

A general framework for hypergraph colouring

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Abstract

The Lovász Local Lemma is a powerful probabilistic technique for proving the existence of combinatorial objects. It is especially useful for colouring graphs and hypergraphs with bounded maximum degree. This paper presents a general theorem for colouring hypergraphs that in many instances matches or slightly improves upon the bounds obtained using the Lovász Local Lemma. Moreover, the theorem directly shows that there are exponentially many colourings. The elementary and self-contained proof is inspired by a recent result for nonrepetitive colourings by Rosenfeld [2020]. We apply our general theorem in the setting of proper hypergraph colouring, proper graph colouring, independent transversals, star colouring, nonrepetitive colouring, frugal colouring, Ramsey number lower bounds, and for k -SAT.

1 Hypergraph Colouring

In their seminal 1975 paper, Erdős and Lovász [25] introduced what is now called the Lovász Local Lemma. This tool is one of the most powerful probabilistic techniques for proving the existence of combinatorial objects. Their motivation was hypergraph colouring. A *hypergraph* G consists of a set $V(G)$ of *vertices* and a set $E(G)$ of *edges*, each of which is a subset of $V(G)$. A *colouring* of a hypergraph G is a function that assigns a ‘colour’ to each vertex of G . A colouring of G is *proper* if no edge of G is monochromatic. The *chromatic number* $\chi(G)$ is the minimum number of colours in a proper colouring of G . The *degree* of a vertex v in a hypergraph G is the number of edges that contain v . A hypergraph is *r -uniform* if each edge has size r . Erdős and Lovász [25] proved (using the Lovász Local Lemma) that $\chi(G) \leq \lceil (4r\Delta)^{1/(r-1)} \rceil$ for every r -uniform hypergraph G with maximum degree Δ . The following result is a consequence of the strengthened Lovász Local Lemma first stated by Spencer [60]; see the book by Molloy and Reed [46] for a comprehensive treatment.

Theorem 1 ([25, 60]). *For every r -uniform hypergraph G with maximum degree Δ ,*

$$\chi(G) \leq \lceil (e(r(\Delta - 1) + 1))^{1/(r-1)} \rceil.$$

This paper presents a general theorem for colouring hypergraphs, which in the special case of proper hypergraph colouring, (slightly) improves the upper bound in Theorem 1.

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Moreover, the proof directly shows that there are exponentially many such colourings. The proof uses a simple counting argument inspired by a recent result for nonrepetitive colourings by Rosenfeld [58], which in turn is inspired by the power series method for pattern avoidance [13, 49, 51].

It is well known that the proof of Theorem 1 works in the setting of list colourings, which we now introduce. Let G be a hypergraph. A *list-assignment* for G is a function L that assigns each vertex v of G a set $L(v)$, whose elements are called *colours*. If $|L(v)| = c$ for each vertex v of G , then L is a *c -list-assignment*. An *L -colouring* of G is a function ϕ such that $\phi(v) \in L(v)$ for each vertex v of G . The *choosability* $\chi_{\text{ch}}(G)$ is the minimum integer c such that G has a proper L -colouring for every c -list-assignment L of G . For a list assignment L of a hypergraph G , let $P(G, L)$ be the number of proper L -colourings of G .

The following theorem is our first contribution.

Theorem 2. *For all integers $r \geq 3$ and $\Delta \geq 1$, and for every r -uniform hypergraph G with maximum degree Δ ,*

$$\chi_{\text{ch}}(G) \leq c := \left\lceil \left(\frac{r-1}{r-2} \right) ((r-2)\Delta)^{1/(r-1)} \right\rceil.$$

Moreover, for every c -list assignment L of G ,

$$P(G, L) \geq ((r-2)\Delta)^{|V(G)|/(r-1)}.$$

We now compare the above-mentioned bounds. Since $(\frac{r-1}{r-2})^{r-2} < e$, it follows that $(\frac{r-1}{r-2})((r-2)\Delta)^{1/(r-1)} < (e(r-1)\Delta)^{1/(r-1)}$, and assuming $\Delta \geq r-1$, the bound in Theorem 2 is slightly better than the bound in Theorem 1. The difference is most evident for small r . For example, if $r = 3$ then the bound in Theorem 2 is $\lceil 2\sqrt{\Delta} \rceil$ compared with $\lceil \sqrt{e(3\Delta-2)} \rceil$ from Theorem 1.

Several researchers have communicated to us that, with a little effort, one can conclude the existence of exponentially many colourings using the Lovász Local Lemma (or other methods), although as far as we are aware no general result of this nature is published. One attraction of our proof is that it gives exponentially many colourings for free. Indeed, this stronger conclusion is a key to enabling the simple proof. See [1, 2, 5, 8, 16, 29, 38, 48, 50] for more results on colouring hypergraphs with given maximum degree or number of edges, and see [21, 22, 39, 43, 61, 62] for other theorems showing the existence of exponentially many colourings in various graph settings.

Theorem 2 is a special case of a more general result that we introduce in the following section. Then, in Section 3, we apply this general result to a variety of colouring problems, including hypergraph colouring, graph colouring, independent transversals, star colouring, nonrepetitive colouring, frugal colouring, Ramsey number lower bounds, and k -SAT. Section 4 concludes by comparing our general result with other techniques including the Lovász Local Lemma and entropy compression.

2 General Framework

For a hypergraph G (allowing parallel edges), let \mathcal{C}_G be the set of all colourings $\phi : V(G) \rightarrow \mathbb{Z}$. (For concreteness, we assume all colours are integers.) For an edge e of G , let \mathcal{C}_e be the set of all colourings $\phi : e \rightarrow \mathbb{Z}$. An *instance* is a pair (G, \mathcal{B}) where G is a hypergraph and $\mathcal{B} = (\mathcal{B}_e \subseteq \mathcal{C}_e : e \in E(G))$. A colouring $\phi \in \mathcal{C}_G$ is *\mathcal{B} -bad* if, for some edge $e \in E(G)$, we have that ϕ restricted to e is in \mathcal{B}_e . Every other colouring in \mathcal{C}_G is *\mathcal{B} -good*. For an integer $c \geq 1$, we say G is *(\mathcal{B}, c) -choosable* if there is a \mathcal{B} -good L -colouring of G for every c -list assignment L of G . For a list assignment L of G , let $P(G, \mathcal{B}, L)$ be the number of \mathcal{B} -good L -colourings of G .

Fix an instance (G, \mathcal{B}) and consider an edge e of G . A subset $S \subseteq e$ *determines* \mathcal{B}_e if any two colourings in \mathcal{B}_e that agree on S are identical. For every vertex v in e , we assume that \mathcal{B}_e is determined by some subset of $e \setminus \{v\}$. (Consider this assumption to be part of the definition of ‘instance’.) Then define the *weight* of (v, e) to be $|e| - 1 - |S|$, where S is a minimum-sized subset of $e \setminus \{v\}$ that determines \mathcal{B}_e . For each vertex v of G , let $E_k(v)$ be the number of pairs (v, e) with weight k .

For example, to model proper colouring in an r -uniform hypergraph G , for each edge e of G , let \mathcal{B}_e be the monochromatic colourings in \mathcal{C}_e . Then a colouring is \mathcal{B} -good if and only if it is proper. For every edge e and every vertex v in e , if w is any vertex in $e \setminus \{v\}$, then $\{w\}$ determines \mathcal{B}_e , implying that (v, e) has weight $r - 2$.

Theorem 3. *Let (G, \mathcal{B}) be an instance. Assume there exist a real number $\beta \geq 1$ and an integer $c \geq 1$ such that for every vertex v of G ,*

$$c \geq \beta + \sum_{k \geq 0} \beta^{-k} E_k(v). \quad (1)$$

Then G is (\mathcal{B}, c) -choosable. Moreover, for every c -list assignment L of G ,

$$P(G, \mathcal{B}, L) \geq \beta^{|V(G)|}.$$

Before proving Theorem 3 we make a couple of minor observations. If $\beta > 1$ then Theorem 3 guarantees exponentially many \mathcal{B} -good colourings. If $\beta = 1$ then Theorem 3 guarantees at least one \mathcal{B} -good colouring. In most applications $\beta > 1$, but on one occasion the case $\beta = 1$ is of interest (see Section 3.1). When applying Theorem 3 it is not necessary to determine the weight of a pair exactly; it suffices to determine a lower bound on the weight (because of the β^{-k} term in (1), where $\beta \geq 1$).

Theorem 3 is an immediate corollary of the following lemma. If (G, \mathcal{B}) is an instance with $\mathcal{B} = (\mathcal{B}_e : e \in E(G))$, and H is a sub-hypergraph of G , then (H, \mathcal{B}) refers to the instance $(H, (\mathcal{B}_e : e \in E(H)))$. Similarly, if L is a list-assignment for G , then we consider L (restricted to $V(H)$) to be a list-assignment for H .

Lemma 4. *Let (G, \mathcal{B}) be an instance. Assume there exist a real number $\beta \geq 1$ and an integer $c \geq 1$ such that (1) holds for every vertex v of G . Then for every c -list assignment L of G , for every induced sub-hypergraph H of G , and for every vertex v of H ,*

$$P(H, \mathcal{B}, L) \geq \beta P(H - v, \mathcal{B}, L).$$

Proof. We proceed by induction on $|V(H)|$. The base case with $|V(H)| = 1$ is trivial. Let H be an induced sub-hypergraph of G , and assume the claim holds for all induced sub-hypergraphs of G with less than $|V(H)|$ vertices. Let v be any vertex of H . Let X be the set of \mathcal{B} -bad L -colourings of H that are \mathcal{B} -good on $H - v$. Then

$$P(H, \mathcal{B}, L) = cP(H - v, \mathcal{B}, L) - |X|. \quad (2)$$

We now find an upper bound for $|X|$. For each L -colouring ϕ in X there is an edge $e \in E(H)$ containing v such that $\phi \in \mathcal{B}_e$ (if there are several options for e , fix a choice arbitrarily). Charge ϕ to (v, e) . Let X_k be the set of colourings in X that are charged to a pair with weight k . Consider ϕ in X_k charged to (v, e) . Let S be a minimum-sized subset of $e \setminus \{v\}$ that determines \mathcal{B}_e . Let $T := e \setminus S$. Then $|T| = k + 1$ and $v \in T$. Since ϕ is \mathcal{B} -good on $H - v$, we know that ϕ is also \mathcal{B} -good on $H - T$. Since S determines \mathcal{B}_e , the number of L -colourings in X_k charged to (v, e) is at most $P(H - T, \mathcal{B}, L)$. By induction,

$$P(H - v, \mathcal{B}, L) \geq \beta^k P(H - T, \mathcal{B}, L).$$

Thus the number of L -colourings in X_k charged to (v, e) is at most $\beta^{-k} P(H - v, \mathcal{B}, L)$. Hence $|X_k| \leq E_k(v) \beta^{-k} P(H - v, \mathcal{B}, L)$, and

$$|X| = \sum_{k \geq 0} |X_k| \leq P(H - v, \mathcal{B}, L) \sum_{k \geq 0} E_k(v) \beta^{-k}.$$

By (2),

$$P(H, \mathcal{B}, L) \geq cP(H - v, \mathcal{B}, L) - P(H - v, \mathcal{B}, L) \sum_{k \geq 0} \beta^{-k} E_k(v).$$

By (1), $P(H, \mathcal{B}, L) \geq \beta P(H - v, \mathcal{B}, L)$, as desired. \square

3 Examples

In this section, we apply Theorem 3 for various types of (hyper)graph colouring problems and for k -SAT. In most cases, Theorem 3 matches or improves on the best known bound on the number of colours (as a function of maximum degree), and in addition shows that there are exponentially many colourings.

3.1 Proper Colouring

First we prove Theorem 2. Let G be an r -uniform hypergraph with maximum degree Δ where $r \geq 3$. For each edge e of G , let \mathcal{B}_e be the monochromatic colourings in \mathcal{C}_e ; then a colouring is \mathcal{B} -good if and only if it is proper. Each pair (v, e) has weight $r - 2$, and $E_{r-2}(v) \leq \Delta$. Observe that (1) holds with $\beta := ((r - 2)\Delta)^{1/(r-1)}$ and $c := \lceil \frac{r-1}{r-2} ((r - 2)\Delta)^{1/(r-1)} \rceil$. Theorem 2 then follows from Theorem 3.

Now consider proper colouring in a graph with maximum degree Δ (the case $r = 2$ in the above). Then every pair (v, e) has weight 0, and $E_0(v) \leq \Delta$. Thus $c := \lceil \Delta + \beta \rceil$

satisfies (1). Theorem 3 with $\beta = 1$ says that every graph G with maximum degree Δ is $(\Delta + 1)$ -choosable. Theorem 3 with $\beta \geq 2$ says that for every $(\Delta + \beta)$ -list assignment L of G there are at least $\beta^{|V(G)|}$ L -colourings. These well-known facts are easily proved by a greedy algorithm. It is interesting that the above general framework includes such statements (the Lovász Local Lemma does not). Note that the Local Action Lemma of Bernshteyn [9] is another general-purpose tool that implies $(\Delta + 1)$ -colourability; also see [10].

See [53, 54] for results about the number of 2-colourings in random hypergraphs and about the number of k -colourings in random graphs.

3.2 Star Colouring

A colouring ϕ of a graph G is a *star colouring* if it is proper and every bichromatic subgraph is a star forest; that is, there is no 2-coloured P_4 (path on four vertices). The *star chromatic number* $\chi_{\text{st}}(G)$ is the minimum number of colours in a star colouring of G . Fertin, Raspaud, and Reed [27] proved (using the Lovász Local Lemma) that $\chi_{\text{st}}(G) \leq O(\Delta^{3/2})$ for every graph G with maximum degree Δ , and that this bound is tight up to a $O(\log \Delta)$ factor. The best known bound is $\chi_{\text{st}}(G) \leq \sqrt{8\Delta^{3/2}} + \Delta$ proved by Esperet and Parreau [26] using entropy compression. Both these methods work for star choosability. We prove the same bound holds with exponentially many colourings.

Theorem 5. *Every graph G with maximum degree Δ is star $\lceil \Delta + \sqrt{8\Delta(\Delta - 1)} \rceil$ -choosable. Moreover, for every $\lceil \Delta + \sqrt{8\Delta(\Delta - 1)} \rceil$ -list assignment L , there are at least $(\sqrt{2\Delta}(\Delta - 1))^{|V(G)|}$ star L -colourings of G .*

Proof. Define the following hypergraph G' with $V(G') = V(G)$. Introduce one edge $e = \{v, w\}$ to G' for each edge vw of G , where \mathcal{B}_e is the set of L -colourings $\phi \in \mathcal{C}_e$ such that $\phi(v) = \phi(w)$, and introduce one edge $e = \{u, v, w, x\}$ to G' for each P_4 subgraph (u, v, w, x) of G , where \mathcal{B}_e is the set of L -colourings $\phi \in \mathcal{C}_e$ such that $\phi(u) = \phi(w)$ and $\phi(v) = \phi(x)$. For any list assignment L of G , note that G is star L -colourable if and only if $P(G', \mathcal{B}, L) \geq 1$. Also, the weight of each 2-element edge is 0, and the weight of each 4-element edge is 1. Thus $E_0(v) \leq \Delta$ and $E_1(v) \leq 2\Delta(\Delta - 1)^2$. Since (1) is satisfied with $\beta := \sqrt{2\Delta}(\Delta - 1)$ and $c := \lceil \Delta + \sqrt{8\Delta(\Delta - 1)} \rceil$, the result follows from Theorem 3. \square

3.3 Nonrepetitive Graph Colouring

Let ϕ be a colouring of a graph G . A path (v_1, \dots, v_{2t}) in G is *repetitively coloured* by ϕ if $\phi(v_i) = \phi(v_{t+i})$ for each $i \in \{1, \dots, t\}$. A colouring ϕ of G is *nonrepetitive* if no path in G is repetitively coloured by ϕ . The *nonrepetitive chromatic number* $\pi(G)$ is the minimum number of colours in a nonrepetitive colouring of G . The *nonrepetitive choice number* $\pi_{\text{ch}}(G)$ is the minimum integer c such that G has a nonrepetitive L -colouring for every c -list assignment L of G . Alon, Grytczuk, Hałuszczak, and Riordan [4] proved that $\pi(G) \leq O(\Delta^2)$ for every graph with maximum degree Δ , and that this bound is tight up to a $O(\log \Delta)$ factor. The proof shows the same bound for π_{ch} . Several authors subsequently improved

the constant in the $O(\Delta^2)$ term: to $36\Delta^2$ by Grytczuk [36], to $16\Delta^2$ by Grytczuk [35], to $(12.2 + o(1))\Delta^2$ by Haranta and Jendroľ [37], and to $10.4\Delta^2$ by Kolipaka, Szegedy, and Xu [44]. All these proofs used the Lovász Local Lemma. Dujmović, Joret, Kozik, and Wood [19] improved the constant to 1, by showing that for every graph G with maximum degree Δ ,

$$\pi(G) \leq \Delta^2 + O(\Delta^{5/3}). \quad (3)$$

The proof of Dujmović et al. [19] uses entropy compression; see [26, 33] for refinements and simplifications to the method. Equation (3) was subsequently proved using the Local Cut Lemma of Bernshteyn [10] and using cluster-expansion [7, 12]. Most recently, Rosenfeld [58] proved (3) with exponentially many colourings. His paper inspired the present work. We now show that the result of Rosenfeld follows from our general framework. Note that all of the above results hold in the setting of choosability.

Theorem 6. *For every graph G with maximum degree Δ , if*

$$\beta := (1 + 2^{1/3}\Delta^{-1/3})(\Delta - 1)^2 \quad \text{and} \quad c := \lceil \beta + 2^{-2/3}\Delta^{5/3}(1 + 2^{1/3}\Delta^{-1/3})^2 \rceil,$$

then G is nonrepetitively c -choosable. Moreover, for every c -list assignment L of G there are at least $\beta^{|V(G)|}$ nonrepetitive L -colourings of G .

Proof. Let G' be the hypergraph with $V(G') = V(G)$, where there is an edge $V(P)$ for each path P in G of even order. Here we consider a path to be a subgraph of G , so that a path and its reverse contribute one edge to G' . For each edge e of G' corresponding to a path P in G of order $2t$, let \mathcal{B}_e be the set of L -colourings $\phi \in \mathcal{C}_e$ such that P is repetitively coloured by ϕ . Thus G is nonrepetitively L -colourable if and only if $P(G', \mathcal{B}, L) \geq 1$.

Consider an edge e of G' corresponding to a path P in G on $2t$ vertices. For each vertex v in P , any colouring $\phi \in \mathcal{B}_e$ is uniquely determined by ϕ restricted to the t vertices in the half of P not containing v . Hence (v, e) has weight $t - 1$. Every vertex of G is in at most $t\Delta(\Delta - 1)^{2t-2}$ paths on $2t$ vertices. So $E_{t-1}(v) \leq t\Delta(\Delta - 1)^{2t-2}$. Equation (1) requires

$$c \geq \beta + \sum_{t \geq 1} t\Delta(\Delta - 1)^{2t-2} \beta^{1-t}.$$

Define $\beta := (1 + \epsilon)(\Delta - 1)^2$ where $\epsilon > 0$ is defined shortly. Equation (1) requires

$$c \geq (1 + \epsilon)(\Delta - 1)^2 + \Delta \sum_{t \geq 1} t(1 + \epsilon)^{-t+1} = (1 + \epsilon)(\Delta - 1)^2 + \epsilon^{-2}(1 + \epsilon)^2\Delta.$$

Define $\epsilon := 2^{1/3}\Delta^{-1/3}$ (to approximately minimise $(1 + \epsilon)(\Delta - 1)^2 + \epsilon^{-2}(1 + \epsilon)^2\Delta$). Then (1) holds with c defined above, and the result follows from Theorem 3. \square

3.4 Frugal Colouring

For an integer $k \geq 1$, a colouring ϕ of a graph G is *k -frugal* if ϕ is proper and $|\{w \in N_G(v) : \phi(w) = i\}| \leq k$ for every vertex v of G and for every colour i , where $N_G(v)$ is the set of neighbours of v in G . A 1-frugal colouring of G corresponds to a proper colouring of G^2 .

Hind, Molloy, and Reed [41] proved that for each integer $k \geq 1$ and sufficiently large Δ , every graph with maximum degree Δ has a k -frugal colouring with $\max\{(k+1)\Delta, \frac{e^3}{k}\Delta^{1+1/k}\}$ colours. An example due to Alon shows that this upper bound is within a constant factor of optimal [41]. In particular, for all $\Delta \geq k \geq 1$, Alon constructed a graph with maximum degree at most Δ that has no k -frugal colouring with $\frac{1}{2k}\Delta^{1+1/k}$ colours. Here we improve the constant in the upper bound without assuming that Δ is sufficiently large, and with exponentially many colourings.

Theorem 7. *For all integers $\Delta > k \geq 2$, let*

$$\beta := \left((k-1)\Delta \binom{\Delta-1}{k} \right)^{1/k} \quad \text{and} \quad c := \Delta + \left\lceil \frac{k\beta}{k-1} \right\rceil.$$

Then every graph G with maximum degree Δ has a k -frugal c -colouring. Moreover, for every c -list-assignment L of G , the number of k -frugal L -colourings of G is at least $\beta^{|V(G)|}$.

Proof. Let G' be the hypergraph with $V(G') = V(G)$, where every edge of G is an edge of G' , and $\{w_1, \dots, w_{k+1}\}$ is an edge of G' for every vertex v of G and set $\{w_1, \dots, w_{k+1}\} \subseteq N_G(v)$. In the latter case, we say the edge is *centred* at v . For every edge $e = \{v, w\}$ of G' , let \mathcal{B}_e be the set of L -colourings $\phi \in \mathcal{C}_e$ such that $\phi(v) = \phi(w)$. For every edge $e = \{w_1, \dots, w_{k+1}\}$ of G' , let \mathcal{B}_e be the set of L -colourings $\phi \in \mathcal{C}_e$ such that $\phi(w_1) = \phi(w_2) = \dots = \phi(w_{k+1})$. Then a colouring of G is k -frugal if and only if it is \mathcal{B} -good.

For each edge $e = \{v, w\}$ of G' , both (v, e) and (w, e) have weight 0. Consider an edge $e = \{w_1, \dots, w_{k+1}\}$ of G' centred at v . For each $i \in \{1, \dots, k+1\}$, the pair (w_i, e) has weight $k-1$, since every colouring $\phi \in \mathcal{B}_e$ is determined by $\{w_j\}$ for any $j \neq i$.

Consider a vertex v of G . Then $E_0(v) \leq \Delta$. Now consider a pair (v, e) with non-zero weight. Then (v, e) has weight $k-1$, and $e = \{w_1, \dots, w_k, v\}$ is centred at some vertex u , for some vertices $w_1, \dots, w_k \in N_G(u) \setminus \{v\}$. There are at most Δ choices for u and at most $\binom{\Delta-1}{k}$ choices for w_1, \dots, w_k . Thus $E_{k-1}(v) \leq \Delta \binom{\Delta-1}{k}$. Hence

$$\beta + \sum_{i \geq 0} E_i(v) \beta^{-i} \leq \beta + \Delta + \Delta \binom{\Delta-1}{k} \beta^{1-k} = \Delta + \frac{k\beta}{k-1} \leq c.$$

The result follows from Theorem 3. □

Since $k(k-1)^{-1+1/k} \rightarrow 1$ and $\binom{\Delta-1}{k}^{1/k} \leq \frac{e}{k}(\Delta-1)$, Theorem 7 implies this:

Corollary 8. *As $\Delta > k \rightarrow \infty$, for every $\lceil (e+o(1))\Delta^{1+1/k}/k \rceil$ -list-assignment L of a graph G with maximum degree Δ , the number of k -frugal L -colourings of G is at least $\beta^{|V(G)|}$.*

Note that Alon's example in [41] shows that Corollary 8 is within a factor of $2e + o(1)$ of optimal.

3.5 Independent Transversals and Constrained Colourings

Consider a hypergraph G . A set $X \subseteq V(G)$ is *independent* if no edge of G is a subset of X . Consider a partition V_1, \dots, V_n of $V(G)$. A *transversal* of V_1, \dots, V_n is a set X such that

$|X \cap V_i| = 1$ for each i . Let $\ell : V(G) \rightarrow \{1, \dots, n\}$ be the function where $\ell(v) := i$ for each vertex $v \in V_i$. For $S \subseteq V(G)$, let $\ell(S) := \{\ell(v) : v \in S\}$. An edge e of G is *stretched* by V_1, \dots, V_n if $|\ell(e)| = |e|$. The following theorem provides a condition that guarantees an independent transversal.

Theorem 9. *Fix integers $r \geq 2$ and $t \geq 1$. For an r -uniform hypergraph G , let V_1, \dots, V_n be a partition of $V(G)$ such that $|V_i| \geq t$ and at most $r^{-r}(r-1)^{r-1}t^{r-1}|V_i|$ stretched edges in G intersect V_i , for each $i \in \{1, \dots, n\}$. Then there exist at least $(\frac{r-1}{r}t)^n$ independent transversals of V_1, \dots, V_n .*

Proof. Non-stretched edges do not influence whether a transversal is independent, so we may assume that every edge is stretched. We may also assume that $|V_i| = t$, since if $|V_i| > t$ and v is a vertex in V_i with maximum degree, then by removing v and its incident edges we obtain another hypergraph satisfying the assumptions. Let X be the hypergraph with $V(X) := \{1, \dots, n\}$, where for each edge $\{v_1, \dots, v_r\}$ of G there is an edge $e = \{\ell(v_1), \dots, \ell(v_r)\}$ in X . By assumption, each vertex i of X has degree at most $r^{-r}(r-1)^{r-1}t^{r-1}|V_i| = r^{-r}(r-1)^{r-1}t^r$. Let L be the list-assignment of X with $L(i) := V_i$ for each $i \in \{1, \dots, n\}$. For each edge e of X corresponding to edge $\{v_1, \dots, v_r\}$ of G , let \mathcal{B}_e be the set consisting of the L -colouring ϕ of e with $\phi(\ell(v_j)) = v_j$ for each $j \in \{1, \dots, r\}$. Thus \mathcal{B} -good L -colourings of X correspond to independent transversals of V_1, \dots, V_n . Since \mathcal{B}_e is determined by \emptyset , each pair (i, e) has weight $r-1$. Define $\beta := \frac{r-1}{r}t$. Then

$$|L(i)| = t = \beta + \frac{(r-1)^{r-1}t^r}{r^r \beta^{r-1}} \geq \beta + \frac{E_{r-1}(i)}{\beta^{r-1}}.$$

Thus (1) holds and the result follows from Theorem 3. \square

Erdős, Gyárfás, and Łuczak [23] study independent transversals in a particular family of sparse hypergraphs. They define an $[n, k, r]$ -hypergraph to be an r -uniform hypergraph G whose vertex set $V(G)$ is partitioned into n sets V_1, \dots, V_n , each with k vertices, such that every edge is stretched by V_1, \dots, V_n and for every r -element subset S of $\{1, 2, \dots, n\}$ there is exactly one edge $e \in E(G)$ such that $\ell(e) = S$. Erdős et al. [23] defined $f_r(k)$ to be the maximum integer n such that every $[n, k, r]$ -hypergraph has an independent transversal. Using the Lovász Local Lemma, they proved that if

$$e \left(\binom{n}{r} - \binom{n-r}{r} \right) < k^r, \quad (4)$$

then $f_r(k) \geq n$. Observe that for every $[n, k, r]$ -hypergraph G with partition V_1, \dots, V_n , for each $i \in \{1, \dots, n\}$, exactly $\binom{n-1}{r-1}$ edges of G intersect V_i . Thus Theorem 9 implies that if

$$\binom{n-1}{r-1} \leq \frac{(r-1)^{r-1}k^r}{r^r}, \quad (5)$$

then $f_r(k) \geq n$. We now compare these last two results. Consider r to be fixed. As k grows, the largest n satisfying (4) or (5) also grows, so we can think of n being large relative to r .

Then

$$\begin{aligned}
& \frac{(r-1)^{r-1} \left[\binom{n}{r} - \binom{n-r}{r} \right]}{r^r \binom{n-1}{r-1}} \\
&= \left(\frac{r-1}{r} \right)^{r-1} \frac{n}{r^2} \left[1 - \frac{(n-r)!^2}{n!(n-2r)!} \right] \\
&= \left(\frac{r-1}{r} \right)^{r-1} \frac{n}{r^2} \left[1 - \prod_{i=0}^{r-1} \frac{n-r-i}{n-i} \right] \\
&\geq \left(\frac{r-1}{r} \right)^{r-1} \frac{n}{r^2} \left[1 - \left(\frac{n-r}{n} \right)^r \right] \\
&= \left(\frac{r-1}{r} \right)^{r-1} \left[1 - \binom{r}{2} \frac{1}{n} + \frac{n}{r^2} \sum_{i=2}^{\lceil r/2 \rceil} \left(\binom{r}{2i-1} \left(\frac{r}{n} \right)^{2i-1} - \binom{r}{2i} \left(\frac{r}{n} \right)^{2i} \right) \right] \\
&\geq \left(\frac{r-1}{r} \right)^{r-1} \left[1 - \frac{r^2}{2n} + \frac{n}{r^2} \sum_{i=2}^{\lceil r/2 \rceil} \binom{r}{2i-1} \left(\frac{r}{n} \right)^{2i-1} \left(1 - \frac{(r-2i+1)r}{2in} \right) \right] \\
&\geq \left(\frac{r-1}{r} \right)^{r-1} \left[1 - \frac{r^2}{2n} \right] \tag{6}
\end{aligned}$$

if $n \geq r^2/4$. Also $(1 - 1/r)^{r-1} > 1/e$. Hence, if n is sufficiently large relative to r , then (6) will exceed $1/e$, and (5) implies (4). In other words, our bound on $f_r(k)$ is better when k is sufficiently large relative to r . Yuster [63, 64] used a different argument to get a better bound in the case of graphs ($r = 2$).

Theorem 9 in the case of graphs says:

Corollary 10. *Fix an integer $t \geq 1$. For a graph G , let V_1, \dots, V_n be a partition of $V(G)$ such that $|V_i| \geq t$ and there are at most $\frac{t}{4}|V_i|$ edges in G with exactly one endpoint in V_i , for each $i \in \{1, \dots, n\}$. Then there exist at least $(\frac{t}{2})^n$ independent transversals of V_1, \dots, V_n .*

Corollary 10 immediately implies the following result (since the average degree out of V_i is at most the maximum degree).

Corollary 11. *For a graph G with maximum degree at most Δ , if V_1, \dots, V_n is a partition of $V(G)$ such that $|V_i| \geq 4\Delta$ for each $i \in \{1, \dots, n\}$, then there exist at least $(2\Delta)^n$ independent transversals of V_1, \dots, V_n .*

We now compare Corollaries 10 and 11 with the literature. Reed and Wood [57] proved the weakening of Corollary 10 with $\frac{t}{4}$ replaced by $\frac{t}{2e}$ and with $(\frac{t}{2})^n$ replaced by 1, and Dvořák, Esperet, Kang, and Ozeki [20] noted that Corollary 10 holds with $(\frac{t}{2})^n$ replaced by 1 (using different terminology). Similarly, Alon [3] proved the weakening of Corollary 11 with 4Δ replaced by $2e\Delta$ and with $(2\Delta)^n$ replaced by 1. The proofs of Reed and Wood [57] and Alon [3] used the Lovász Local Lemma, while the proof of Dvořák et al. [20] used the Local Cut Lemma. Using a different method, Haxell [40] proved the strengthening of Corollary 11 with 4Δ replaced by 2Δ , but with $(2\Delta)^n$ replaced by 1. The bound here of 2Δ is best possible [15, 64]. It is open whether $\frac{t}{4}$ in Corollary 10 can be improved to $\frac{t}{2}$; see [42]. See [32, 45, 63] for more on independent transversals in graphs.

These results are related to the following ‘constrained colouring’ conjecture of Reed [55]:

Conjecture 12 ([55]). *Let L be a $(k+1)$ -list assignment of a graph G such that for each vertex v of G and colour $c \in L(v)$, there are at most k neighbours $w \in N_G(v)$ with $c \in L(w)$. Then there exists a proper L -colouring of G .*

Haxell [40] observed the following connection between constrained colourings and independent transversals. Consider an $f(k)$ -list-assignment L of a graph G . Let H be the graph with $V(H) := \{(v, c) : v \in V(G), c \in L(v)\}$, where $(v, c)(w, c) \in E(H)$ for each edge $vw \in E(G)$ and colour $c \in L(v) \cap L(w)$. Let $H_v := \{(v, c) : c \in L(v)\}$. Then $(H_v : v \in V(G))$ is a partition of H with each $|H_v| \geq f(k)$ such that proper L -colourings of G correspond to independent transversals of $(H_v : v \in V(G))$. Now if we assume that for each vertex v and colour $c \in L(v)$ there are at most k neighbours $w \in N_G(v)$ with $c \in L(w)$, then H has maximum degree at most k . Hence the above-mentioned result of Alon [3] proves Conjecture 12 with $k+1$ replaced by $2ek$ (also proved by Reed [55]), and the above-mentioned result of Haxell [40] proves Conjecture 12 with $k+1$ replaced by $2k$. Bohman and Holzman [14] disproved Conjecture 12. The best asymptotic result, due to Reed and Sudakov [56], says that for each $\epsilon > 0$ there exists k_0 such that Conjecture 12 holds with $k+1$ replaced by $(1+\epsilon)k$ for all $k \geq k_0$. None of these results conclude that there are exponentially many colourings. Corollary 10 and the above connection by Haxell [40] implies the following result:

Corollary 13. *Fix an integer $t \geq 2$. Let L be a t -list assignment of a graph G such that for each vertex v of G ,*

$$4 \sum_{w \in N_G(v)} |L(v) \cap L(w)| \leq t^2.$$

Then there exist at least $(\frac{t}{2})^{|V(G)|}$ proper L -colourings of G .

Taking $t = 4k$ we obtain the following result in the direction of Conjecture 12:

Corollary 14. *Let L be a $4k$ -list assignment of a graph G such that for each vertex v of G and colour $c \in L(v)$, there are at most k neighbours $w \in N_G(v)$ such that $c \in L(w)$. Then there exist at least $(2k)^{|V(G)|}$ proper L -colourings of G .*

The following stronger result can also be proved using a variant of Theorem 3.

Theorem 15. *Let L be a list-assignment of a graph G such that for every vertex v of G ,*

$$|L(v)| \geq 4 \sum_{w \in N_G(v)} \frac{|L(v) \cap L(w)|}{|L(w)|}. \quad (7)$$

Then there exist at least $\prod_{v \in V(G)} \frac{|L(v)|}{2}$ proper L -colourings.

Proof. We proceed by induction on $|V(H)|$ with the following hypothesis: for every induced subgraph H of G , and for every vertex v of H ,

$$P(H, L) \geq \frac{|L(v)|}{2} P(H - v, L).$$

(The proof is very similar to that of Lemma 4 except that β depends on v ; in particular, $\beta_v = \frac{|L(v)|}{2}$.) The base case with $|V(H)| = 1$ is trivial. Let H be an induced subgraph of G , and assume the claim holds for all induced subgraphs of G with less than $|V(H)|$ vertices. Let v be any vertex of H . Let X be the set of improper L -colourings of H that are proper on $H - v$. Then

$$P(H, L) = |L(v)| P(H - v, L) - |X|. \quad (8)$$

We now find an upper bound for $|X|$. For $w \in N_G(v)$, let X_w be the set of colourings ϕ in X such that $\phi(v) = \phi(w)$. Each L -colouring in X is in some X_w . Thus

$$|X| \leq \sum_{w \in N_G(v)} |X_w| \leq \sum_{w \in N_G(v)} P(H - v - w, L) |L(v) \cap L(w)|.$$

By induction, $P(H - v, L) \geq \frac{|L(w)|}{2} P(H - v - w, L)$. Hence

$$|X| \leq \sum_{w \in N_G(v)} \frac{2|L(v) \cap L(w)|}{|L(w)|} P(H - v, L).$$

By (8)

$$P(H, L) \geq |L(v)| P(H - v, L) - \sum_{w \in N_G(v)} \frac{2|L(v) \cap L(w)|}{|L(w)|} P(H - v, L).$$

By (7), $P(H, L) \geq \frac{|L(v)|}{2} P(H - v, L)$, as desired. \square

Note that Theorem 15 immediately implies Corollary 13, taking $|L(v)| = 2t$ for each v .

3.6 Ramsey Numbers

For integers $k, c \geq 2$, let $R_c(k)$ be the minimum integer n such that every edge c -colouring of K_n contains a monochromatic K_k . Ramsey [52] and Erdős and Szekeres [24] independently proved that $R_c(k)$ exists. The best asymptotic lower bound on $R_2(k)$ is due to Spencer [59, 60] who proved that

$$R_2(k) \geq \left(\frac{\sqrt{2}}{e} - o(1) \right) k 2^{k/2}. \quad (9)$$

More precisely, Spencer [59, 60] proved that if

$$e \binom{k}{2} \left(\binom{n-2}{k-2} + 1 \right) < 2^{\binom{k}{2}-1}, \quad (10)$$

then there exists an edge 2-colouring of K_n with no monochromatic K_k , implying $R_2(k) > n$. Theorem 3 leads to an analogous result with the same asymptotics, but with slightly better lower order terms. For a graph G and integer $k \geq 2$, let $D_k(G)$ be the maximum, taken over all edges $vw \in E(G)$, of the number of k -cliques in G containing v and w .

Theorem 16. Fix integers $k \geq 3$ and $c \geq 2$. Let $m := \binom{k}{2} - 1$. Then for every graph G with

$$D_k(G) \leq \frac{(m-1)^{m-1} c^m}{m^m}, \quad (11)$$

there exists an edge c -colouring of G with no monochromatic K_k . In fact, there exists at least $(D_k(G)(m-1))^{|E(G)|/m}$ such colourings.

Proof. Let G' be the hypergraph with $V(G') := E(G)$, where $S \subseteq E(G)$ is an edge of G' whenever S is the edge-set of a K_k subgraph in G . For each edge vw of G , let $L(vw) := \{1, \dots, c\}$. For each edge S of G' , let \mathcal{B}_S be the set of monochromatic L -colourings of S . Thus \mathcal{B} -good L -colourings of G' correspond to edge c -colourings of G with no monochromatic K_k . Each pair (v, e) has weight $m-1$, and $E_{m-1}(v) \leq D_k(G)$. Thus (1) holds if

$$c \geq \beta + D_k(G) \beta^{1-m}. \quad (12)$$

To minimise the right-hand side of this expression, define $\beta := (D_k(G)(m-1))^{1/m}$. Then (11) implies (12), so the result follows from Theorem 3. \square

Applying Theorem 16 to a complete graph gives the following corollary.

Corollary 17. For every integer $k \geq 3$ and $c \geq 2$, if $m := \binom{k}{2} - 1$ and

$$\frac{m^m}{(m-1)^{m-1}} \binom{n-2}{k-2} \leq c^m$$

then there exists an edge c -colouring of K_n with no monochromatic K_k , and $R_c(k) > n$.

Since $\frac{m^m}{(m-1)^{m-1}} < em = e(\binom{k}{2} - 1)$, Corollary 17 is slightly stronger than (10). While this improvement only changes the implicit lower order term in (9), we consider it to be of interest, since it suggests a new approach for proving lower bounds on $R_c(k)$.

3.7 k -SAT

The k -SAT problem takes as input a Boolean formula ψ in conjunctive normal form, where each clause has exactly k distinct literals, and asks whether there is a satisfying truth assignment for ψ . The Lovász Local Lemma proves that if each variable is in at most $\frac{2^k}{ke}$ clauses, then there exists a satisfying truth assignment; see [30] for a thorough discussion of this topic. The following result (slightly) improves upon this bound (since $(\frac{k-1}{k})^{k-1} > \frac{1}{e}$), and moreover, guarantees exponentially many truth assignments.

Theorem 18. Let ψ be a Boolean formula in conjunctive normal form, with variables v_1, \dots, v_n and clauses c_1, \dots, c_m , each with exactly k literals. Assume that each variable is in at most $\Delta := \frac{2^k}{k} \left(\frac{k-1}{k}\right)^{k-1}$ clauses. Then there exists a satisfying truth assignment for ψ . In fact, there are at least $(2 - \frac{2}{k})^n$ such truth assignments.

Proof. Let G be the hypergraph with $V(G) = \{v_1, \dots, v_n\}$ and $E(G) = \{e_1, \dots, e_m\}$, where edge e_i consists of those variables in clause c_i . So G is k -uniform. Let $L(v_i) = \{0, 1\}$

for each vertex v_i . Let \mathcal{B}_{e_i} be the set of L -colourings of e_i such that c_i is not satisfied. Satisfying truth assignments for ψ correspond to \mathcal{B} -good L -colourings of G . Each pair (v, e) has weight $k - 1$. Thus $E_{k-1}(v) \leq \Delta$ and $E_i(v) = 0$ for all $i \neq k - 1$. Then (1) holds with $\beta := 2 - \frac{2}{k}$ and $c := 2$. The result follows from Theorem 3. \square

Note that Gebauer, Szabó, and Tardos [31] proved that if each variable is in at most $(1 - o(1))\frac{2^{k+1}}{ke}$ clauses, then there exists a satisfying truth assignment, and that this bound is best possible up to the $o(1)$ term; see Harris [38] for further improvements. These results improve upon the bound in Theorem 18 by a factor of 2. However, Theorem 18 may still be of interest since it gives exponentially many satisfying assignments and is an immediate corollary of our general framework.

See [17] for bounds on the number of satisfying truth assignments in random k -SAT formulas.

4 Reflection

We now reflect on Theorem 3, which provides a general framework for colouring hypergraphs of bounded degree.

First we discuss minimising the number of colours in Theorem 3. To do so, one needs to minimise the right hand side of (1), which is a Laurent series $Q(\beta)$ with nonnegative integer coefficients. We assume that at least one edge has positive weight, since otherwise $Q(\beta)$ is linear. We also assume that the coefficients in $Q(\beta)$ grow slowly enough that it and its first two derivatives converge for all $\beta > R$ for some real number R . For example, when the weight of edges is bounded (which is true in every example in this paper outside of Section 3.3), we are optimising a Laurent polynomial, and may take $R = 0$. Now, $Q''(\beta) > 0$ for all $\beta > R$, so we expect a unique minimum for $Q(\beta)$ on the interval $[R, \infty)$, say at $\beta = \beta_0$. Since $Q'(1) \leq 0$ (or $R > 1$), we must have $\beta_0 \geq 1$. Even using a value of $\beta \neq \beta_0$, one still obtains a non-trivial result from Theorem 3. In fact, choosing $\beta > \beta_0$ may be desirable if one wants to find conditions under which there are more colourings than are guaranteed by taking $\beta = \beta_0$.

Compared with the Lovász Local Lemma, Theorem 3 has the advantage of directly proving the existence of exponentially many colourings, and often gives slightly better bounds. The proof of Theorem 3 is elementary, and as discussed above, (1) is often easier to optimise than the General Lovász Local Lemma.

Theorem 3 should also be compared with entropy compression, which is a method that arose from the algorithmic proof of the Lovász Local Lemma due to Moser and Tardos [47]. See [11, 18, 19, 26, 34] for examples of the use of entropy compression in the context of graph colouring. We expect that the results in Section 3 can be proved using entropy compression. For example, see [33, Theorem 12] for a generic graph colouring lemma in a similar spirit to our Theorem 3 that is proved using entropy compression. However, we consider the proof of Theorem 3 and the proofs of results that apply Theorem 3 to be simpler than their entropy compression counterparts, which require non-trivial analytic techniques from enumerative combinatorics. On the other hand, entropy compression has

the advantage that it provides an explicit algorithm to compute the desired colouring, often with polynomial expected time complexity.

It is also likely that our results in Section 3 can be proved using the Local Cut Lemma [10] or via cluster expansion [12]. The advantage of Theorem 3 is the simplicity and elementary nature of its proof. See [6, 28] for results connecting the Lovász Local Lemma, entropy compression, and cluster expansion.

Finally, we mention a technical advantage of the Lovász Local Lemma and of entropy compression. In the setting of hypergraph colouring, the Lovász Local Lemma and entropy compression need only bound the number of edges that intersect a given edge, whereas Theorem 3 requires a bound on the number of edges that contain a given vertex (because the proof is by induction on the number of vertices).

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