Hypergraph Colouring and Degeneracy

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Abstract

A hypergraph is *d-degenerate* if every subhypergraph has a vertex of degree at most d. A greedy algorithm colours every such hypergraph with at most $d+1$ colours. We show that this bound is tight, by constructing an r-uniform d-degenerate hypergraph with chromatic number $d+1$ for all $r \geq 2$ and $d \geq 1$. Moreover, the hypergraph is triangle-free, where a *triangle* in an r-uniform hypergraph consists of three edges whose union is a set of $r+1$ vertices.

1 Introduction

Erdős and Lovász [7] proved the following fundamental result about colouring hypergraphs[1]

Theorem 1 ([7]). *For fixed r, every r-uniform hypergraph with maximum degree Δ has chromatic number at most $O(\Delta^{1/(r-1)})$.*

Theorem 1 implies that every r-uniform hypergraph with maximum degree Δ has an independent set of size at least $\Omega(n/\Delta^{1/(r-1)})$. Spencer [10] proved the following stronger bound.

Theorem 2 ([10]). *For fixed r, every r-uniform hypergraph with n vertices and average degree d has an independent set of size at least $\Omega(n/d^{1/(r-1)})$.*

A hypergraph is *d-degenerate* if every subhypergraph has a vertex of degree at most d. A minimum-degree-greedy algorithm colours every d-degenerate

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[1] A *hypergraph* G consists of a set $V(G)$ of *vertices* and a set $E(G)$ of subsets of $V(G)$ called *edges*. A hypergraph is *r-uniform* if every edge has size r. A *graph* is a 2-uniform hypergraph. A hypergraph H is a *subhypergraph* of a hypergraph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A *colouring* of a hypergraph G assigns one colour to each vertex in $V(G)$ such that no edge in $E(G)$ is monochromatic. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number of colours in a colouring of G. A colouring of G can be thought of as a partition of $V(G)$ into *independent sets*, each containing no edge. The *degree* of a vertex v is the number of edges that contain v. See the textbook of Berge [3] for other notions of degree in a hypergraph.

hypergraph with at most $d+1$ colours. This bound is tight for graphs ($r=2$) since the complete graph on $d+1$ vertices is d-degenerate, and of course, has chromatic number $d+1$. However, this observation does not generalise for $r \geq 3$. In particular, for the complete r-uniform hypergraph on n vertices, every vertex has degree $\binom{n-1}{r-1}$, yet the chromatic number is $\lceil \frac{n}{r-1} \rceil$. Thus for $r \geq 3$, the degeneracy is much greater than the chromatic number.

Given Theorems 1 and 2, it seems plausible that for $r \geq 3$, every r-uniform d-degenerate hypergraph is $o(d)$-colourable. It even seems possible that every r-uniform d-degenerate hypergraph is $O(d^{1/(r-1)})$-colourable. This natural strengthening of Theorems 1 and 2 would (roughly) say that G can be partitioned into independent sets, whose average size is that guaranteed by Theorem 2.

This note rules out these possibilities, by showing that the naive upper bound $\chi \leq d+1$ is tight for all r. This is the main conclusion of this paper. Moreover, we prove it for triangle-free hypergraphs, where a *triangle* in an r-uniform hypergraph consists of three edges whose union is a set of $r+1$ vertices. Observe that this definition with $r=2$ is equivalent to the standard notion of a triangle in a graph (although there are other notions of a triangle in a hypergraph [4]).

Theorem 3. *For all $r \geq 2$ and $d \geq 1$ there is a triangle-free d-degenerate r-uniform hypergraph with chromatic number $d+1$.*

Theorem 3 and its proof is a generalisation of a result of Alon et al. [2] who proved it for graphs ($r=2$). Of course, the complete graph K_{d+1} is d-degenerate with chromatic number $d+1$. The triangle-free property was the main conclusion of their result. See [1, 9] for other related results.

2 Proof

Theorem 3 is a corollary of the following:

Lemma 4. *Fix $r \geq 2$. For all $d \geq 1$ there is a triangle-free d-degenerate r-uniform hypergraph G_d with chromatic number $d+1$, such that in every $(d+1)$-colouring of G_d each colour is assigned to at least $r-1$ vertices.*

Proof. We proceed by induction on d. First consider the base case $d=1$. Let $n := r(r-1)$. Let $V(G_1) := \{v_1, \dots, v_n\}$ and $E(G_1) := \{e_i : 1 \leq i \leq n-r+1\}$, where $e_i := \{v_i, v_{i+1}, \dots, v_{i+r-1}\}$. If $S \subseteq V(G_1)$ and i is minimum such that $v_i \in S$, then v_i has degree at most 1 in the subhypergraph induced by S. Thus G_1 is 1-degenerate. If e_i, e_j, e_k are three edges in G_1 with $i<j<k$, then $e_i \cup e_j \cup e_k$ includes the $r+2$ distinct vertices $v_i, v_{i+1}, \dots, v_{i+r-1}, v_{j+r-1}, v_{k+r-1}$. Hence G_1 is triangle-free. Consider a 2-colouring of G_1. Clearly, G_1 contains $r-1$ pairwise disjoint edges, each of which contains vertices of both colours. Hence each colour is assigned to at least $r-1$ vertices. This completes the base case.

Now assume that G_{d-1} is a triangle-free $(d-1)$-degenerate r-uniform hypergraph with chromatic number d, such that in every d-colouring of G_{d-1} each colour is assigned to at least $r-1$ vertices.

Initialise G_d to consist of $d+r-2$ disjoint copies $H_1, \dots, H_{d+r-2}$ of G_{d-1}. Let S be a set of $(r-1)d$ vertices in $H_1 \cup \dots \cup H_{d+r-2}$ such that $|S \cap V(H_i)| \in \{0, r-1\}$ for $1 \leq i \leq d+r-2$. That is, S contains exactly $r-1$ vertices from exactly d of the H_i, and contains no vertices from the other $r-2$. Now, for each such set S, add $r-1$ *new* vertices $v_1, \dots, v_{r-1}$ to G_d and add the *new* edge $(S \cap V(H_i)) \cup \{v_j\}$ to G_d whenever $|S \cap V(H_i)| = r-1$. Thus each new vertex has degree d. Since $H_1 \cup \dots \cup H_{d+r-2}$ is d-degenerate, G_d is also d-degenerate.

Suppose on the contrary that G_d contains a triangle T. Since G_{d-1} is triangle-free, at least one edge in T is a new edge, which is contained in $V(H_i) \cup \{v\}$ for some $i \in [1, d+r-2]$ and some new vertex v. Each vertex in a triangle is in at least two of the edges of the triangle. However, by construction, v is contained in only one edge contained in $V(H_i) \cup \{v\}$. Thus G_d is triangle-free.

Since $H_1 \cup \dots \cup H_{d+r-2}$ is d-colourable, and no edge contains only new vertices, assigning all the new vertices a $(d+1)$-th colour produces a $(d+1)$-colouring of G_d. Thus $\chi(G_d) \leq d+1$.

Suppose on the contrary that G_d has a $(d+1)$-colouring with at most $r-2$ vertices of some colour, say 'blue'. Say the other colours are $1, \dots, d$. At most $r-2$ copies of the H_i contain blue vertices. Hence, without loss of generality, $H_1, \dots, H_d$ contain no blue vertices. That is, $H_1, \dots, H_d$ are d-coloured with colours $1, \dots, d$. By induction, H_i contains a set S_i of $r-1$ vertices coloured i for $1 \leq i \leq d$. By construction, there are $r-1$ vertices $v_1, \dots, v_{r-1}$ in G_d, such that $S_i \cup \{v_j\}$ is an edge of G_d for $1 \leq i \leq d$ and $1 \leq j \leq r-1$. Since each such edge is not monochromatic, each vertex v_j is coloured blue. In particular, there are at least $r-1$ blue vertices, which is a contradiction. Therefore, in every $(d+1)$-colouring of G_d, each colour class has at least $r-1$ vertices, as claimed. (In particular, G_d has no d-colouring.) □

3 An Open Problem

We conclude with an open problem. The *girth* of a graph (that contains some cycle) is the length of its shortest cycle. Erdős [5] proved that there exists a graph with chromatic number at least k and girth at least g, for all $k \geq 3$ and $g \geq 4$. (Erdős and Hajnal [6] proved an analogous result for hypergraphs). Theorem 3 strengthens this result for triangle-free graphs (that is, with girth $g = 4$). This leads to the following question: Does there exist a d-degenerate graph with chromatic number $d+1$ and girth g, for all $d \geq 2$ and $g \geq 4$? Odd cycles prove the $d = 2$ case. An affirmative answer would strengthen the above result of Erdős [5]. A negative answer would also be interesting—this would provide a non-trivial upper bound on the chromatic number of d-degenerate graphs with girth g.

Note

After this paper was written the author discovered the beautiful paper by Kostochka and Nešetřil [8] which proves a strengthening of Theorem 3 and includes the positive solution of the above open problem.

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