)| = 2$, *and* $A^+ \cong C_3$, *then at least one* e *or* f *is a true edge, where* e *and* f *are the two edges of* A *with an end not on* $V(A \cap B)$.
(10) *if* (A, B) *is a flap of* J, *with* $X \subseteq V(B)$, $|V(A \cap B)| = 2$, *and* $A^+ \cong C_4$, *then* e *is a true edge, where* e *is the unique edge of* A *not incident to* $V(A \cap B)$,
(11) *if* (A, B) *is a flap of* J *such that* A^+ *is* 3*-connected, then all edges of* A *with neither end in* $V(A \cap B)$ *are true.*

Then for every n*-vertex graph* G *embeddable in* Σ *and every clique* Y *in* G *with* $|Y| = |X|$, *there are at most* $c_{5.1}(j, g)n$ *images of* $J' - \mathcal{P}$ *in* G *with* X *fixed at* Y *which extend to an image of* J' *in* G.

Proof of Lemma 5.1. Let G be an n-vertex graph embedded in a surface Σ of Euler genus g and Y be a clique in G with $|Y| = |X|$. We begin by proving the lemma when J is small. The lemma clearly holds if $|V(J)| = 1$ or $|V(J)| = 2 = |X|$. If $|V(J)| = 2$ and $|X|$=1, let y be the vertex of J not in X. Since there are at most $n-1$ vertices of G to send y to, we are done. Similarly, we are done if $J \in \{P_3, C_3\}$ and $|X| = 2$. If $J \in \{P_3, C_3\}$ and $|X| = 1$ let e be the unique edge of J not incident to X. Note that e exists in the case that $J \cong P_3$, since the vertex in X cannot be the middle vertex of J by (1). By (3) and (4), e is a true edge, so there are at most $|E(G)| \leqslant 3(g+1)n$ edges of G to send e to. Each edge gives at most two images of $J' - \mathcal{P}$ with X fixed at Y in G, so there are at most $6(g+1)n$ such images. Suppose $|V(J)| = 4$. Note that $J \ncong P_4$, by (1). If $J \cong C_4$, then $|X| = 2$ by (1). Let e be the unique edge of J not incident to a vertex of X. By (5), e is a true edge, so again there are at most $3(g+1)n$ edges of G to send e to. Each edge gives at most four images of $J' - \mathcal{P}$ with X fixed at Y in G, so there are at most $12(g+1)n$ such images.

In summary, by the above discussion we may assume that $|V(J)| \geqslant 4$, and $J \ncong P_4, C_4$ in case $|V(J)| = 4$.

Let T_J be the SPQRK tree of J. Suppose $V(T_J) = \{a\}$. If a is a K-node, then we are done since $|V(J)| \leqslant 2$. If a is an S-node, then by (1), $J \cong C_3$ or $J \cong C_4$, so we are done. By the preceding remarks, we may assume that J is 3-connected, or $|V(J)| \geqslant 4$ and $|V(T_J)| \geqslant 2$. A clique X' of J is a *true clique* if $|X'| = 1$, or $|X'| = 2$ and the edge of X' is a true edge. If J is 3-connected, we have the following easy claim.

Claim 5.2. *If J is 3-connected, then there exists a true clique X' in J such that for all $w \in V(J) \setminus (X \cup X')$, there are three internally disjoint paths in J from w to $X \cup X'$, whose ends in $X \cup X'$ are distinct.*

Proof. Let X' be an edge of J with neither end in X. By (6), X' is a true edge. Since J is 3-connected, we are done by Menger's theorem. □

We now suppose that $|V(J)| \geqslant 4$ and $|V(T_J)| \geqslant 2$, and we prove that Claim 5.2 also holds in this case. Let W be the set of K-, S-, and R-nodes of $V(T_J)$. If U is a non-empty proper subset of W, we define $H_U := \bigcup_{a \in U} H_a$, $\mathrm{bd}(U) := V(H_U \cap H_{W \setminus U})$, $\lambda(U) := |\,\mathrm{bd}(H_U)|$, and $\mathrm{sep}(U) := (H_U, H_{W \setminus U})$. The next two claims follow from (1).

Claim 5.3. *T_J is a path and there is a leaf ℓ of T_J such that $X \subseteq V(H_\ell)$ and $X \setminus \mathrm{bd}(\{\ell\}) \neq \emptyset$.*

Claim 5.4. *Let r be the other leaf of T_J. Then for all non-empty $U \subseteq W \setminus \{\ell, r\}$ such that U is not a single K-node, $\lambda(U) \geqslant 3$.*

The next claim also follows from (1). For completeness, we include the proof.

Claim 5.5. *Let $S := \{s \in V(J) \setminus X \mid \deg_J(s) \leqslant 2\}$. Then $|S| \leqslant 2$, $S \subseteq V(H_r)$, and if $|S| = 2$, then the two vertices in S are adjacent in J.*

Proof. Since $|V(J)| \geqslant 4$, for each $s \in S$, $(\delta(s), J - s)$ is a flap with $X \subseteq V(J - s)$, where $\delta(s)$ is the subgraph of J induced by the edges incident to s. Thus, by (1), S is a clique in J, and therefore $|S| \leqslant 3$. Moreover, $|S| = 3$ is impossible, since $|V(J)| \geqslant 4$ and J is connected. Thus, $|S| \leqslant 2$. Since $(A, B) = \mathrm{sep}(\{r\})$ is a flap with $X \subseteq V(B)$, (1) also implies $S \subseteq V(H_r)$. □

Claim 5.6. *J has at most two cut-vertices. Moreover, if J has two cut-vertices, then they are the vertex set of some K-node of T_J.*

Proof. Suppose c and d are distinct cut-vertices of J. Let $W_{cd} \subseteq W$ be the set of K-, S-, and R-nodes of $V(T_J)$ strictly between the Q-nodes corresponding to c and d in T_J. Note that $\mathrm{sep}(W_{cd})$ is a 2-separation of J, unless W_{cd} is just a single K-node. Moreover, if W_{cd} is not a single K-node, then $\mathrm{sep}(\{r\})$ and $\mathrm{sep}(W_{cd})$ would contradict (1). It follows that J has at most two cut-vertices, and that $\{c, d\}$ is the vertex set of some K-node of T_J. □

Claim 5.7. *There exists a true clique X' in J such that for all $w \in V(J) \setminus (X \cup X')$, there are three internally disjoint paths in J from w to $X \cup X'$, whose ends in $X \cup X'$ are distinct.*

Proof. If r is an R-node, then all edges of H_r^- are true by (11). In this case we let X' be any edge of H_r^- such that $X' \cap \mathrm{bd}(\{r\}) = \emptyset$. If r is an S-node, then by (1), either $H_r \cong C_3$, or $H_r \cong C_4$ and $|\mathrm{bd}(\{r\})| = 2$. In either case, by (8), (9), and (10), we can choose X' to be a true edge such that $V(H_r) \setminus (X' \cup \mathrm{bd}(\{r\})) = \emptyset$. Lastly, suppose r is a K-node. Then $H_r \cong K_2$, say H_r consists of the edge uv with $v \in \mathrm{bd}(\{r\})$. If $v \notin S$, we let $X' := \{u\}$. If $v \in S$, we let $X' := \{u, v\}$; note that uv is a true edge by (7) in this case.

Suppose the claim is false for the above choice of X' for some vertex $w \in V(J) \setminus (X \cup X')$, and let $X^+ := X \cup X'$. Note that $|X^+| \geqslant 3$ by our choice of X', since if $|X| = 1$ then J is 2-connected by (1). (Indeed, if J is not 2-connected then J has a flap (A, B) which is a 1-separation with $X \subseteq B$, but then $(B, A \cup X)$ is also a flap, contradicting (1).) Thus, by Menger's theorem, there is a $(\leqslant 2)$-separation (J_1, J_2) of J with $w \in V(J_1) \setminus V(J_2)$ and $X^+ := X \cup X' \subseteq V(J_2)$.

Let a be an S-node of T_J. Observe that every 2-separation of the cycle H_a lifts to a 2-separation of J. We say that a 2-separation of J is *rooted at a* if it is a lift of a 2-separation of H_a. Since the SPQRK tree T_J of J 'displays' all the $(\leqslant 2)$-separations of J, every $(\leqslant 2)$-separation (A, B) of J

- is equal to $\mathrm{sep}(U)$ for some $U \subseteq W$, or

- is rooted at some S-node a of T_J, or
- is obtained from a 1-separation (A', B') by adding an isolated vertex to A' or B' (which is thus in $A \cap B$).

Suppose $(J_1, J_2) = \mathrm{sep}(U)$ for some $U \subseteq W$. Since $X^+ \subseteq V(J_2)$, $X \setminus \mathrm{bd}(\{\ell\}) \neq \emptyset$, and $X' \setminus \mathrm{bd}(\{r\}) \neq \emptyset$, we have $U \subseteq W \setminus \{\ell, r\}$. This is a contradiction since $\lambda(U) \geqslant 3$ by Claim 5.4. Similarly, (J_1, J_2) cannot be rooted at an S-node, unless $\deg_J(x) = 2$ for some $x \in V(J_1) \setminus V(J_2)$. However, by Claim 5.5 and our choice of X', $\deg_J(x) \geqslant 3$, for all $x \in V(J) \setminus X^+$, so this is also impossible.

Finally, suppose the third possibility holds for some 1-separation (A', B') of J. Observe that every 1-separation (A, B) of J has $X' \subseteq V(A)$ and $X \subseteq V(B)$; or $X \subseteq V(A)$ and $X' \subseteq V(B)$. By swapping the order of (A', B') we may assume that $X' \subseteq V(A')$ and $X \subseteq V(B')$. Moreover, $(A', B' \cup \{a\}) \in \{(J_1, J_2), (J_2, J_1)\}$ for some $a \in V(A') \setminus V(B')$; or $(A' \cup \{b\}, B') \in \{(J_1, J_2), (J_2, J_1)\}$ for some $b \in V(B') \setminus V(A')$. If $(A' \cup \{b\}, B') = (J_1, J_2)$, then (A', B') is a 1-separation of J such that $X \cup X' \subseteq V(B')$. However, no such separation exists (by the proof that $(J_1, J_2) \neq \mathrm{sep}(U)$ for all $U \subseteq W$). Similarly, $(A', B' \cup \{a\}) = (J_2, J_1)$ is impossible. If $(A' \cup \{b\}, B') = (J_2, J_1)$, then (A', B') and $(B', A' \cup \{b\})$ contradict (1). The remaining case is $(A', B' \cup \{a\}) = (J_1, J_2)$. Let c be the unique vertex in $V(A') \cap V(B')$. Recall that by the choice of X', if r is an R-node or an S-node, then $|X'| = 2$ and $c \notin X'$. However, this contradicts $X' \subseteq V(J_2)$. Thus, r is a K-node. Note that $(A', B') = \mathrm{sep}(\{r\})$ is impossible, because $V(A') \setminus (V(B' \cup \{a\}))$ would be empty, and hence $(A', B' \cup \{a\})$ is not a 2-separation. Thus $c \notin V(H_r)$. Let v be the cut-vertex of J in $V(H_r)$. By Claim 5.6, c and v are adjacent and $v \in S = \{s \in V(J) \setminus X \mid \deg_J(s) \leqslant 2\}$. In this case, by our choice of X', we have $|X'| = 2$ and $c \notin X'$, so we again have a contradiction. $\square$

Let X' be the true clique of J given by Claim 5.2 or by Claim 5.7, depending whether J is 3-connected, or $|V(J)| \geqslant 4$ and $|V(T_J)| \geqslant 2$. Suppose $|X'| = 1$. For each $y \in V(G)$ we let c_y be the number of images of $J' - \mathcal{P}$ in G with X fixed at Y and X' fixed at y, which extend to an image of J' in G. Suppose $|X'| = 2$. For each $f \in E(G)$ we let c_f be the number of images of $J' - \mathcal{P}$ in G with X fixed at Y and X' fixed at f, which extend to an image of J' in G.

We claim that if $|X'| = 1$ then $c_y \leqslant c_{3.1}(j, c_{3.4}(j, 2g+3))$ for all $y \in V(G)$, if $|X'| = 2$ then $c_f \leqslant c_{3.1}(j, c_{3.4}(j, 2g+3))$ for all $f \in E(G)$. We will prove both inequalities simultaneously, since the proof is the same. Arguing by contradiction, suppose y or $f := uv$ is a counterexample, and set $Y^+ = Y \cup \{y\}$ if $|X'| = 1$ and $Y^+ = Y \cup \{u, v\}$ if $|X'| = 2$. Then, there exists a collection $\mathcal{J}_1$ of more than $c_{3.1}(j, c_{3.4}(j, 2g+3))$ images of J' in G with X fixed at Y and X' fixed at y (respectively, X' fixed at f) such that the restrictions of these images to $J' - \mathcal{P}$ are all distinct.

By Lemma 3.1, $\mathcal{J}_1$ contains a coherent subfamily $\mathcal{J}_2$ of size at least $c_{3.4}(j, 2g+3)$. Let $\mathcal{V}$ be the collection of vertex sets of $\mathcal{J}_2$. Note that by coherence, $|\mathcal{V}| = |\mathcal{J}_2| \geqslant c_{3.4}(j, 2g+3)$. By Lemma 3.4, $\mathcal{V}$ contains an s-sunflower $\mathcal{F}$, where $s \geqslant 2g+3$. Let Z

be the kernel of $\mathcal{F}$. By construction, $Y^+ \subseteq Z$. Let I be the set of internal vertices of all $P \in \mathcal{P}$. Since the restrictions of each copy of J' in $\mathcal{F}$ to $J' - \mathcal{P}$ are all distinct, for all $F \in \mathcal{F}$ there must be a vertex $w_F \in F \setminus Z$ such that w_F is the image of a vertex in $V(J') \setminus I$. By coherence, we may assume that each w_F corresponds to the same vertex in $V(J') \setminus I$. By Claim 5.2 and Claim 5.7, there are three internally disjoint paths from w_F to Y^+ in $G[F]$ whose ends in Y^+ are distinct. For each $F \in \mathcal{F}$, let Z_F be the set consisting of the first vertices of Z on each of these three paths. By coherence, we may assume Z_F is the same for all $F \in \mathcal{F}$. Thus, G contains a subdivision of $K_{3,2g+3}$. However, this is impossible, since $K_{3,2g+3}$ does not embed in Σ.

It follows that $c_y, c_f \leqslant c_{3.1}(j, c_{3.4}(j, 2g+3))$ for all $y \in V(G)$ and $f \in E(G)$. The proof is complete by summing over all possible $y \in V(G)$ if $|X'| = 1$, and summing over all possible $f \in E(G)$ if $|X'| = 2$. ☐

The final ingredient we need is the following 'flap reduction' lemma.

Lemma 5.8. *Let H be a connected graph with flap-number $k \geqslant 1$. Let A be a subgraph of H which is maximal (under the subgraph relation) subject to the following conditions:*

- *A has no isolated vertices, and*
- *there exists a flap (A, B) of H and a set $\mathcal{F}$ of k independent flaps in H with $(A, B) \in \mathcal{F}$.*

Then B^+ has flap-number $k-1$. Moreover, A is connected and A^+ does not contain independent flaps (C, D) and (C', D') such that $V(A \cap B) \subseteq V(D \cap D')$.

Proof. We first show that B^+ has flap-number at least $k-1$. To see this, let $\mathcal{F}$ be a set of k independent flaps in H such that $(A, B) \in \mathcal{F}$. Every flap $(C, D) \in \mathcal{F} \setminus \{(A, B)\}$ corresponds to a flap (C, D') in B^+, unless $k = 2$, H is planar, and $\mathcal{F} = \{(A, B), (B, A)\}$. In either case, B^+ has flap-number at least $k-1$.

We now prove the upper bound. Towards a contradiction, let $(C_1, D_1), \dots, (C_k, D_k)$ be independent flaps in B^+. Let $X := V(A \cap B)$. If $X \subseteq V(S)$ for a subgraph S of B^+, we let $S +_X A$ be the subgraph of H obtained by gluing A to S along X, and deleting the edge between the ends of X in B^+ if the edge does not exist in H. If X is contained in $V(D_\ell)$ for every $\ell \in [k]$, then $(A, B), (C_1, D_1 +_X A), \dots, (C_k, D_k +_X A)$ are $k+1$ pairwise independent flaps in H. Thus X is not contained in $V(D_\ell)$ for some $\ell \in [k]$. By relabelling, we may assume $\ell = 1$. Since $(V(C_1) \setminus V(D_1)) \cap X \neq \emptyset$ and $(V(C_1) \setminus V(D_1)) \cap (V(C_i) \setminus V(D_i)) = \emptyset$ for all $i > 1$, we have $X \subseteq V(D_i)$ for all $i > 1$. Then, $(C_1 +_X A, D_1), (C_2, D_2 +_X A), \dots (C_k, D_k +_X A)$ are k independent flaps in H. Since A is a proper subgraph of $C_1 +_X A$, this contradicts the maximality of A.

Finally, we show that the last sentence of the lemma holds. Suppose A is disconnected and $A_1, \dots, A_c$ are the connected components of A. Since H is connected, A_i contains a vertex of $V(A \cap B)$ for all $i \in [c]$. Thus, $c = 2$ and A_1 and A_2 each contain exactly one vertex of $V(A \cap B)$. Since neither A_1 nor A_2 is an isolated vertex, there exist B_1 and B_2 such that (A_1, B_1) and (A_2, B_2) are independent flaps of H. Thus, $\mathcal{F} \setminus \{(A, B)\} \cup \{(A_1, B_1), (A_2, B_2)\}$ is a set of $k + 1$ independent flaps of H, which contradicts that H has flap-number k. Similarly, A^+ does not contain independent flaps (C, D) and (C', D') such that $V(A \cap B) \subseteq V(D \cap D')$. □

We call a subgraph A of H a *half-flap* if (A, B) is a flap of H for some B. Two half-flaps A and C are *independent* if there exist B and D such that (A, B) and (C, D) are independent flaps. We say that A is a *full-half-flap* if A satisfies the conditions of Lemma 5.8.

We now complete the proof of the upper bound in Theorem 1.2.

Theorem 5.9. *Let $c_{5.9}(h, g) := 6(g+1)c_{4.1}(h, g)c_{3.2}(h, g)c_{5.1}(h, g+2)^h$. Then for every graph H with h vertices and every surface Σ of Euler genus g in which H embeds,*

$$C(H, \Sigma, n) \leqslant I(H, \Sigma, n) \leqslant c_{5.9}(h, g)n^{f(H)}.$$

Proof. Let $k := f(H)$. Since $c_{5.9}(h_1, g) \cdot c_{5.9}(h_2, g) \leqslant c_{5.9}(h, g)$ whenever $h_1 + h_2 = h$, we may assume that H is connected by induction on $|V(H)|$. A *reduction sequence* of H is a sequence of graphs $H_k, \ldots, H_j$ for some $j \leqslant k$, where $H_k := H$, and for all $i > j$, $H_{i-1} := B_i^+$, where (A_i, B_i) is a flap in H_i satisfying the conditions of Lemma 5.8. By Lemma 5.8, every reduction sequence satisfies the following properties.

Claim 5.10. *Let $H_k, \ldots, H_j$ be a reduction sequence of H, with corresponding flaps (A_ℓ, B_ℓ) in H_ℓ. Then for all $\ell \in \{k, \ldots, j + 1\}$, A_ℓ is connected and A_ℓ^+ does not contain independent flaps (C, D) and (C', D') such that $V(A_\ell) \cap V(B_\ell) \subseteq V(D) \cap V(D')$. Moreover, for all $\ell \in \{k, \ldots, j\}$, H_ℓ has flap-number ℓ.*

We now establish further properties of reduction sequences. Let $H_k, \ldots, H_j$ be a reduction sequence of H, with corresponding flaps (A_ℓ, B_ℓ) in H_ℓ. If $V(A_\ell) \cap V(B_\ell) := \{u, v\}$ and $uv \notin E(H_\ell)$, we declare uv to be a *fake edge* of $H_{\ell-1}$. An edge of H_ℓ is a fake edge if it is a fake edge of $H_{\ell'}$ for some $\ell' \geqslant \ell$, and it is a *true edge* if it is not a fake edge.

Claim 5.11. *Let $H_k, \ldots, H_j$ be a reduction sequence of H, with corresponding flaps (A_ℓ, B_ℓ) in H_ℓ. Then for all $\ell \in \{k, \ldots, j\}$, if (C, D) is a flap in H_ℓ such that C^+ is 3-connected, then all edges of C with neither end in $V(C \cap D)$ are true.*

Proof. Let (C, D) be a flap of H_ℓ which is a counterexample with $|E(C)| - |V(D)|$ maximum. Let $X := V(C \cap D)$. By the maximality of $|E(C)| - |V(D)|$, note that if H_ℓ contains an edge f whose ends are X, then $f \in E(C)$. We claim that each $x \in X$ is incident to an edge of D. If not, then x must be an isolated vertex of D. But now, $(C, D - x)$ contradicts the maximality of $|E(C)| - |V(D)|$. For all $i \in \{k, \ldots, \ell + 1\}$ set $X_i := V(A_i \cap B_i)$. Let $\mathcal{I}$ be the set of indices $i \in \{k, \ldots, \ell+1\}$ such that X_i is the set of ends of a fake edge of C^-. Let s be the smallest index in $\mathcal{I}$ and e be the corresponding fake edge. Since $H_k, \dots, H_j$ is a reduction sequence, we can choose a collection $\mathcal{F}_s$ of s independent flaps of H_s such that $(A_s, B_s) \in \mathcal{F}_s$ and $\sum_{(A',B')\in\mathcal{F}_s} |V(A')| + |E(A')|$ is minimum. Let (A, B) be an arbitrary flap in $\mathcal{F}_s \setminus \{(A_s, B_s)\}$.

Suppose $V(A)$ is a proper subset of $V(C)$. Since C^+ is 3-connected and $V(C)\setminus V(A) \neq \emptyset$, for some $Y \in \{X_s, X\}$, one vertex y of Y is in $V(A)\setminus V(B)$ and the other vertex of Y is in $V(B)\setminus V(A)$. Observe that $V(A_s) \subseteq V(B)$ because (A_s, B_s) and (A, B) are independent flaps of H_s. In particular, $X_s \subseteq V(B)$. By the minimality of $\sum_{(A',B')\in\mathcal{F}_s} |V(A')|+|E(A')|$, if H_s contains an edge f whose ends are X, then $f \in E(B)$. Since $V(A)$ is a proper subset of $V(C)$ and each $x \in X$ is incident to an edge of D, this implies $X \subseteq V(B)$. Therefore, for either choice of Y, we have $y \in V(B)$. Thus, $y \in V(A \cap B)$, which contradicts that $y \in V(A) \setminus V(B)$.

Suppose $V(A) = V(C)$ and $|X_s \cup X| \geqslant 3$. Observe that $X_s \cup X \subseteq V(A)$. Since $V(A_s) \subseteq V(B)$, $X_s \subseteq V(B)$. Moreover, since $V(D) \subseteq V(B)$, $X \subseteq V(B)$. Therefore, $X_s \cup X \subseteq V(A\cap B)$. Since $|X_s \cup X| \geqslant 3$, this implies that (A, B) is a $(\geqslant 3)$-separation of H_s, which contradicts that (A, B) is a flap.

Suppose $A \cap C = S$, where S is a stable set and S contains a vertex $x \notin X$. If $x \notin X_s$, then x is an isolated vertex of A. If $x \in X_s$, then since (A, B) and (A_s, B_s) are independent flaps, $V(A \cap A_s) \setminus X_s = \emptyset$. Thus, x is an isolated vertex of A in this case as well. But now, replacing (A, B) by $(A - x, B)$ contradicts the minimality of $\sum_{(A',B')\in\mathcal{F}_s} |V(A')| + |E(A')|$.

Suppose $A\cap C$ consists of just a single edge xy. Since C^+ is 3-connected, $\deg_{C^+}(x) \geqslant 3$ and $\deg_{C^+}(y) \geqslant 3$. For $z \in \{x, y\}$, let $d(z)$ be the number of $Y \in \{X_s, X\}\setminus\{\{x, y\}\}$ such that $z \in Y$. Observe that $\deg_{C^+}(z) \leqslant \deg_{A\cap C}(z)+\deg_{B\cap C}(z)+d(z)$ for both $z \in \{x, y\}$. Since $\deg_{A\cap C}(z) = 1$ for both $z \in \{x, y\}$, this yields $\deg_{B\cap C}(z) \geqslant 2 - d(z)$ for both $z \in \{x, y\}$. If $\{x, y\} \subseteq X\cup X_s$, then $d(x) \leqslant 1$ and $d(y) \leqslant 1$. Therefore, $\deg_{B\cap C}(x) \geqslant 1$ and $\deg_{B\cap C}(y) \geqslant 1$, which implies $\{x, y\} \subseteq V(A\cap B)$. But now, $(A-xy, B\cup\{xy\})$ is a $(\leqslant 2)$-separation, which contradicts the minimality of $\sum_{(A',B')\in\mathcal{F}_s} |V(A')| + |E(A')|$. By symmetry, we may assume $y \notin X_s\cup X$. Thus, $d(y) = 0$, which gives $\deg_{B\cap C}(y) \geqslant 2$. In particular, $y \in V(A \cap B)$. Moreover, since $y \notin X_s \cup X$, we have $y \in V(C) \setminus V(D)$, and thus $\deg_A(y) = \deg_{A\cap C}(y) = 1$. Observe that $|V(A - y) \cap V(B \cup \{xy\})| \leqslant 2$ because $V(A - y) \cap V(B \cup \{xy\}) \subseteq (V(A \cap B) \setminus \{y\}) \cup \{x\}$. Therefore $(A - y, B \cup \{xy\})$ is a $(\leqslant 2)$-separation. But now $A - y$ is a half-flap contained in A, which contradicts the minimality of $\sum_{(A',B')\in\mathcal{F}_s} |V(A')| + |E(A')|$.

Suppose $V(C)$ is a proper subset of $V(A)$. Clearly, $X_s \subseteq V(A)$. Also, $X_s \subseteq V(B)$ since (A, B) and (A_s, B_s) are independent flaps. Therefore, $V(A \cap B) = X_s$, since $|X_s| = 2$. Moreover, $A \cap D$ contains at least one vertex not in X, since $V(C)$ is a proper subset of $V(A)$. If $A \cap D$ meets $B \cap D$ at a vertex $x \notin X$, then $x \notin X_s$ and $x \in V(A \cap B)$, which contradicts that $V(A \cap B) = X_s$. Thus, $V(A \cap D) \cap V(B \cap D) \subseteq X$. It follows that $A \cap D$ is a half-flap, since $|X| \leqslant 2$. However, this contradicts the minimality of $\sum_{(A',B')\in\mathcal{F}_s} |V(A')| + |E(A')|$ since $|V(A \cap D)| < |V(A)|$.

Suppose $3 \leqslant |V(A \cap C)| < |V(C)|$ and $V(A) \setminus V(C) \neq \emptyset$. Since C^+ is 3-connected, for some $Y \in \{X_s, X\}$, one vertex y of Y is in $V(A) \setminus V(B)$ and the other vertex of Y is in $V(B) \setminus V(A)$. Since (A, B) and (A_s, B_s) are independent flaps, we have $A_s \subseteq B$. Thus, $X_s \subseteq V(B)$, and so $Y = X$. Let B^* be obtained from B by replacing A_s by an edge whose ends are X_s, and adding the edge with ends X. Since C^+ is 3-connected, $(C^+ \cap A, C^+ \cap B^*)$ is a ($\geqslant 3$)-separation of C^+. It follows that $A \cap C$ meets $B \cap C$ in at least two vertices of $V(C) \setminus X$, and thus exactly two since $|V(A \cap B)| \leqslant 2$. In particular, $V(A \cap B) \subseteq V(C) \setminus X$, and hence $V(A \cap D) \cap V(B \cap D) = \emptyset$. It follows that $A \cap D$ is a half-flap. However, this contradicts the minimality of $\sum_{(A',B') \in \mathcal{F}_s} |V(A')| + |E(A')|$ since $|V(A \cap D)| < |V(A)|$.

By the previous cases, there are only two cases left to consider for (A, B), namely (1) $V(A \cap C) \subseteq X$ and $A \cap C$ has no edges, or (2) $V(A) = V(C)$ and $|X_s \cup X| \leqslant 2$. Since (A, B) is an arbitrary flap of $\mathcal{F}_s \setminus \{(A_s, B_s)\}$, we may assume that either $V(A' \cap C) \subseteq X$ and $A' \cap C$ has no edges for all $(A', B') \in \mathcal{F}_s \setminus \{(A_s, B_s)\}$; or that $V(A) = V(C)$ and $|X_s \cup X| \leqslant 2$.

Suppose $V(A' \cap C) \subseteq X$ and $A' \cap C$ has no edges for all $(A', B') \in \mathcal{F}_s \setminus \{(A_s, B_s)\}$. By replacing (A_s, B_s) by $(A_s +_{X_s} C, B_s - (V(C) \setminus X))$ in $\mathcal{F}_s$, we contradict that A_s is a full-half-flap.

The last case is $V(A) = V(C)$ and $|X_s \cup X| \leqslant 2$. Since X_s is the set of ends of a fake edge of C^-, this implies that $X_s \neq X$. Thus, we must have $|X| = 1$ and $X \subseteq X_s$. Let t be the smallest index in $\mathcal{I}$ which is larger than s and let f be the corresponding fake edge. Note that t exists since C^- contains a fake edge with neither end in X. Let $\mathcal{F}_t$ be a collection of t independent flaps of H_t such that $(A_t, B_t) \in \mathcal{F}_t$ and $\sum_{(A',B') \in \mathcal{F}_t} |V(A')| + |E(A')|$ is minimum. Let (A^*, B^*) be an arbitrary flap in $\mathcal{F}_t \setminus \{(A_t, B_t)\}$. Let $A_s^* \subseteq H_t$ be obtained from A_s by reversing all the flap reductions for all $j \in [s, t]$. The remainder of the proof is essentially the same as the previous cases with (A^*, B^*) taking the role of (A, B) and $\{X_s, X_t\}$ taking the role of $\{X_s, X\}$. For completeness, we include all the details.

Suppose $V(A^*)$ is a proper subset of $V(C)$. Since C is 3-connected, for some $Y \in \{X_s, X_t\}$, $V(A^*) \setminus V(B^*)$ contains one vertex y of Y and $V(B^*) \setminus V(A^*)$ contains the other vertex of Y. By Lemma 5.8, A_s^* and A_t are both connected. Since $V(A^*) \subseteq V(C)$, this implies $X_s \cup X_t \subseteq V(B^*)$. Thus, for either choice of Y, we have $y \in V(A^* \cap B^*)$, which contradicts that $y \in V(A^*) \setminus V(B^*)$.

Suppose $V(A^*) = V(C)$. Clearly, $X_s \cup X_t \subseteq V(C) = V(A^*)$. Since A_s^* and A_t are both connnected by Lemma 5.8, $X_s \cup X_t \subseteq V(B^*)$. Therefore, $X_s \cup X_t \subseteq V(A^* \cap B^*)$. Since $|X_s \cup X_t| \geqslant 3$, this implies that (A^*, B^*) is a ($\geqslant 3$)-separation of H_t, which contradicts that (A^*, B^*) is a flap.

Suppose $A^* \cap C = S$, where S is a stable set and S contains a vertex $x \notin X_s$. If $x \notin X_t$, then x is an isolated vertex of A. If $x \in X_t$, then since (A^*, B^*) and (A_t, B_t) are independent flaps, $V(A^* \cap A_t) \setminus X_t = \emptyset$. Thus, x is an isolated vertex of A^* in this case as well. But now, replacing (A^*, B^*) by $(A^* - x, B^*)$ contradicts the minimality of $\sum_{(A',B') \in \mathcal{F}_t} |V(A')| + |E(A')|$.

Suppose $A^* \cap C$ consists of just a single edge xy. Since C is 3-connected, $\deg_C(x) \geqslant 3$ and $\deg_C(y) \geqslant 3$. For $z \in \{x, y\}$, let $d(z)$ be the number of $Y \in \{X_s, X_t\} \setminus \{\{x, y\}\}$ such that $z \in Y$. Observe that $\deg_C(z) \leqslant \deg_{A^* \cap C}(z) + \deg_{B^* \cap C}(z) + d(z)$ for both $z \in \{x, y\}$. Since $\deg_{A^* \cap C}(z) = 1$ for both $z \in \{x, y\}$, this yields $\deg_{B^* \cap C}(z) \geqslant 2 - d(z)$ for both $z \in \{x, y\}$. If $\{x, y\} \subseteq X_s \cup X_t$, then $d(x) \leqslant 1$ and $d(y) \leqslant 1$. Therefore, $\deg_{B^* \cap C}(x) \geqslant 1$ and $\deg_{B^* \cap C}(y) \geqslant 1$, which implies $\{x, y\} \subseteq V(A^* \cap B^*)$. But now, $(A^* - xy, B^* \cup \{xy\})$ is a $(\leqslant 2)$-separation, which contradicts the minimality of $\sum_{(A',B') \in \mathcal{F}_t} |V(A')| + |E(A')|$. By symmetry, we may assume $y \notin X_s \cup X_t$. Thus, $d(y) = 0$, which gives $\deg_{B^* \cap C}(y) \geqslant 2$. In particular, $y \in V(A^* \cap B^*)$. Moreover, since $y \notin X_s \cup X_t$, we have $y \in V(C) \setminus V(D)$, and thus $\deg_{A^*}(y) = \deg_{A^* \cap C}(y) = 1$. Observe that $|V(A^* - y) \cap V(B^* \cup \{xy\})| \leqslant 2$ because $V(A^* - y) \cap V(B^* \cup \{xy\}) \subseteq (V(A^* \cap B^*) \setminus \{y\}) \cup \{x\}$. Therefore $(A^* - y, B^* \cup \{xy\})$ is a $(\leqslant 2)$-separation. But now $A^* - y$ is a half-flap contained in A^*, which contradicts the minimality of $\sum_{(A',B') \in \mathcal{F}_t} |V(A')| + |E(A')|$.

Suppose $V(C)$ is a proper subset of $V(A^*)$. Clearly, $X_t \subseteq V(A^*)$. Also, $X_t \subseteq V(B^*)$ since (A^*, B^*) and (A_t, B_t) are independent flaps. Therefore, $V(A^* \cap B^*) = X_t$, since $|X_t| = 2$. Moreover, $A^* \cap (D \cup A_s^*)$ contains at least one vertex not in X_t, since $V(C)$ is a proper subset of $V(A^*)$. If $A^* \cap (D \cup A_s^*)$ meets $B^* \cap (D \cup A_s^*)$ at a vertex $x \notin X_s$, then $x \in V(A \cap B)$, which contradicts that $V(A \cap B) = X_s$. Thus, $V(A^* \cap (D \cup A_s^*)) \cap V(B^* \cap (D \cup A_s^*)) \subseteq X_s$. It follows that $A^* \cap (D \cup A_s^*)$ is a half-flap, since $|X_s| \leqslant 2$. However, this contradicts the minimality of $\sum_{(A',B') \in \mathcal{F}_t} |V(A')| + |E(A')|$ since $|V(A^* \cap (D \cup A_s^*))| < |V(A)|$.

Suppose $3 \leqslant |V(A^* \cap C)| < |V(C)|$ and $V(A^*) \setminus V(C) \neq \emptyset$. Since C is 3-connected, for some $Y \in \{X_s, X_t\}$, one vertex y of Y is in $V(A^*) \setminus V(B^*)$ and the other vertex of Y is in $V(B^*) \setminus V(A^*)$. Since (A^*, B^*) and (A_t, B_t) are independent flaps, we have $A_t \subseteq B^*$. Thus, $X_t \subseteq V(B^*)$, and so $Y = X_s$. Let β be obtained from B^* by replacing A_t by an edge whose ends are X_t and adding the edge with ends X_s. Since C is 3-connected, $(C \cap A^*, C \cap \beta)$ is a $(\geqslant 3)$-separation of C. It follows that $A \cap C$ meets $B \cap C$ in at least two vertices of $V(C) \setminus X$, and thus exactly two since $|V(A \cap B)| \leqslant 2$.

Thus, $A^* \cap C$ meets $B^* \cap C$ in at least two vertices of $V(C) \setminus X_s$, and thus exactly two since $V(A^* \cap B^*)| \leqslant 2$. In particular, $V(A^* \cap B^*) \subseteq V(C) \setminus X_s$, and hence $V(A^* \cap (D \cup A_s^*)) \cap V(B^* \cap (D \cup A_s^*)) = \emptyset$. It follows that $A \cap (D \cup A_s^*)$ is a half-flap. However, this contradicts the minimality of $\sum_{(A',B') \in \mathcal{F}_s} |V(A')| + |E(A')|$ since $|V(A \cap D)| < |V(A)|$.

Since (A^*, B^*) is an arbitrary flap of $\mathcal{F}_t \setminus \{(A_t, B_t)\}$, by the previous cases, we may assume $V(A' \cap C) \subseteq X_s$ and $A' \cap C$ has no edges for all $(A', B') \in \mathcal{F}_t \setminus \{(A_t, B_t)\}$. Therefore, by replacing (A_t, B_t) by $(A_t +_{X_t} C, B_t - (V(C) \setminus X_t))$ in $\mathcal{F}_t$, we contradict that A_t is a full-half-flap. $\square$

We will need to pick reduction sequences that satisfy a few additional properties. Let $H_k, \dots, H_j$ be a reduction sequence of H, with corresponding flaps (A_ℓ, B_ℓ) in H_ℓ. We say that $H_k, \dots, H_j$ is a *good reduction sequence* if for all $\ell \in \{k, \dots, j\}$, H_ℓ is not a cycle, and for all $\ell \in \{k, \dots, j+1\}$

(1) if $A_\ell \cong P_3$ and $V(A_\ell) \cap V(B_\ell) = \{x\}$, then e is a true edge of H_ℓ, where e is the unique edge of A_ℓ not incident to x,
(2) if $A_\ell \cong C_3$ and $V(A_\ell) \cap V(B_\ell) = \{x\}$, then e is a true edge of H_ℓ, where e is the unique edge of A_ℓ not incident to x,
(3) if $A_\ell \cong C_4$ and $V(A_\ell) \cap V(B_\ell) = \{x, y\}$, then e is a true edge of H_ℓ, where e is the unique edge of A_ℓ not incident to either x or y.

Note that, by Lemma 5.8, if $A_\ell \cong P_3$ and $V(A_\ell) \cap V(B_\ell) = \{x\}$ then x cannot be the center of the P_3, hence edge e is well defined in case (1) above. Similarly, if $A_\ell \cong C_4$ and $V(A_\ell) \cap V(B_\ell) = \{x, y\}$, then x and y cannot be opposite vertices of the C_4, and thus e is also well defined in case (3).

We now give conditions under which a good reduction sequence can be extended to a longer good reduction sequence. Let H' be a graph where some edges are fake, and let $u, v \in V(H')$. A *u-culdesac* of H' is a cycle C of H' such that $u \in V(C)$, $\deg_{H'}(u) \geqslant 3$, and $\deg_{H'}(w) = 2$ for all $w \in V(C) \setminus \{u\}$. A *u-alley* of H' is a path P of H' such that u is an end of P, $|V(P)| \geqslant 2$, $\deg_{H'}(u) \geqslant 3$, and $\deg_P(w) = \deg_{H'}(w)$ for all $w \in V(P) \setminus \{u\}$. A *u–v-alley* of H' is a path P in H' such that u and v are the ends of P, $|V(P)| \geqslant 3$, $\deg_{H'}(u), \deg_{H'}(v) \geqslant 3$, and $\deg_{H'}(w) = 2$ for all $w \in V(P) \setminus \{u, v\}$.

Claim 5.12. *Let C be a u-culdesac of H' with $|V(C)| \geqslant 4$, and let $e := uv$ be an edge of C incident to u. Let P and Q be the 3- and 4-vertex paths of C containing e and ending at u, respectively. If $|V(C)|$ is even, then P is a full-half flap of H', and if $|V(C)|$ is odd, then Q is a full-half-flap.*

Proof. Let $\mathcal{F}$ be a collection of $f(H')$ independent flaps in H', and let $\mathcal{F}'$ be the collection of flaps $(F_1, F_1') \in \mathcal{F}$ such that $E(F_1 \cap C) \neq \emptyset$. Since C contains a set of $m := \lfloor \frac{|V(C)|}{2} \rfloor$ independent half-flaps of H', we must have $|\mathcal{F}'| \geqslant m$; otherwise, $|\mathcal{F}|$ is not maximum. If $\mathcal{F}'$ contains a flap (F_1, F_1') such that $F_1 \cap C = uv$, then we replace (F_1, F_1') by $(F_1 - v, F_1' \cup \{uv\})$. Similarly, we may assume that $\mathcal{F}'$ does not contain a flap (F_1, F_1') such that $F_1 \cap C = tu$, where t is the other neighbour of u in C. In particular, $|E(F_1 \cap C)| \geqslant 2$ for all $(F_1, F_1') \in \mathcal{F}'$. This implies that $|\mathcal{F}'| \leqslant m$, and hence $|\mathcal{F}'| = m$. Observe that C contains a set $\mathcal{H}$ of m independent half-flaps with $P \in \mathcal{H}$ if $|V(C)|$ is even, and $Q \in \mathcal{H}$ if $|V(C)|$ is odd. It follows that P can be extended to a full-half-flap P' if $|V(C)|$ is even, and Q can be extended to a full-half-flap Q' if $|V(C)|$ is odd. Since every collection of $f(H')$ independent flaps of H' must contain at least m flaps which use an edge of C, and $\bigcup_{F \in \mathcal{H}} F = C$, it follows that $C \cap P' = P$. If P' is not contained in C, then P' contains two independent half-flaps, which is a contradiction. Thus, $P' = P$. The same argument gives $Q' = Q$. ☐

The same proof also establishes the following claim for u–v-alleys.

Claim 5.13. *Let P be a u–v-alley of H' with $|V(P)| \geqslant 5$. If $|V(P)|$ is even, let P' be the 4-vertex subpath of P ending at u. If $|V(P)|$ is odd, let P' be 3-vertex subpath of P ending at u. Then P' is a full-half-flap of H'.*

An edge e is *distance* 2 *from a vertex* u if e is not incident to u and e is incident to an edge which is incident to u. Let C be a u-culdesac. We say that C is *tame* if C has a real edge which is distance 2 from u. Let P be a u–v-alley. We say that P is *tame* if P has a real edge which is distance 2 from u or v. Let P be a u-alley. We say that P is *tame* if P has a real edge which is distance 2 from u. Finally, we say that H' is *tame* if for all $u, v \in V(H')$ all u-culdesacs, u-alleys, and u–v-alleys of H' are tame.

Claim 5.14. *Let $H_k, \dots, H_j$ be a good reduction sequence of a planar graph such that $j \geqslant 3$ and H_j is tame. Then $H_k, \dots, H_j$ can be extended to a good reduction sequence $H_k, \dots, H_{j-1}$ such that H_{j-1} is tame.*

Proof. Suppose that H_j contains a u-culdesac C with $|V(C)| \geqslant 4$. Since H_j is tame, C contains a real edge e which is distance 2 from u. If $|V(C)|$ is even (respectively, odd), let P be the 3-vertex (respectively, 4-vertex) path of C such that one end of P is u and $e \notin E(P)$. By Claim 5.12, P is a full-half-flap of H_j. Letting H_{j-1} be obtained from H_j by applying Lemma 5.8 with $A = P$, we have that $H_\ell, \dots, H_{j-1}$ is a good reduction sequence and H_{j-1} is tame. Thus, we may assume that all culdesacs of H_j are triangles. By Claim 5.13, we may also assume that for all $u, v \in V(H_j)$, all u–v-alleys of H_j have at most four vertices.

Let (A_j, B_j) be a flap of H_j such that A_j is a full-half-flap and let $H_{j-1} = B_j^+$. Suppose H_{j-1} is a cycle. If (A_j, B_j) is a 1-separation, then $H_{j-1} \cong C_3$, since all culdesacs of H_j are triangles. Since $f(C_3) = 1$, this contradicts $j \geqslant 3$. If (A_j, B_j) is a 2-separation, then $H_{j-1} \in \{C_3, C_4\}$ since all alleys of H_j have at most four vertices. In either case, $f(H_j) = 2$, which contradicts $j \geqslant 3$. Thus H_{j-1} is not a cycle.

Since H_j is tame, it follows that $H_k, \dots, H_{j-1}$ is a good reduction sequence. It only remains to show that H_{j-1} is tame. Towards a contradiction, suppose H_{j-1} contains a u-culdesac C which is not tame. The cases that H_{j-1} contains a u-alley which is not tame, or a u–v-alley which is not tame are similar and are omitted. We assume that (A_j, B_j) is a 2-separation (the case that (A_j, B_j) is a 1-separation is easier and is omitted). Since H_j is tame, there must be an edge $xy \in E(C)$ such that $V(A_j) \cap V(B_j) = \{x, y\}$. Let P_{xu} and P_{yu} be the x–u and y–u paths in C such that $V(P_{xu}) \cap V(P_{yu}) = \{u\}$. Since all alleys of H_j have at most four vertices, $|V(P_{xu})|, |V(P_{yu})| \leqslant 4$. Moreover, if $|V(P_{xu})| \in \{2, 4\}$, then adding the edge of P_{xu} incident to x to A_j contradicts that A_j is a full-half-flap. Thus, $|V(P_{xu})|, |V(P_{yu})| \in \{1, 3\}$. In particular, this implies that C is not a triangle. For each fake edge $e \in E(C)$ let $\ell(e)$ be the smallest index such that $V(A_{\ell(e)}) \cap V(B_{\ell(e)})$ is equal to the set of ends of e. Among all fake edges of $C \setminus \{xy\}$ in H_{j-1}, let f be such that $\ell(f)$ is smallest. Since C is not a triangle, there are two edges of C which are distance 2 from u (both of which are fake). Therefore, f exists. Let g be the unique edge of C such that $f \cup g$ is P_{xu} or P_{yu}. Then adding g to $A_{\ell(f)}$ contradicts that $A_{\ell(f)}$ is a full-half flap. □

Observe that if H is nonplanar, then for every reduction sequence $H_k, \dots, H_j$ of H, each H_i is nonplanar, and in particular cannot be a cycle. Therefore, the previous proof gives the following result for nonplanar graphs.

Claim 5.15. *Let $H_k, \dots, H_j$ be a good reduction sequence of a nonplanar graph such that $j \geqslant 1$ and H_j is tame. Then $H_k, \dots, H_j$ can be extended to a good reduction sequence $H_k, \dots, H_{j-1}$ such that H_{j-1} is tame.*

The final ingredient we need is the existence of a certain collection of paths in H. Let $H_k, \dots, H_j$ be a reduction sequence of H with corresponding flaps (A_ℓ, B_ℓ) in H_ℓ, and for each $\ell \in \{k, \dots, j\}$ let F_ℓ be the set of fake edges of H_ℓ. For each fake edge e we define a set of indices $\mathcal{I}(e)$ recursively as follows. Let ℓ be the largest index such that e is a fake edge of H_ℓ and recursively define $\mathcal{I}(e) = \{\ell + 1\} \cup \bigcup_{f \in F_{\ell+1} \cap E(A_{\ell+1})} \mathcal{I}(f)$.

Claim 5.16. *Let $H_k, \dots, H_j$ be a reduction sequence of H with corresponding flaps (A_ℓ, B_ℓ) in H_ℓ, and for each $\ell \in \{k, \dots, j\}$ let F_ℓ be the set of fake edges of H_ℓ. Then for all $\ell \in \{k, \dots, j\}$, there is a collection of paths $\mathcal{P}_\ell = \{P_f \mid f \in F_\ell\}$ in H such that for all $f \in F_\ell$, P_f has the same ends as f, and $P_f \subseteq \bigcup_{i \in \mathcal{I}(f)} A_i$. Moreover, letting $H'_\ell := (H_\ell \setminus F_\ell) \cup \mathcal{P}_\ell$, we have that $(H'_\ell, \mathcal{P}_\ell)$ is a partially subdivided graph, H'_ℓ is a subgraph of H, and $H'_\ell/\mathcal{P}_\ell$ is isomorphic to H_ℓ.*

Proof. We proceed by reverse induction. Since $H_k = H$ does not contain any fake edges, we may take $\mathcal{P}_k := \emptyset$. Suppose the claim is true for some $\ell > j$, and consider $\ell - 1$. If $|V(A_\ell) \cap V(B_\ell)| = 2$ then let $\{a, b\} := V(A_\ell) \cap V(B_\ell)$. We are done by induction if $|V(A_\ell) \cap V(B_\ell)| \leqslant 1$ or $e := ab$ is a true edge of $H_{\ell-1}$. Thus, we may assume that e is a fake edge of $H_{\ell-1}$. By the final part of Lemma 5.8, there is path P'_e in A_ℓ between a and b. By induction, for each fake edge f in P'_e, there is a path $P'_f \subseteq \bigcup_{i \in \mathcal{I}(f)} A_i$. Moreover, note that if f_1 and f_2 are distinct fake edges of $H_{\ell-1}$, then $\mathcal{I}(f_1) \cap \mathcal{I}(f_2) = \emptyset$. Therefore, replacing each fake edge f of P'_e with P'_f, we obtain a path P_e contained in $\bigcup_{i \in \mathcal{I}(e)} A_i$. Every other fake edge e' of $H_{\ell-1}$ is a fake edge of $H_{\ell'}$ for some $\ell' > \ell$. By induction, for each such fake edge e', there is a path $P_{e'}$ contained in $\bigcup_{i \in \mathcal{I}(e')} A_i$. Note that P_{f_1} and P_{f_2} are internally-disjoint for all distinct $f_1, f_2 \in F_{\ell-1}$, since $\mathcal{I}(f_1) \cap \mathcal{I}(f_2) = \emptyset$ Thus, $\mathcal{P}_{\ell-1} := \{P_f \mid f \in F_{\ell-1}\}$ is the required set of paths. □

We now prove the theorem in the case that H is non-planar.

Claim 5.17. *Suppose H is non-planar and $H_k, \dots, H_0$ is a good reduction sequence of H such that H_ℓ is tame for all $\ell \in \{k, \dots, 0\}$. For each $\ell \in \{0, \dots, k\}$, let F_ℓ be the set of fake edges of H_ℓ. Then for all $\ell \in \{0, \dots, k\}$, there are at most $c_{4.1}(h, g) c_{5.1}(h, g+2)^\ell \cdot n^\ell$ images of $H_\ell \setminus F_\ell$ in G which extend to an image of H in G.*

Before proceeding with the proof, we quickly show that Claim 5.17 does indeed prove Theorem 5.9 when H is nonplanar. First note that $H_k, \ldots, H_0$ exist by Claim 5.15. Next, applying Claim 5.17 for $\ell = k$, we get that there are at most $c_{4.1}(h, g)c_{5.1}(h, g+2)^k \cdot n^k \leqslant c_{5.9}(h, g)n^k$ images of H in G.

Proof. We proceed by induction on ℓ. When $\ell = 0$, there are at most $c_{4.1}(h, g)$ images of $H_0 \setminus F_0$ in G which extend to an image of H in G, by Theorem 4.1. For the inductive step, suppose there are at most $c_{4.1}(h, g)c_{5.1}(h, g+2)^\ell \cdot n^\ell$ images of $H_\ell \setminus F_\ell$ in G which extend to an image of H in G, and consider $\ell + 1$.

Let $\phi : V(H_\ell \setminus F_\ell) \to V(G)$ be a fixed copy of $H_\ell \setminus F_\ell$ in G which extends to an image $\psi : V(H) \to V(G)$ of H in G. For every subgraph S of H, let S^ψ be the subgraph of G obtained by restricting ψ to S. Let $(A_{\ell+1}, B_{\ell+1})$ be the flap in $H_{\ell+1}$ such that $H_\ell = B^+_{\ell+1}$, and let $F_{A_{\ell+1}}$ be the set of fake edges of $H_{\ell+1}$ contained in $E(A_{\ell+1})$. By Claim 5.16, there is a collection of paths $\mathcal{P}^\psi := \{P^\psi_f \mid f \in F_{A_{\ell+1}}\}$ in H^ψ such that for all $f \in F_{A_{\ell+1}}$, P^ψ_f has the same ends as f^ψ and $((A_{\ell+1} \setminus F_{A_{\ell+1}})^\psi \cup \mathcal{P}^\psi, \mathcal{P}^\psi)$ is a partially subdivided subgraph of H^ψ.

Let $G_{\ell+1}$ be the minor of G obtained by contracting all but one edge from each path in $\mathcal{P}^\psi$. Since $((A_{\ell+1} \setminus F_{A_{\ell+1}})^\psi \cup \mathcal{P}^\psi)/\mathcal{P}^\psi$ is isomorphic to $A_{\ell+1}$, $(A_{\ell+1} \setminus F_{A_{\ell+1}})^\psi \cup \mathcal{P}^\psi$ becomes an image of $A_{\ell+1}$ in $G_{\ell+1}$.

Let $X := V(A_{\ell+1}) \cap V(B_{\ell+1})$ and $Y := \phi(X)$. If $X := \{a, b\}$ and ab is a fake edge of $H_{\ell+1}$, then add a handle to Σ and use the handle to draw an edge between the vertices in Y to obtain a graph $G'_{\ell+1} \supset G_{\ell+1}$ embedded in a surface of Euler genus $g+2$. Otherwise, let $G'_{\ell+1} := G_{\ell+1}$.

Note that by Claim 5.10, Claim 5.11, goodness of the reduction sequence, and tameness of $H_{\ell+1}$, the conditions of Lemma 5.8 are satisfied, with $J = A^+_{\ell+1}$. Therefore, by Lemma 5.1, there are at most $c_{5.1}(|V(A^+_{\ell+1})|, g+2)n$ images of $A^+_{\ell+1}$ with X rooted at Y in $G'_{\ell+1}$. Hence, there are at most $c_{5.1}(|V(A^+_1)|, g+2)n$ images of $H_{\ell+1} \setminus F_{\ell+1}$ in G which extend ϕ and also extend to an image of H in G. By induction, there are at most $c_{4.1}(h, g)c_{5.1}(h, g+2)^\ell \cdot n^\ell$ possibilities for ϕ, so there are at most

$$c_{5.1}(|V(A^+_1)|, g+2) \cdot c_{4.1}(h, g)c_{5.1}(h, g+2)^\ell \cdot n^{\ell+1} \leqslant c_{4.1}(h, g)c_{5.1}(h, g+2)^{\ell+1} \cdot n^{\ell+1}$$

images of $H_{\ell+1} \setminus F_{\ell+1}$ in G which extend to an image of H in G. □

The case when H is planar is similar, except that we must handle the case when H is a cycle separately, which we do now. We make a case distinction depending if the cycle has even or odd length.

Suppose $H \cong C_{2k}$. Note that $f(C_{2k}) = k$. Let M be a perfect matching of C_{2k}. For each copy ϕ of C_{2k} in G, let $\phi(M)$ be the subset of $E(G)$ that ϕ maps M to. Since there are at most $\binom{|E(G)|}{k}$ choices for $\phi(M)$, and each such choice corresponds to at most 2^k images of C_{2k} in G, there are at most $2^k\binom{|E(G)|}{k} \leqslant c_{5.9}(2k, g)n^k$ images of C_{2k} in G.

Suppose $H \cong C_{2k+1}$. Note that $f(C_{2k+1}) = k$. We prove by induction on $n = |V(G)|$ that there are at most $6^k\binom{6(g+1)}{2}(g+1)^k \cdot n^k$ images of C_{2k+1} in G. For each vertex a of C_{2k+1} let $N_{C_{2k+1}}(a)$ be the two neighbours of a, and let M_a be the unique perfect matching of $C_{2k+1} - (N_{C_{2k+1}}(a) \cup \{a\})$. Since G is embedded in a surface of Euler genus g, G has a vertex x of degree at most $6(g+1)$. For each copy ϕ of C_{2k+1} in G containing x, there are at most $\binom{|E(G-x)|}{k-1}$ choices for $\phi(M_x)$. Since x has degree at most $6(g+1)$ in G, there are at most $\binom{6(g+1)}{2}$ choices for $N_{C_{2k+1}}(x)$. Each choice of M_x and $N_{C_{2k+1}}(x)$ yields at most 2^k images of C_{2k+1}, so there are at most $2^k\binom{6(g+1)}{2}\binom{3(g+1)(n-1)}{k-1}$ images of C_{2k+1} in G containing x. By induction there are at most $6^k\binom{6(g+1)}{2}(g+1)^k \cdot (n-1)^k$ images of C_{2k+1} in $G-x$. Summing these two bounds, we conclude that there are at most $6^k\binom{6(g+1)}{2}(g+1)^k \cdot n^k$ images of C_{2k+1} in G. Note that $6^k\binom{6(g+1)}{2}(g+1)^k \leqslant c_{5.9}(2k+1, g)$.

If H is planar and $f(H) = 1$, then there are at most $c_{3.2}(h, g)n \leqslant c_{5.9}(h, g)n$ images of H in G. Therefore, the last remaining case is when H is planar, $f(H) \geqslant 2$, and H is not a cycle. This is handled by the following claim.

Claim 5.18. *Suppose H is a planar graph, $f(H) \geqslant 2$, and H is not a cycle. Let $H_k, \dots, H_2$ be a good reduction sequence of H such that H_ℓ is tame for all $\ell \in \{k, \dots, 2\}$ Let $H_1 := uv$, where uv is a true edge of H_2 such that there do not exist independent flaps (C, D) and (C', D') of H_2 such that $\{u, v\} \subseteq V(D \cap D')$. For all $\ell \in \{1, \dots, k\}$, let F_ℓ be the set of fake edges of H_ℓ. Then for each $\ell \in \{1, \dots, k\}$, there exist at most $6(g+1)c_{5.1}(h, g+2)^{\ell-1} \cdot n^\ell$ images of $H_\ell \setminus F_\ell$ in G which extend to an image of H in G.*

Before proceeding with the proof we note that $H_k, \dots, H_2$ exist by Claim 5.14. The true edge uv exists, by considering a leaf node of the SPQRK tree of H_2 and using Claim 5.11 and the tameness of H_2. Again, Theorem 5.9 follows by taking $\ell = k$.

Proof. We proceed by induction on ℓ. For $\ell = 1$, $H_1 = uv$ is a true edge. Therefore, there are at most $6(g+1)n$ images of H_1 in G. For $\ell = 2$, we apply Lemma 5.1, with $J' = (H_2 \setminus F_2) \cup \mathcal{P}_2, \mathcal{P} = \mathcal{P}_2, J = H_2$, and $X = \{u, v\}$, where $\mathcal{P}_2$ is defined in Claim 5.16. Note that conditions (1) and (2) of Lemma 5.1 hold since uv is a true edge and there do not exist independent flaps (C, D) and (C', D') of H_2 such that $\{u, v\} \subseteq V(D \cap D')$. Conditions (3)-(6) hold vacuously. Conditions (7), (9), and (10) hold by goodness of the reduction sequence. Condition (8) holds since H_2 is tame. Finally, (11) holds by Claim 5.11. Therefore, each of the at most $6(g+1)n$ images of H_1 in G extends to an image of J' in at most $c_{5.1}(h, g)n$ ways. Therefore, there are at most $6(g+1)c_{5.1}(h, g)n^2 \leqslant 6(g+1)c_{5.1}(h, g+2) \cdot n^2$ images of $H_2 \setminus F_2$ which extend to an image of H in G. For $\ell \geqslant 3$, the inductive step is exactly as in the nonplanar case. Therefore, we conclude that for each $\ell \in \{1, \dots, k\}$ there are at most $6(g+1)c_{5.1}(h, g+2)^{\ell-1} \cdot n^\ell$ images of $H_\ell \setminus F_\ell$ in G which extend to an image of H in G. □

This completes the proof of Theorem 5.9. □

6. Copies of Complete Graphs

This section studies the maximum number of copies of a given complete graph K_s in an n-vertex graph that embeds in a given surface Σ. The flap-number of K_s equals 1 if $s \leqslant 4$ and equals 0 if $s \geqslant 5$. Thus Theorem 1.2 implies that $C(n, K_s, \Sigma) = \Theta(n)$ for $s \leqslant 4$ and $C(n, K_s, \Sigma) = \Theta(1)$ for $s \geqslant 5$. The bounds obtained in this section are much more precise than those given by Theorem 1.2. Our method follows that of Dujmović et al. [16], who characterised the n-vertex graphs that embed in a given surface Σ and with the maximum number of complete subgraphs (in total), and then derived an upper bound on this maximum.

A *triangulation* of a surface Σ is an embedding of a graph in Σ in which each facial walk has three vertices and three edges with no repetitions. Let G be a triangulation of Σ. An edge vw of G is *reducible* if vw is in exactly two triangles in G. And G is *irreducible* if no edge of G is reducible [8, 9, 14, 38–40, 50, 59–61]. Barnette and Edelson [8, 9] proved that each surface has a finite number of irreducible triangulations. For $\mathbb{S}_h$ with $h \leqslant 2$ and $\mathbb{N}_c$ with $c \leqslant 4$ the list of all irreducible triangulations is known [39, 40, 59, 61]. In general, the best known upper bound on the number of vertices in an irreducible triangulation of a surface with Euler genus $g \geqslant 1$ is $13g - 4$, due to Joret and Wood [38].

Let vw be a reducible edge of a triangulation G of Σ. Let vwx and vwy be the two faces incident to vw in G. As illustrated in Figure 3, let G/vw be the graph obtained from G by *contracting* vw; that is, delete the edges vw, wy, wx, and identify v and w into v. G/vw is a simple graph since x and y are the only common neighbours of v and w. Indeed, G/vw is a triangulation of Σ. Conversely, we say that G is obtained from G/vw by *splitting* the path xvy at v. If, in addition, $xy \in E(G)$, then we say that G is obtained from G/vw by *splitting* the triangle xvy at v. Note that xvy need not be a face of G/vw. In the case that xvy is a face, splitting xvy is equivalent to adding a new vertex adjacent to each of x, v, y.

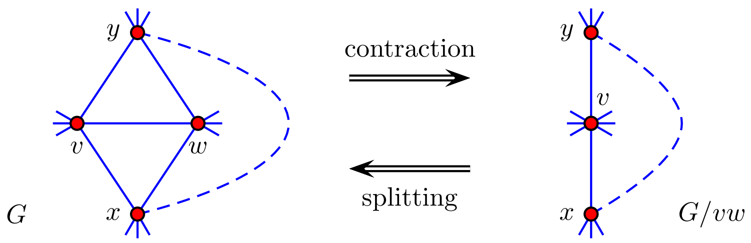

Figure 3. Contracting a reducible edge.

6.1. Copies of Triangles. In this section we consider $C(K_3, \Sigma, n)$, and define the *excess* of a graph G to be $C(K_3, G) - 3|V(G)|$.

Lemma 6.1. *For each surface Σ, every graph embeddable in Σ with maximum excess is a triangulation of Σ.*

Proof. Let G be a graph embedded in Σ that maximises the excess. We claim that G is a triangulation. Suppose on the contrary that F is a non-triangular facial walk in G.

Suppose that two vertices in F are not adjacent. Then there are vertices v and w at distance 2 in the subgraph induced by F. Thus adding the edge vw 'across' the face increases the number of triangles and the excess. This contradicts the choice of G. Now assume that F induces a clique.

Suppose that F has at least four distinct vertices. Let G' be the embedded graph obtained from G by adding one new vertex 'inside' the face adjacent to four distinct vertices of F. Thus G' is embeddable in Σ, has $|V(G)|+1$ vertices, has at least $C(K_3, G) + \binom{4}{2} = C(K_3, G) + 6$ triangles, and thus has excess at least the excess of G plus 3. This contradicts the choice of G. Now assume that F has at most three distinct vertices.

By Lemma 6.2 below, $F = (u, v, w, u, v, w)$. Let G' be the graph obtained from G by adding two new adjacent vertices p and q, where p is adjacent to the first u, v, w sequence in F, and q is adjacent to the second u, v, w sequence in F. So G' is embeddable in Σ and has $|V(G)|+2$ vertices. If S is a non-empty subset of $\{p, q\}$ and $T \subseteq \{u, v, w\}$ with $|S|+|T| = 3$, then $S \cup T$ is a triangle of G' but not of G. There are $\binom{2}{1}\binom{3}{2} + \binom{2}{2}\binom{3}{1} = 6 + 3 = 9$ such triangles. Thus $C(K_3, G') \geqslant C(K_3, G) + 9$ and the excess of G' is at least the excess of G plus 3, which contradicts the choice of G. Hence no face of G has repeated vertices, and G is a triangulation of Σ. □

Lemma 6.2. *Let F be a facial walk in an embedded graph, such that F has exactly three distinct vertices that are pairwise adjacent. Then $F = (u, v, w)$ or $F = (u, v, w, u, v, w)$.*

Proof. Say u, v, w are three consecutive vertices in F. Then $u \neq v$ and $v \neq w$ (since there are no loops). And $u \neq w$, since if $u = w$ then $\deg(v) = 1$ (since there are no parallel edges), which is not possible since v is adjacent to the two other vertices in F. So any three consecutive vertices in F are pairwise distinct. If F has no repeated vertex, then F is the 3-cycle (u, v, w). Otherwise, $F = (u, v, w, u, \dots)$. Again, since any three consecutive vertices in F are pairwise distinct, $F = (u, v, w, u, \dots)$. Repeating this argument, $F = (u, v, w, u, v, w, \dots)$. Each edge is traversed at most twice; see [49, Sections 3.2 and 3.3]. Thus $F = (u, v, w, u, v, w)$. □

Theorem 6.3. *Let ϕ be the maximum excess of an irreducible triangulation of Σ. Let X be the set of irreducible triangulations of Σ with excess ϕ. Then the excess of every graph G embeddable in Σ is at most ϕ. Moreover, the excess of G equals ϕ if and only if G is obtained from some graph in X by repeatedly splitting triangles.*

Proof. We proceed by induction on $|V(G)|$. By Lemma 6.1, we may assume that G is a triangulation of Σ. If G is irreducible, then the claim follows from the definition of X and ϕ. Otherwise, some edge vw of G is in exactly two triangles vwx and vwy. By induction, the excess of G/vw is at most ϕ. Moreover, the excess of G/vw equals ϕ if and only if G is obtained from some graph $H \in X$ by repeatedly splitting triangles.

Observe that every triangle of G that is not in G/vw is in $\{A \cup \{w\} : A \subseteq \{x, v, y\}, |A| = 2\}$. Thus $C(K_3, G) \leqslant C(K_3, G/vw) + 3$. Moreover, equality holds if and only if xvy is a triangle. It follows from the definition of excess that the excess of G is at most ϕ. If the excess of G equals ϕ, then the excess of G/vw equals ϕ, and xvy is a triangle and G is obtained from H by repeatedly splitting triangles.

Conversely, if G is obtained from some $H \in X$ by repeatedly splitting triangles, then xvy is a triangle and G/vw is obtained from H by repeatedly splitting triangles. By induction, the excess of G/vw equals ϕ, implying the excess of G equals ϕ. ☐

In general, since every irreducible triangulation of a surface Σ with Euler genus g has $O(g)$ vertices [38, 50], Theorem 6.3 implies that $C(K_3, \Sigma, n) \leqslant 3n + O(g^3)$. We now show that $C(K_3, \Sigma, n) = 3n + \Theta(g^{3/2})$.

The following elementary fact will be useful. For integers $s \geqslant 2$ and $m \geqslant 2$,

$$\sum_{i \geqslant m} \frac{1}{i^s} \leqslant \int_{m-1}^{\infty} i^{-s} di = \frac{1}{(s-1)(m-1)^{s-1}}. \tag{1}$$

Theorem 6.4. *For every surface Σ of Euler genus g,*

$$3n + (\sqrt{6} - o(1))g^{3/2} \leqslant C(K_3, \Sigma, n) \leqslant 3n + \frac{21}{2} g^{3/2} + O(g \log g),$$

where the lower bound holds for all $n \geqslant \sqrt{6g}$ and the upper bound holds for all n.

Proof. First we prove the lower bound. Because of the $o(1)$ term we may assume that $g \geqslant 4$. Let $p := \lfloor \frac{1}{2}(7 + \sqrt{24g+1}) \rfloor$. Note that $p \geqslant 8$ and $p - \frac{5}{2} > \sqrt{6g}$. The Map Colour Theorem [58] says that K_p embeds in Σ. To obtain a graph with n vertices embedded in Σ repeat the following step $n - p$ times: choose a face f and add a new vertex 'inside' f adjacent to all the vertices on the boundary of f. Each new vertex creates at least three new triangles. Thus $C(K_3, \Sigma, n) \geqslant 3(n-p) + \binom{p}{3}$ for $n \geqslant p$. Since $p \geqslant 8$ we have $\binom{p}{3} - 3p \geqslant \frac{1}{6}(p - \frac{5}{2})^3 \geqslant \sqrt{6} g^{3/2}$. Thus $C(K_3, \Sigma, n) \geqslant 3n + \sqrt{6} g^{3/2}$.

To prove the upper bound, by Lemma 6.1, it suffices to consider an n-vertex triangulation G of Σ. First suppose that $n > 13g$. Then G contains an edge e so that G/e is another triangulation [38]. Then $C(K_3, G) \leqslant C(K_3, G/e) + 3$. Since G/e has $n - 1$ vertices, the result follows by induction. Now assume that $n \leqslant 13g$. Let $v_1, \dots, v_n$ be a vertex ordering of G, where v_i has minimum degree in $G_i := G[\{v_1, \dots, v_i\}]$. By Euler's formula, $i \cdot \deg_{G_i}(v_i) \leqslant 2|E(G_i)| \leqslant 6(i + g)$, implying

$$\deg_{G_i}(v_i) \leqslant 6\left(1 + \frac{g}{i}\right).$$

Let $m := \lceil 3\sqrt{g} \rceil$. The number of triangles $v_a v_b v_i$ with $a < b < i \leqslant m$ is at most $\binom{m}{3} \leqslant \binom{3\sqrt{g}+1}{3} \leqslant \frac{9}{2} g^{3/2}$. Charge each triangle $v_a v_b v_i$ with $a < b < i$ and $i \geqslant m + 1$ to vertex v_i. For $m+1 \leqslant i \leqslant n$, the number of triangles charged to v_i is at most

$$\binom{\deg_{G_i}(v_i)}{2} < 18\left(1+\frac{g}{i}\right)^2 = 18\left(1+\frac{2g}{i}+\frac{g^2}{i^2}\right).$$

Thus

$$\begin{aligned} C(K_3,G) &\leqslant \frac{9}{2}g^{3/2} + 18\sum_{i=m+1}^{n}\left(1+\frac{2g}{i}+\frac{g^2}{i^2}\right) \\ &\leqslant \frac{9}{2}g^{3/2} + 18n + 36g(\ln(n)+1) + 18g^2\sum_{i\geqslant m+1}\frac{1}{i^2}. \end{aligned}$$

By (1) with $s=2$,

$$C(K_3,G) \leqslant \frac{9}{2}g^{3/2} + 18n + 36g + 36g\ln(n) + \frac{18g^2}{m}.$$

Since $m \geqslant 3\sqrt{g}$ and $n \leqslant 13g$,

$$C(K_3,G) \leqslant \frac{9}{2}g^{3/2} + 270g + 36g\ln(13g) + 6g^{3/2} = \frac{21}{2}g^{3/2} + 270g + 36g\ln(13g). \quad \square$$

6.2. Copies of K_4. In this section, we consider the case $H = K_4$, and define the *excess* of a graph G to be $C(K_4,G) - |V(G)|$.

Lemma 6.5. *For each surface Σ, every graph embeddable in Σ with maximum excess is a triangulation of Σ.*

Proof. Let G be a graph embedded in Σ with maximum excess. We claim that G is a triangulation.

Suppose that some facial walk F contains non-adjacent vertices v and w. Let G' be the graph obtained from G by adding the edge vw. Thus $C(K_4,G') \geqslant C(K_4,G)$. If two common neighbours of v and w are adjacent, then $C(K_4,G+vw) > C(K_4,G)$, implying that the excess of $G+vw$ is greater than the excess of G, which contradicts the choice of G. Now assume that no two common neighbours of v and w are adjacent. Let $G'' := G'/vw$. Every K_4 subgraph in G' is also in G''. Thus $C(K_4,G'') \geqslant C(K_4,G') \geqslant C(K_4,G)$. Since $|V(G'')| < |V(G)|$, the excess of G'' is greater than the excess of G, which contradicts the choice of G. Now assume that every facial walk induces a clique in G.

Suppose that some facial walk F has at least four distinct vertices. Let G' be the embedded graph obtained from G by adding one new vertex 'inside' the face adjacent to four distinct vertices of F. Thus G' is embeddable in Σ, has $|V(G)|+1$ vertices, has at least $C(K_4,G)+\binom{4}{3} = C(K_4,G)+4$ triangles, and thus has excess at least the excess of G plus 3. This contradicts the choice of G. Now assume that every facial walk in G has at most three distinct vertices.

Suppose that some facial walk F is not a triangle. By Lemma 6.2, $F = (u, v, w, u, v, w)$. Let G' be the graph obtained from G by adding two new adjacent vertices p and q, where p is adjacent to the first u, v, w sequence in F, and q is adjacent to the second u, v, w sequence in F. So G' is embeddable in Σ and has $|V(G)| + 2$ vertices. If S is a non-empty subset of $\{p, q\}$ and $T \subseteq \{u, v, w\}$ with $|S| + |T| = 4$, then $S \cup T$ induces a copy of K_4 in G' but not in G. There are $\binom{2}{2}\binom{3}{2} + \binom{2}{1}\binom{3}{3} = 3 + 2 = 5$ such copies. Thus $C(K_4, G') \geqslant C(K_4, G) + 5$ and the excess of G' is at least the excess of G plus 3, which contradicts the choice of G. Therefore G is a triangulation of Σ. □

Theorem 6.6. *Let ϕ be the maximum excess of an irreducible triangulation of Σ. Let X be the set of irreducible triangulations of Σ with excess ϕ. Then the excess of every graph G embeddable in Σ is at most ϕ. Moreover, the excess of G equals ϕ if and only if G is obtained from some graph in X by repeatedly splitting triangles.*

Proof. We proceed by induction on $|V(G)|$. By Lemma 6.5, we may assume that G is a triangulation of Σ. If G is irreducible, then the claim follows from the definition of X and ϕ. Otherwise, some edge vw of G is in exactly two triangles vwx and vwy. By induction, the excess of G/vw is at most ϕ. Moreover, the excess of G/vw equals ϕ if and only if G is obtained from some graph $H \in X$ by repeatedly splitting triangles.

Observe that every clique of G that is not in G/vw is in $\{A \cup \{w\} : A \subseteq \{x, v, y\}\}$. Thus $C(K_4, G) \leqslant C(K_4, G/vw) + 1$. Moreover, equality holds if and only if xvy is a triangle. It follows from the definition of excess that the excess of G is at most ϕ. If the excess of G equals ϕ, then the excess of G/vw equals ϕ, and xvy is a triangle, and G is obtained from H by repeatedly splitting triangles.

Conversely, if G is obtained from some $H \in X$ by repeatedly splitting triangles, then xvy is a triangle and G/vw is obtained from H by repeatedly splitting triangles. By induction, the excess of G/vw equals ϕ, implying the excess of G equals ϕ. □

Since every irreducible triangulation of a surface Σ with Euler genus g has $O(g)$ vertices [38, 50], Theorem 6.6 implies that $C(K_4, \Sigma, n) \leqslant n + O(g^4)$. We now show that $C(K_4, \Sigma, n) = n + \Theta(g^2)$.

Theorem 6.7. *For every surface Σ of Euler genus g,*

$$n + \frac{3}{2}g^2 \leqslant C(K_4, \Sigma, n) \leqslant n + \frac{283}{24}g^2 + O(g^{3/2}),$$

where the lower bound holds for $g \geqslant 1$ and $n \geqslant \sqrt{6g}$, and the upper bound holds for all n.

Proof. First we prove the lower bound. If $\Sigma = \mathbb{N}_2$ then let $p := 6$. Otherwise, let $p := \lfloor \frac{1}{2}(7 + \sqrt{24g + 1}) \rfloor$. Since $g \geqslant 1$ we have $p \geqslant 6$. The Map Colour Theorem [58] says that K_p embeds in Σ. To obtain a graph with n vertices embedded in Σ repeat the following step $n - p$ times: choose a face f and add a new vertex 'inside' f adjacent to all the vertices on the boundary of f. Each new vertex creates at least one new copy of K_4 (since the boundary of each face is always a clique on at least three vertices). Thus $C(K_4,\Sigma,n) \geqslant n-p+\binom{p}{4}$ for $n \geqslant p$. Since $\binom{p}{4}-p \geqslant \frac{1}{24}(p-\frac{5}{2})^4$ and $p-\frac{5}{2} > \sqrt{6g}$ we have $C(K_4,\Sigma,n) \geqslant n+\frac{1}{24}(\sqrt{6g})^4 = n+\frac{3}{2}g^2$.

Now we prove the upper bound. The claim is trivial for $g=0$, so now assume that $g \geqslant 1$. By Lemma 6.5, it suffices to consider an irreducible triangulation G. Joret and Wood [38] proved that $n := |V(G)| \leqslant 13g$. Let $v_1,\dots,v_n$ be a vertex ordering of G, where v_i has minimum degree in $G_i := G[\{v_1,\dots,v_i\}]$. By Euler's formula,

$$i \cdot \deg_{G_i}(v_i) \leqslant 2|E(G_i)| \leqslant 6(i+g),$$

and

$$\deg_{G_i}(v_i) \leqslant 6\left(1+\frac{g}{i}\right).$$

Define $m := \lceil 4\sqrt{g}\rceil$. The number of copies $v_av_bv_cv_i$ with $a<b<c<i\leqslant m$ is at most $\binom{m}{4} \leqslant \binom{4\sqrt{g}+1}{4} \leqslant \frac{32}{3}g^2$. Charge each copy $v_av_bv_cv_i$ with $a<b<c<i$ and $i \geqslant m+1$ to vertex v_i. For $m+1 \leqslant i \leqslant n$, the number of copies charged to v_i is at most

$$\binom{\deg_{G_i}(v_i)}{3} < 36\left(1+\frac{g}{i}\right)^3 = 36\left(\left(\frac{g}{i}\right)^3+3\left(\frac{g}{i}\right)^2+3\left(\frac{g}{i}\right)+1\right).$$

In total,

$$C(K_4,G) \leqslant \frac{32}{3}g^2 + 36\sum_{i=m+1}^{n}\left(\frac{g}{i}\right)^3+3\left(\frac{g}{i}\right)^2+3\left(\frac{g}{i}\right)+1.$$

By (1) with $s=2$ and $s=3$,

$$C(K_4,G) \leqslant \frac{32}{3}g^2 + 36\left(\frac{g^3}{2m^2}+\frac{3g^2}{m}+3g(\ln n+1)+n\right).$$

Since $m \geqslant 4\sqrt{g}$ and $n \leqslant 13g$,

$$\begin{aligned} C(K_4,G) &\leqslant \frac{32}{3}g^2 + 36\left(\frac{g^2}{32}+\frac{3g^{3/2}}{4}+3g(\ln(13g)+1)+13g\right) \\ &= \frac{283}{24}g^2+27g^{3/2}+108g(\ln(13g)+1)+468g. \end{aligned}$$

□

6.3. General Complete Graph. Now consider the case when $H=K_s$ for some $s \geqslant 5$. Theorem 1.2 shows that $C(K_s,\Sigma,n)$ is bounded for fixed s and Σ. We now show how to determine $C(K_s,\Sigma,n)$ more precisely.

Theorem 6.8. *For every integer $s \geqslant 5$ and surface Σ there is an irreducible triangulation G such that $C(K_s,G) = \max_n C(K_s,\Sigma,n)$.*

Proof. Let $q := \max_n C(K_s,\Sigma,n)$. Let G_0 be a graph embedded in Σ with $C(K_s,G_0) = q$. As described in the proof of Lemma 6.1 we can add edges and vertices to G_0 to create a triangulation G of Σ. Adding edges and vertices does not remove copies of K_s. Thus $C(K_s,G)=q$. If G is irreducible, then we are done. Otherwise, some edge vw of G is in exactly two triangles vwx and vwy. Let $G' := G/vw$. Then G' is another triangulation of Σ. Observe that every clique of G that is not in G' is in $\{A \cup \{w\} : A \subseteq \{x, v, y\}\}$. Each such clique has at most four vertices. Thus $C(K_s, G') = C(K_s, G) = q$. Repeat this step to G' until we obtain an irreducible triangulation G'' with $C(K_s, G'') = q$. $\square$

We now prove a precise bound on $C(K_s, \Sigma, n)$, making no effort to optimise the constant 300.

Theorem 6.9. *For every integer $s \geqslant 5$ and surface Σ of Euler genus g and for all n,*

$$\left(\frac{\sqrt{6g}}{s}\right)^s \leqslant C(K_s, \Sigma, n) \leqslant \left(\frac{300\sqrt{g}}{s}\right)^s,$$

where the lower bound holds for all $n \geqslant \sqrt{6g} \geqslant s$ and the upper bound holds for all n.

Proof. For the lower bound, it follows from the Map Colour Theorem [58] that K_p embeds in Σ where $p := \lceil \sqrt{6g} \rceil$. Thus, for $n \geqslant p \geqslant s$,

$$C(K_s, \Sigma, n) \geqslant \binom{\sqrt{6g}}{s} \geqslant \left(\frac{\sqrt{6g}}{s}\right)^s.$$

Now we prove the upper bound. The claim is trivial for $g = 0$, so assume that $g \geqslant 1$. By Theorem 6.8, it suffices to consider an irreducible triangulation G of Σ. Joret and Wood [38] proved that $n := |V(G)| \leqslant 13g$. Let $v_1, \dots, v_n$ be a vertex ordering of G, where v_i has minimum degree in $G_i := G[\{v_1, \dots, v_i\}]$. By Euler's formula,

$$i \cdot \deg_{G_i}(v_i) \leqslant 2|E(G_i)| \leqslant 6(i + g) \leqslant 6(n + g) \leqslant 84g.$$

Define $m := \lceil \sqrt{g} \rceil$. The number of copies of K_s in $G[\{v_1, \dots, v_m\}]$ is at most

$$\binom{m}{s} \leqslant \left(\frac{2e\sqrt{g}}{s}\right)^s \leqslant \left(\frac{2e}{s}\right)^s g^{s/2}.$$

Charge every other copy X of K_s to the rightmost vertex in X (with respect to $v_1, \dots, v_n$). For $m + 1 \leqslant i \leqslant n$, the number of copies of K_s charged to v_i is at most

$$\binom{\deg_{G_i}(v_i)}{s-1} \leqslant \left(\frac{e \deg_{G_i}(v_i)}{s-1}\right)^{s-1} \leqslant \left(\frac{84eg}{i(s-1)}\right)^{s-1}.$$

In total,

$$C(K_s, G) \leqslant \left(\frac{2e}{s}\right)^s g^{s/2} + \left(\frac{84eg}{s-1}\right)^{s-1} \sum_{i \geqslant m+1} \frac{1}{i^{s-1}}.$$

By (1),

$$C(K_s, G) \leqslant \left(\frac{2e}{s}\right)^s g^{s/2} + \left(\frac{84eg}{s-1}\right)^{s-1} \frac{1}{(s-2)m^{s-2}}.$$

Since $m \geqslant \sqrt{g}$,

$$C(K_s, G) \leqslant \left(\frac{2e}{s}\right)^s g^{s/2} + \left(\frac{84eg}{s-1}\right)^{s-1} \frac{1}{(s-2)\, g^{(s-2)/2}} \leqslant \left(\frac{300\sqrt{g}}{s}\right)^s. \qquad \square$$

6.4. Computational Results. For $\Sigma \in \{\mathbb{S}_0, \mathbb{S}_1, \mathbb{S}_2, \mathbb{N}_1, \mathbb{N}_2, \mathbb{N}_3, \mathbb{N}_4\}$, we use Lemmas 6.1 and 6.5 and Theorem 6.8, the lists of all irreducible triangulations [39, 40, 59, 61], and an elementary computer program to count cliques to obtain the exact results for $C(K_s, \Sigma, n)$ shown in Table 1.

Table 1. The maximum number of copies of K_s in an n-vertex graph embeddable in surface Σ.

Σ	$s=0$	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$	$s=8$	total
$\mathbb{S}_0$	1	n	$3n-6$	$3n-8$	$n-3$					$8n-16$
$\mathbb{S}_1$	1	n	$3n$	$3n+14$	$n+28$	21	7	1		$8n+72$
$\mathbb{S}_2$	1	n	$3n+6$	$3n+38$	$n+68$	58	28	8	1	$8n+208$
$\mathbb{N}_1$	1	n	$3n-3$	$3n+2$	$n+9$	6	1			$8n+16$
$\mathbb{N}_2$	1	n	$3n$	$3n+12$	$n+21$	12	2			$8n+48$
$\mathbb{N}_3$	1	n	$3n+3$	$3n+24$	$n+40$	27	8	1		$8n+104$
$\mathbb{N}_4$	1	n	$3n+6$	$3n+39$	$n+71$	61	29	8	1	$8n+216$

Let $C(G)$ be the total number of complete subgraphs in a graph G; that is $C(G) = \sum_{s\geqslant 0} C(K_s, G)$. For a surface Σ, let $C(\Sigma, n)$ be the maximum of $C(G)$ taken over all n-vertex graphs G embeddable in Σ. Dujmović et al. [16] proved that $C(\Sigma, n) - 8n$ is bounded for fixed Σ, which is implied by Theorems 6.4, 6.7 and 6.9. The following conjectures have been verified for each of $\mathbb{S}_0$, $\mathbb{S}_1$, $\mathbb{S}_2$, $\mathbb{N}_1$, $\mathbb{N}_2$, $\mathbb{N}_3$, $\mathbb{N}_4$.

Conjecture 6.10. *For every surface Σ and integer n,*

$$C(\Sigma, n) = \sum_{s\geqslant 0} C(K_s, \Sigma, n).$$

Conjecture 6.11. *If $C(G) = C(\Sigma, n)$ for some n-vertex graph G embeddable in a surface Σ, then for $s \geqslant 0$,*

$$C(K_s, G) = C(K_s, \Sigma, n).$$

Conversely, we conjecture that maximising the number of triangles is equivalent to maximising the total number of complete subgraphs. More precisely:

Conjecture 6.12. *If $C(K_3, G) = C(K_3, \Sigma, n)$ for some n-vertex graph G embeddable in a surface Σ, then*

$$C(G) = C(\Sigma, n).$$

Note that K_3 cannot be replaced by some arbitrary complete graph in Conjecture 6.12. For example, every graph embeddable in $\mathbb{N}_3$ contains at most one copy of K_7, but there are irreducible triangulations G of $\mathbb{N}_3$ that contain K_7 and do not maximise the total number of cliques (that is, $C(G) < 8|V(G)| + 104$). Similarly, every graph embeddable in $\mathbb{N}_4$ contains at most 8 copies of K_7, but there are irreducible triangulations G of $\mathbb{N}_4$ for which $C(K_7, G) = 8$ and $C(G) < 8|V(G)| + 216$.

7. Minor-Closed Classes

Consider the following natural open problem extending our results for graphs on surfaces: For graphs H and X and an integer n, what is the maximum number of copies of H in an n-vertex X-minor-free graph? This problem has been extensively studied when H and X are complete graphs [21, 22, 41, 54, 56, 65]. Eppstein [17] proved the following result when X is a complete bipartite graph and H is highly connected.

Theorem 7.1 ([17]). *Fix positive integers $s \leqslant t$ and a $K_{s,t}$-minor-free graph H with no $(\leqslant s-1)$-separation. Then every n-vertex $K_{s,t}$-minor-free graph contains $O(n)$ copies of H.*

What happens when H is not highly connected? We have the following lower bound. Fix positive integers $s \leqslant t$ and a $K_{s,t}$-minor-free graph H. If H has no $(\leqslant s-1)$-separation, then let $k := 1$; otherwise, let k be the maximum number of pairwise independent $(\leqslant s-1)$-separations in H. The construction in Section 2 generalises to give n-vertex $K_{s,t}$-minor-free graphs containing $\Theta(n^k)$ copies of H.

The following question naturally arises: Does every n-vertex $K_{s,t}$-minor-free graph contain $O(n^k)$ copies of H? By Theorem 7.1, the answer is 'yes' if $k = 1$. The methods presented in this paper show the answer is 'yes' if $s \leqslant 3$. We omit the proof, since it is essentially the same as for graphs embedded on a surface, except that in the $k = 1$ case we use Theorem 7.1 instead of the additivity of Euler genus (Theorem 3.3).

When H is a tree, this problem specialises as follows: Fix a tree T and positive integers $s \leqslant t$. Let $\beta(T)$ be the size of the largest stable set of vertices in T, each with degree at most $s-1$. The construction in Corollary 1.3 generalises to give n-vertex $K_{s,t}$-minor-free graphs containing $\Omega(n^{\beta(T)})$ copies of T. Does every n-vertex $K_{s,t}$-minor-free graph contain $O(n^{\beta(T)})$ copies of T?

8. Homomorphism Inequalities

This section reinterprets the results of this paper in terms of homomorphism inequalities, and presents some open problems that arise from this viewpoint.

For two graphs H and G, a *homomorphism* from H to G is a function $\phi : V(H) \to V(G)$ that preserves adjacency; that is, $\phi(v)\phi(w)$ is an edge of G for each edge vw of H. Let $\hom(H, G)$ be the number of homomorphisms from H to G. For example, $\hom(H, K_t) > 0$ if and only if H is t-colourable. In the other direction, $\hom(K_1, G)$ is the number of vertices in G, and $\hom(K_2, G)$ is twice the number of edges in G, and $\hom(K_3, G)$ is 6 times the number of triangles in G.

Homomorphism inequalities encode bounds on the number of copies of given graphs in a host graph. Much of extremal graph theory can be written in terms of homomorphism inequalities, and a beautiful theory has recently developed that greatly simplifies the task of proving such inequalities; see [42].

Consider the following concrete example. Mantel [47] proved that every n-vertex graph with more than $\frac{n^2}{4}$ edges has a triangle, which is tight for the complete bipartite graph $K_{n/2,n/2}$. Goodman [27] strengthened Mantel's Theorem by providing a lower bound of $\frac{m}{3}(\frac{4m}{n}-n)$ on the number of triangles in an n-vertex m-edge graph. Goodman's Theorem can be rewritten as the following homomorphism inequality:

$$\hom(K_1,G)\hom(K_3,G) \geqslant \hom(K_2,G)(2\hom(K_2,G)-\hom(K_1,G)^2). \tag{2}$$

In a celebrated application of the flag algebra method, Razborov [55] generalised (2) by determining the minimum number of triangles in an n-vertex m-edge graph. The minimum number of copies of K_r in an n-vertex m-edge graph (the natural extension of Turan's Theorem) was a notoriously difficult question [43, 44], recently solved for $r = 4$ by Nikiforov [53] and in general by Reiher [57]. All of these results can be written in terms of homomorphism inequalities.

The results of this paper show that for every fixed graph H with flap-number k, and for every graph G that embeds in a fixed surface Σ,

$$\hom(H,G) \leqslant c_1 \hom(K_1,G)^k;$$

and if H embeds in Σ, then $\hom(H,G) \geqslant c_2 \hom(K_1,G)^k$ for infinitely many graphs G that also embed in Σ.

Here is another example of a homomorphism inequality for graphs on surfaces. Euler's Formula implies[3] that the number of triangles in an n-vertex m-edge graph with Euler genus g is at least $2(m-2n+4-2g)$. This result is an analogue of Goodman's Theorem for graphs G of Euler genus g, and can be written as the following homomorphism inequality:

$$\hom(K_3,G) \geqslant 6\hom(K_2,G) - 24\hom(K_1,G) + 48 - 24g.$$

We consider it an interesting line of research to prove similar homomorphism inequalities in other minor-closed classes. The following open problems naturally arise.

- Is there a method (akin to flag algebras [55] or graph algebras [42]) for systematically proving homomorphism inequalities in minor-closed classes?
- Hatami and Norine [37] proved that it is undecidable to test the validity of a linear homomorphism inequality. In which minor-closed classes is it decidable to test the validity of a linear homomorphism inequality?

These questions are open even for forests; see [12, 13, 15] for related results.

[3]Let G be a graph with n vertices, m edges and c components. Let Σ be a surface with Euler genus g. Assume that G embeds in Σ with t triangular faces and f non-triangular faces. By Euler's formula, $n-m+t+f = 1+c-g$. Double-counting edges, $3t+4f \leqslant 2m$. Thus $4(m-n-t+1+c-g) = 4f \leqslant 2m-3t$ and $t \geqslant 2m-4n+4+4c-4g \geqslant 2(m-2n+4-2g)$, as claimed.

Closely related to the study of graph homomorphisms is the theory of graph limits and graphons [42]. While this theory focuses on dense graphs, a theory of graph limits for sparse graphs is emerging. For example, results are known for bounded degree graphs [11, 36], planar graphs [10, 28], and bounded tree-depth graphs [52]. The above questions regarding graph homomorphisms parallel the theory of graph limits in sparse classes.

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(T. Huynh and D.R. Wood)
School of Mathematics
Monash University
Melbourne, Australia

Email address: {tony.bourbaki@gmail.com, david.wood@monash.edu}

(G. Joret)
Département d'Informatique
Université libre de Bruxelles
Brussels, Belgium

Email address: gjoret@ulb.ac.be